KKL observer design for non observable systems

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Contents

Standard observation problem

We consider a nonlinear dynamical system

$$\dot{x} = f(x, u) \quad , \quad y = h(x, u) \tag{1}$$

with state x in $\mathbb{R}^{d_x}$, input $u : \mathbb{R} \to \mathbb{R}^{d_u}$, output $y : \mathbb{R} \to \mathbb{R}^{d_y}$.

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Observation problem

Whatever u in $\mathcal{U}$,
whatever the initial condition $x_0 \in \mathcal{X}_0$,
find an estimate $\hat{x}(t)$ of $x(t)$ at each time $t \geq 0$, based on $u_{[0,t]}$ and $y_{[0,t]}$ and such that

$$\lim_{t \to +\infty} \hat{x}(t) - x(t) = 0 \ .$$

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=> Observer design

Luenberger design : history

In *Observing the state of a linear system*, D. Luenberger, 1964

x-coordinates in $\mathbb{R}^{d_x}$ $\qquad$ ξ-coordinates in $\mathbb{R}^{d_\xi}$

$$\begin{array}{rcl} \dot{x} & = & F\,x \\ y & = & H\,x \end{array} \quad \xrightarrow{\quad T \quad} \quad \dot{\xi} = A\,\xi + B\,y$$

A Hurwitz

$$\hat{x} = \ ? \quad \xleftarrow{\quad T^{-1} \quad} \quad \dot{\hat{\xi}} = A\,\hat{\xi} + B\,y$$

We have

$$\lim \hat{\xi} - \xi = 0$$

and if T invertible,

$$\lim \hat{x} - x = 0$$

Goal : find T invertible such that $\xi = Tx$ verifies $\dot{\xi} = A\xi + B\,y$ with A Hurwitz

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- unique solution if F and A have no common eigenvalue
- invertible if (F, H) observable and (A, B) controllable

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=> T must verify the "Luenberger equation"

$$TF = AT + BH$$

- unique solution if F and A have no common eigenvalue
- invertible if (F, H) observable and (A, B) controllable

=> **Luenberger observer** :

$$\dot{\hat{\xi}} = A\hat{\xi} + B\,y \qquad , \qquad \hat{x} = T^{-1}\hat{\xi}$$

$$\iff$$

$$\dot{\hat{x}} = F\,\hat{x} + T^{-1}B(y - H\hat{x})$$

Extension to nonlinear systems ?

Local analysis :

A. Shoshitaishvili, *On control branching systems with degenerate linearization*, NOLCOS, 1992

N. Kazantzis, C. Kravaris, *Nonlinear observer design using Lyapunov's auxiliary theorem*, SCL, 1998

A. Krener and M. Xiao, *Nonlinear observer design in the Siegel domain*, SIAM, 2003

Global analysis (autonomous) :

G. Kreisselmeier, R. Engel, *Nonlinear observers for autonomous Lipschitz continuous systems*, TAC, 2003

V. Andrieu, L. Praly, *On the existence of a Kazantzis–Kravaris/Luenberger observer*, SIAM, 2006

V. Andrieu, *Convergence speed of nonlinear Luenberger observers*, SIAM, 2014

Global analysis (time-varying) :

R. Engel, *Nonlinear observers for Lipschitz continuous systems with inputs* , IJC, 2007

P. Bernard, V. Andrieu, *Luenberger observers for non autonomous nonlinear systems*, TAC, 2019

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A Hurwitz

Theorem 1 (KKL observer, general statement).

Let u such that the trajectories of interest remain in $\mathcal{X}$ and the system is backward-distinguishable on $\mathcal{X}$.

