Stabilization of non-uniformly observable control systems

Lucas Brivadis

Joint work with Jean-Paul Gauthier, Ludovic Sacchelli and Ulysse Serres

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Mini-workshop: Défauts d'observabilité des systèmes non-linéaires

L2S Laboratoire Signaux & Systèmes

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Introduction

Dynamic output feedback stabilization

Consider a nonlinear observation-control system:

$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x) \end{cases}$$

$x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ is the input and $y \in \mathbb{R}^m$ is the output.

Semi-global dynamic output feedback stabilization:

For each compact set $K \subset \mathbb{R}^n$, find a dynamic output feedback

$$\begin{cases} \dot{\hat{x}} = \nu(\hat{x}, u, y) \\ u = \varpi(\hat{x}, y) \end{cases}$$

and a compact set $\hat{K}$ such that $(x^\star, \hat{x}^\star)$ is an asymptotically stable equilibrium with basin of attraction containing $K \times \hat{K}$ in closed-loop.

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$$\begin{cases} \dot{\hat{x}} = \nu(\hat{x}, u, y) \\ u = \varpi(\hat{x}, y) \end{cases}$$

and a compact set $\hat{K}$ such that $(0, 0)$ is an asymptotically stable equilibrium with basin of attraction containing $K \times \hat{K}$ in closed-loop.

Dynamic output feedback stabilization

State feedback stabilization: Find a feedback λ such that the origin is a globally asymptotically stable equilibrium point of the vector field $x \mapsto f(x, \lambda(x))$.

Observer-based strategy: Design an observer system

$$\dot{\hat{x}} = \nu(\hat{x}, u, y)$$

such that $\hat{x} - x \to 0$ for all initial conditions in $K \times \hat{K}$ and use the control $u = \lambda(\hat{x})$ with λ globally stabilizing.

Closed-loop:

$$\begin{cases} \dot{x} = f(x, \lambda(\hat{x})) \\ y = h(x) \end{cases}, \qquad \begin{cases} \dot{\hat{x}} = \nu(\hat{x}, u, y) \\ u = \lambda(\hat{x}) \end{cases}$$

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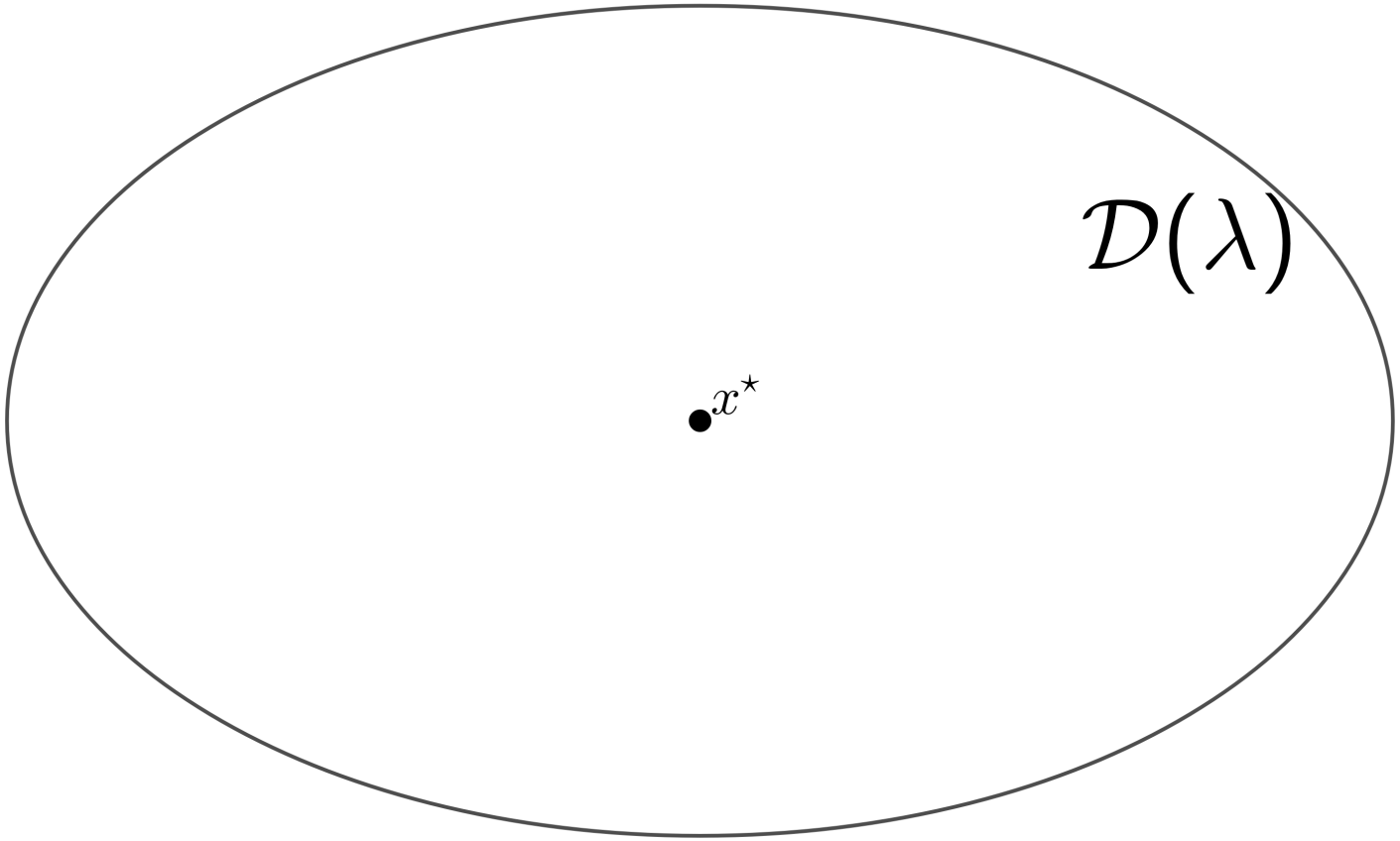