Then, there exists a set S of zero-Lebesgue measure in $\mathbb{C}^{d_x+1}$ such that for any diagonalizable Hurwitz matrix $A_0 \in \mathbb{C}^{(d_x+1)\times(d_x+1)}$ and any $B_0 \in \mathbb{C}^{d_x+1}$ such that $\mathrm{eig}(A_0) \notin S$ and (A_0, B_0) controllable,

there exists T_u such that

- *$\xi(t) = T_u(x(t),t)$ is solution to $\dot{\xi} = A\xi + By$ with $A = A_0 \otimes I_{d_y}$, $B = B_0 \otimes I_{d_y}$*
- *$x \mapsto T_u(x,t)$ becomes **injective** on $\mathcal{X}$, at least after some time.*

1. Introduction
2. Non observable system
   - Sensorless PMSM with unknown resistance
   - Indistinguishable trajectories
   - KKL observer design
3. Functional KKL observer
   - Existence of T
   - Injectivity of T with respect to q
   - Left-inverse and convergence
4. KKL and indistinguishability

Contents

Sensorless PMSM with unknown resistance

PMSM model in a fixed $\alpha\beta$-frame :

$$\dot{\Psi} = u - R\, i$$

where

- $\Psi \in \mathbb{R}^2$: total flux generated by the windings and the permanent magnet
- $u \in \mathbb{R}^2$, $i \in \mathbb{R}^2$: voltage and intensity of the current in the fixed frame
- R : stator winding resistance

For a non-salient PMSM :

$$\Psi = Li + \Phi \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

where L is the inductance, Φ the magnet's flux, and θ the electrical phase.

"Sensorless" PMSM model :

$$\dot{\Psi} = u - R\,i$$

$$\Psi = Li + \Phi \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \iff \begin{cases} |\Psi - Li|^2 - \Phi^2 = 0 \\ \theta = \arg(\Psi - Li) \end{cases}$$

Measured input signals
Parameters assumed known

Goal : estimate θ with R unknown $\iff$ estimate Ψ with R unknown

Model :

$$\begin{cases} \dot{\Psi} &= u - R\,i \\ \dot{R} &= 0 \end{cases} \quad , \quad y = |\Psi - Li|^2 - \Phi^2 = 0 \qquad \text{"virtual measurement"}$$

Observability analysis

For inputs (u, i), the model is

$$\begin{cases} \dot{x} = u - x_3\, i \\ \dot{x}_3 = 0 \\ \\ y = 0 = x^\top x - 2Li^\top x + L^2|i|^2 - \Phi^2 \end{cases}$$

with state $x = (x_1, x_2) \in \mathbb{R}^2$ and $x_3 \in \mathbb{R}$.

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Let (Ψ, R) a particular solution, with the corresponding θ such that

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and $\omega := \dot{\theta}$.

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and $\omega := \dot{\theta}$.

Denote $i_{dq} = (i_d, i_q) = \mathcal{R}(-\theta) i_{dq}$ and $u_{dq} = (u_d, u_q) = \mathcal{R}(-\theta) u_{dq}$.

Does there exist another solution $t \mapsto (x(t), x_3(t))$ giving the same output ?

Theorem 2 (Finite number of indistinguishable trajectories).

There exist at most 6 indistinguishable solutions, unless

a) $\omega(t) = 0$ *for all* t,

or

b) *there exists* t *such that* $\omega(t) \neq 0$, *but for all* t, $i_d(t) = 0$, *and* $\frac{\omega}{i_q}$ *is constant,*

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In particular, at each time t, how many solutions in (x, x_3) may the following equation have

$$H_3(x, x_3, t) = 0 \ = \ \left(h(x,t) \ , \ \dot{\overbrace{h(x,t)}} \ , \ \ddot{\overbrace{h(x,t)}} \right) \ ?$$

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If $\omega(t) \neq 0$ and $i_d(t) \neq 0$, there are as many solutions (x, x_3) as the number of distinct real roots of a polynomial of degree six :

$$P(x_3, t) = \omega(t)^6 \Phi^6 \left[\left(\mathbf{1} - \frac{(R - x_3)}{\omega(t)\Phi} \left(\dot{\overbrace{\left(\frac{i_d}{\omega}\right)}}(t) - 2 i_q(t) \right) + \frac{(R - x_3)^2}{\omega(t)^2 \Phi^2} \mu(t) |i(t)|^2 \right)^2 \right.$$
$$\left. - \left(\mathbf{1} + \frac{(R - x_3)}{\omega(t)\Phi} 2 i_q + \frac{(R - x_3)^2}{\omega(t)^2 \Phi^2} |i(t)|^2 \right)^3 \right]$$

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$$\left. - \left(1 + \frac{(R - x_3)}{\omega(t)\Phi} 2 i_q + \frac{(R - x_3)^2}{\omega(t)^2 \Phi^2} |i(t)|^2 \right)^3 \right]$$

If besides $i_q(t) \neq -\dot{\overbrace{\left(\frac{i_d}{\omega}\right)}}(t)$, then $P(x_3, t)$ admits at least two distinct real roots.

Particular case : ω, i_d and i_q constant

Theorem 3 (Case of constant ω, i_d and i_q).

Assume $\omega \neq 0$ and $i_d \neq 0$.

There exist exactly two indistinguishable solutions (x, x_3), of the form (Ψ, R) and (Ψ_δ, R_δ) with

$$R_\delta = R + \frac{2\Phi\omega\, i_q}{|i|^2} \quad , \quad i_{d,\delta} = i_d \quad , \quad i_{q,\delta} = -i_q \ ,$$
$$\omega_\delta = \dot\theta_\delta = \dot\theta = \omega \ .$$

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$$\omega_\delta = \dot{\theta}_\delta = \dot{\theta} = \omega \ .$$

Besides, if $(\hat{R}, \hat{\theta})$ is one of the solutions (R, θ) or $(R_\delta, \theta_\delta)$, then the other solution is

$$\left(\hat{R} + \frac{2\Phi\hat{\omega}\,\widehat{i_q}}{|i|^2} \ , \ \hat{\theta} + \arctan_2\left(2\frac{\widehat{i_q}\,\widehat{i_d}}{|i|^2}, 1 - 2\frac{\widehat{i_q}^{\,2}}{|i|^2} \right) \right) \ .$$

Observer design

Find a transformation $(x, x_3, t) \mapsto T_\lambda(x, x_3, t)$ mapping

$$\left\{ \begin{array}{rcl} \dot{x} & = & u - x_3\, i \\ \dot{x}_3 & = & 0 \end{array} \right.$$

into a Hurwitz form

$$\dot{\xi}_\lambda = -\lambda \xi_\lambda + y \qquad , \quad y = x^\top x - 2Li^\top x + L^2|i|^2 - \Phi^2$$

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We find

$$T_\lambda(x, x_3, t) = \lambda^2\, x^\top x + \lambda\, c_\lambda(t)^\top x + \lambda\, x_3\, b_\lambda(t)^\top x + a_\lambda(t) x_3 + d_\lambda(t) x_3^2 - e_\lambda$$

with

$$\begin{array}{rcl} \dot{a_\lambda} & = & -\lambda\,(a_\lambda - c_\lambda^\top i + b_\lambda^\top u) \\ \dot{b_\lambda} & = & -\lambda\,(b_\lambda - 2i) \\ \dot{c_\lambda} & = & -\lambda\,(c_\lambda + 2u + 2\lambda L i) \\ \dot{d_\lambda} & = & -\lambda\,(d_\lambda - b_\lambda^\top i) \\ \dot{e_\lambda} & = & -\lambda\,(e_\lambda - c_\lambda^\top u + \lambda^2 L^2 |i|^2 - \lambda^2 \Phi^2)\ . \end{array}$$

Take

$$\begin{aligned} T(x, x_3, t) &= \begin{pmatrix} T_{\lambda_1}(x, x_3, t) \\ T_{\lambda_2}(x, x_3, t) \\ T_{\lambda_3}(x, x_3, t) \end{pmatrix} \\ &= m(\Lambda)\, x^\top x + M(\Lambda)\,(c(t) + x_3\, b(t))\, x + a(t)\, x_3 + d(t)\, x_3^2 - e(t) \end{aligned} \tag{2}$$

for $\lambda_k \in \mathbb{C}$ with $\Re(\lambda_k) > 0$.

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$$T(\hat{x}, \hat{x}_3, t) = 0$$

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Number of solutions proved to be equal to the number of solutions to $H_3(x, x_3, t) = 0$ in the differential observability analysis for sufficiently fast eigenvalues !

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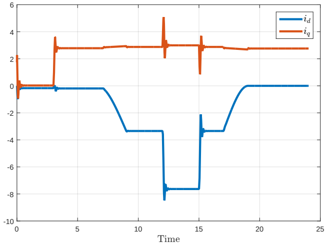
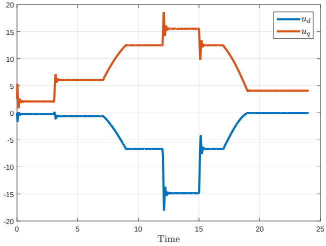
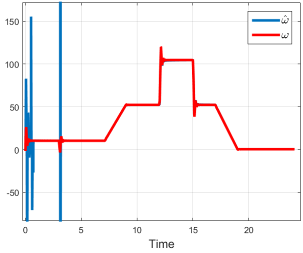
$\Longrightarrow$ Elimination of $x^\top x$, then x and finally grid in x_3 !



(a) $i_{dq} = \mathcal{R}(-\theta)i$

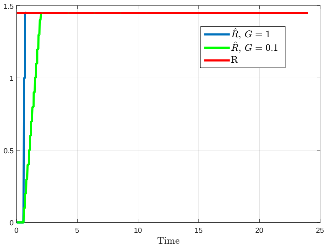


(b) $u_{dq} = \mathcal{R}(-\theta)u$

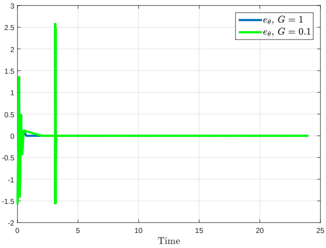


(c) Rotation speed $\omega = \dot{\theta}$ and estimated rotation speed $\hat{\omega} = \dot{\hat{\theta}}$.

Additional knowledge : $i_q > 0$ because motor mode $\Rightarrow$ removes indistinguishability

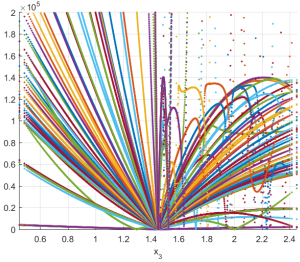


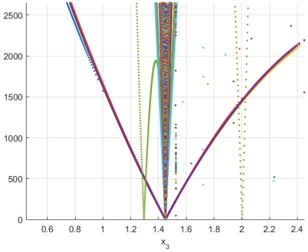
(d) $\hat{R}$



(e) $e_\theta = \hat{\theta} - \theta$ [rad]

Figure – Results of the observer algorithm with $\lambda_1 = 20$, $\lambda_2 = 30$, $\lambda_3 = 40$, $dt_R = 0.1$, and two grids with amplitude $G = 1$ and $G = 0.1$ respectively. The estimation starts at $\bar{t} = 0.5$.





(b) Zoom around $R = 1.45$

Figure – Plot of the criterion $|J(\cdot, t)|$ on the grid with $G = 1$ at each iteration where $\hat{R}$ is updated, i.e. every $dt_R = 0.1$.

Experimental data with constant i_d, i_q and ω

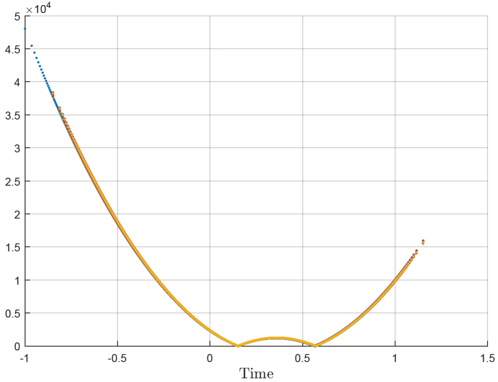


Figure – Real data : Plot of the criterion $|J(\cdot, t)|$ on the grid with $G = 1$ at each iteration where $\hat{R}$ is updated, i.e. every $dt_R = 0.1$.

Conclusion

- KKL observer design possible for a non observable system
- the inversion of the transformation enables to recover the different possibilities predicted by differential observability analysis
- comparison with high gain :
    - transformation independent from input derivatives
    - non observability prevents having a triangular normal form (set-valued last line)

Contents

KKL design

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\end{tikzcd}
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A Hurwitz

Existence of a left-inverse $\mathcal{T}$ requires observability !

$\Rightarrow$ What if the system is not observable ?

$\Rightarrow$ Estimate only the observable part !

Functional KKL design

Consider a continuous map $q : \mathbb{R}^{d_x} \to \mathbb{R}^{d_q}$.

Goal : estimate $z := q(x)$

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\end{tikzcd}
$$

A Hurwitz

Existence of T doesn't change!

Can we show existence of $\mathcal{T}$ such that $\lim_{t\to+\infty} |q(x) - \hat{z}| = 0$ under observability of $q(x)$?

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$$\begin{tikzcd}
\begin{array}{rcl} \dot{x} & = & f(x) \\ y & = & h(x) \end{array} \arrow[r, "T"] \arrow[d, dashed] & \dot{\xi} = A\xi + B\,y \arrow[d] \\
\hat{z} = \ ? & \dot{\hat{\xi}} = A\hat{\xi} + B\,y \arrow[l, "\mathcal{T}"']
\end{tikzcd}$$

A Hurwitz

Existence of T doesn't change!

Can we show existence of $\mathcal{T}$ such that $\lim_{t\to+\infty} |q(x) - \hat{z}| = 0$ under observability of $q(x)$?

Assumption 1 (Bounded trajectories).

The trajectories of $\dot{x} = f(x)$ to be estimated evolve in a compact set $\mathcal{X}$.

Existence of T

Goal : Find a transformation $x \mapsto T(x)$ such that $\xi = T(x)$ verifies

$$\dot{\xi} = A\xi + B\,y \quad , \quad A \text{ Hurwitz}$$

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Theorem 4 (Existence of T).

There exists $\ell > 0$ such that for any $A \in \mathbb{R}^{d_\xi \times d_\xi}$ with $\mathrm{Re}(\mathrm{eig}(A)) < -\ell$, and any $B \in \mathbb{R}^{d_\xi \times d_y}$, the map

$$T(x) = \int_{-\infty}^{0} e^{-As} B\,h(\check{X}(x,s))ds$$

is C^1 and verifies PDE (3), where $s \mapsto \check{X}(x,s)$ is the solution initialized at x of $\dot{x} = \check{f}(x)$ with $\check{f}$ bounded equal to f on $\mathcal{X}$.

Injectivity of T with respect to q

Let an open set $\mathcal{S}$ such that $\mathcal{X} \subset \mathcal{S}$.

Assumption 2 (Backward $\mathcal{O}$-distinguishability with respect to q).

There exist $\delta_d > \delta_\Upsilon$*, such that for each pair* (x_a, x_b) *in* $\mathcal{O} + \delta_\Upsilon$ *verifying* $q(x_a) \neq q(x_b)$*, there exists* $t \in (\max\{\sigma^-_{\mathcal{O}+\delta_d}(x_a), \sigma^-_{\mathcal{O}+\delta_d}(x_b)\}, 0]$*, such that*

$$h(X(x_a, t)) \neq h(X(x_b, t)).$$

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Theorem 5 (Injectivity with respect to q).

Assume the system is backward $\mathcal{O}$-distinguishable with respect to q.

Then, there exists $\ell > 0$ and a set S of zero-Lebesgue measure in $\mathbb{C}_\ell^{d_x+1}$ with

$$\mathbb{C}_\ell = \{z \in \mathbb{C} \ : \ \mathrm{Re}(z) < -\ell\}$$

such that for any diagonalizable matrix $A_0 \in \mathbb{C}^{(d_x+1)\times(d_x+1)}$ and any $B_0 \in \mathbb{C}^{d_x+1}$ such that $\mathrm{eig}(A_0) \in \mathbb{C}_\ell^{d_x+1} \setminus S$ and (A_0, B_0) controllable,

the map T defined for $A = A_0 \otimes I_{d_y}$ and $B = B_0 \otimes I_{d_y}$ is injective with respect to q on $\mathcal{X}$, i.e, for all $(x_a, x_b) \subset \mathcal{X} \times \mathcal{X}$,

$$T(x_a) = T(x_b) \implies q(x_a) = q(x_b) \ .$$

Left-inverse and convergence

Taking real and imaginary parts, $T : \mathbb{R}^{d_x} \to \mathbb{R}^{d_\xi}$ with $d_\xi = 2(d_x + 1)d_y$, such that for all $(x_a, x_b) \in \mathcal{X} \times \mathcal{X}$,

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Therefore, there exists $\mathcal{T} : \mathbb{R}^{d_\xi} \to \mathbb{R}^{d_x}$ and ρ of class $\mathcal{K}$ such that

- $\mathcal{T}(T(x)) = q(x)$ for all $x \in \mathcal{X}$
- $|\mathcal{T}(\xi_a) - \mathcal{T}(\xi_b)| \leq \rho(|\xi_a - \xi_b|)$ for all $(\xi_a, \xi_b) \in \mathbb{R}^{d_\xi} \times \mathbb{R}^{d_\xi}$

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Since $x(t) \in \mathcal{X}$ for all $t \geq 0$, taking $\xi_a = \hat{\xi}$ and $\xi_b = \xi = T(x)$,

$$|\mathcal{T}(\hat{\xi}(t)) - x(t)| \leq \rho(|\hat{\xi}(t) - \xi(t)|)$$

so that

$$\lim_{t \to +\infty} |\hat{z}(t) - q(x(t))| = 0$$

Functional KKL

$$\dot{x} = f(x) \qquad , \qquad y = h(x)$$

Theorem 6 (Functional KKL).

Assume

- *the trajectories of interest remain in a compact set* $\mathcal{X}$
- *the system is backward distinguishable with respect to* q.

Then, there exists a map $\mathcal{T} : \mathbb{R}^{d_\xi} \to \mathbb{R}^{d_x}$, *a matrix* $A \in \mathbb{R}^{d_\xi \times d_\xi}$ *and* $B \in \mathbb{R}^{d_\xi \times d_y}$ *with* $d_\xi = 2(d_x + 1)d_y$ *such that*

$$\dot{\hat{\xi}} = A\hat{\xi} + B\,y$$
$$\hat{z} = \mathcal{T}(\hat{\xi})$$

is a functional KKL observer with respect to q, *i.e., for any* $\hat{\xi}(0)$,

$$\lim_{t\to+\infty} |\hat{z}(t) - q(x(t))| = 0$$

Remarks :

- It is actually sufficient to take $d_\xi = (2d_x + 1)d_y$ and the result holds for "almost any" choice of Hurwitz $A_0 \in \mathbb{R}^{(2d_x+1)\times(2d_x+1)}$ and any B making (A, B) controllable.

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- The result extends to **time-varying systems** with $q(x, t)$ continuous but for $d_\xi = (2d_x + 2)d_y$ and with time-varying $T(x, t)$ and $\mathcal{T}(x, t)$. But uniform injectivity in time not guaranteed for convergence.

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- In presence of **inputs**, the time-varying paradigm applies, but with $T_u(x, t)$ and $\mathcal{T}_u(x, t)$ depending implicitly on the the input :
  - -> either we can compute this dependence explicitly (for instance via filters as in the context of electrical machines)
  - -> or we design T and $\mathcal{T}$ for a particular class of inputs generated by

$$\dot{w} = s(w) \quad , \quad u = l(w)$$

via a functional observer on the extended system with state (x, w) and extended output $(h(x, l(w)), l(w))$, leading to

$$\dot{\hat{\xi}} = A\hat{\xi} + B\begin{pmatrix} y \\ u \end{pmatrix}$$

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- When the maps T and $\mathcal{T}$ are not known explicitly, we can compute them numerically via neural networks ⇒ "numerical KKL"

L. Da Costa Ramos, F. Di Meglio, L. F. Figueira Da Silva, V. Morgenthaler, P. Bernard, *Numerical design of Luenberger observers for nonlinear systems*, CDC, 2020

Contents

Differential non-observability

$$\dot{x} = f(x,t) \qquad , \qquad y = h(x,t)$$

The system is always instantaneously backward-distinguishable with respect to the map

$$q(x,t) = \begin{pmatrix} h(x,t) \\ L_f h(x,t) \\ \vdots \\ L_f^{m-1} h(x,t) \end{pmatrix}$$

for any $m \in \mathbb{N}$.

$\Rightarrow$ $q(x,t)$ can always be estimated via functional KKL observer !

$$\dot{\hat{\xi}} = A\hat{\xi} + B\,y \qquad , \qquad \hat{z} = \mathcal{T}(\hat{\xi},t)$$

$\Rightarrow$ Recover all possible $\hat{x}$ such that

$$\hat{x} \in \mathrm{Argmin}_{x\in\mathcal{X}} \, |q(x,t) - \hat{z}|^2$$

$\Rightarrow$ No need for normal triangular form unlike in high-gain design !

Indistinguishability

Given $\breve{f}$ bounded such that $\breve{f} = f$ on an open set $\mathcal{O}$ containing $\mathcal{X}$:

Consider the set $\breve{\mathcal{I}} \subset \mathbb{R}^{d_x} \times \mathbb{R}^{d_x}$ of pairs (x_a, x_b) backward-indistinguishable for $\dot{x} = \breve{f}(x)$, i.e., such that

$$h(\breve{X}(x_a), t)) = h(\breve{X}(x_b), t)) \quad \forall t \leq 0$$

Then, by definition, for all $(x_a, x_b) \in \breve{\mathcal{I}}$,

$$T(x_a) = \int_{-\infty}^{0} e^{-As} B \, h(\breve{X}(x_a, s)) ds = \int_{-\infty}^{0} e^{-As} B \, h(\breve{X}(x_b, s)) ds = T(x_b)$$

$\Rightarrow$ Indistinguishable by T !

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Conversely, applying the proof of KKL on the open set $(\mathbb{R}^{d_x} \times \mathbb{R}^{d_x}) \setminus \breve{\mathcal{I}}$ instead of

$$\{(x_a, x_b) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_x} \;:\; x_a \neq x_b\}$$

should show that for "almost all" choice of (A, B) of dimension $(2d_x + 1)d_y$,

$$T(x_a) = T(x_b) \implies (x_a, x_b) \in \breve{\mathcal{I}}$$

i.e., T is injective with respect to the distinguishable states !

Conclusion

- KKL design well suited for non observable systems since the dynamics are well-defined and everything is in the inversion of the transformation
- very general theoretical answer to the observation problem under weak assumptions
- **problem** : numerical computation of T and $\mathcal{T}$ challenging in practice !
- how generic is the case of "finite-indistinguishability" ?