Dynamic output feedback stabilization diagram

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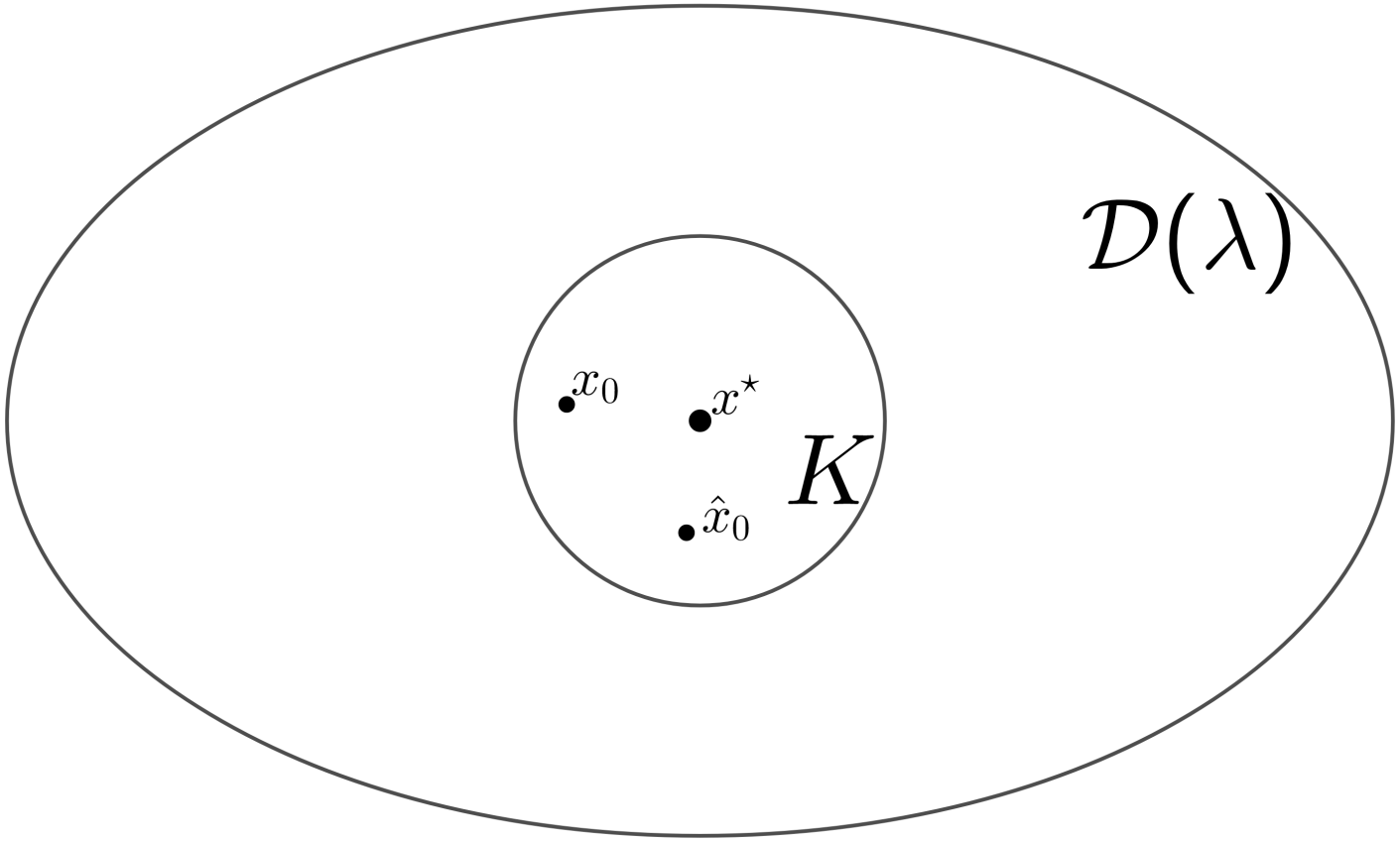

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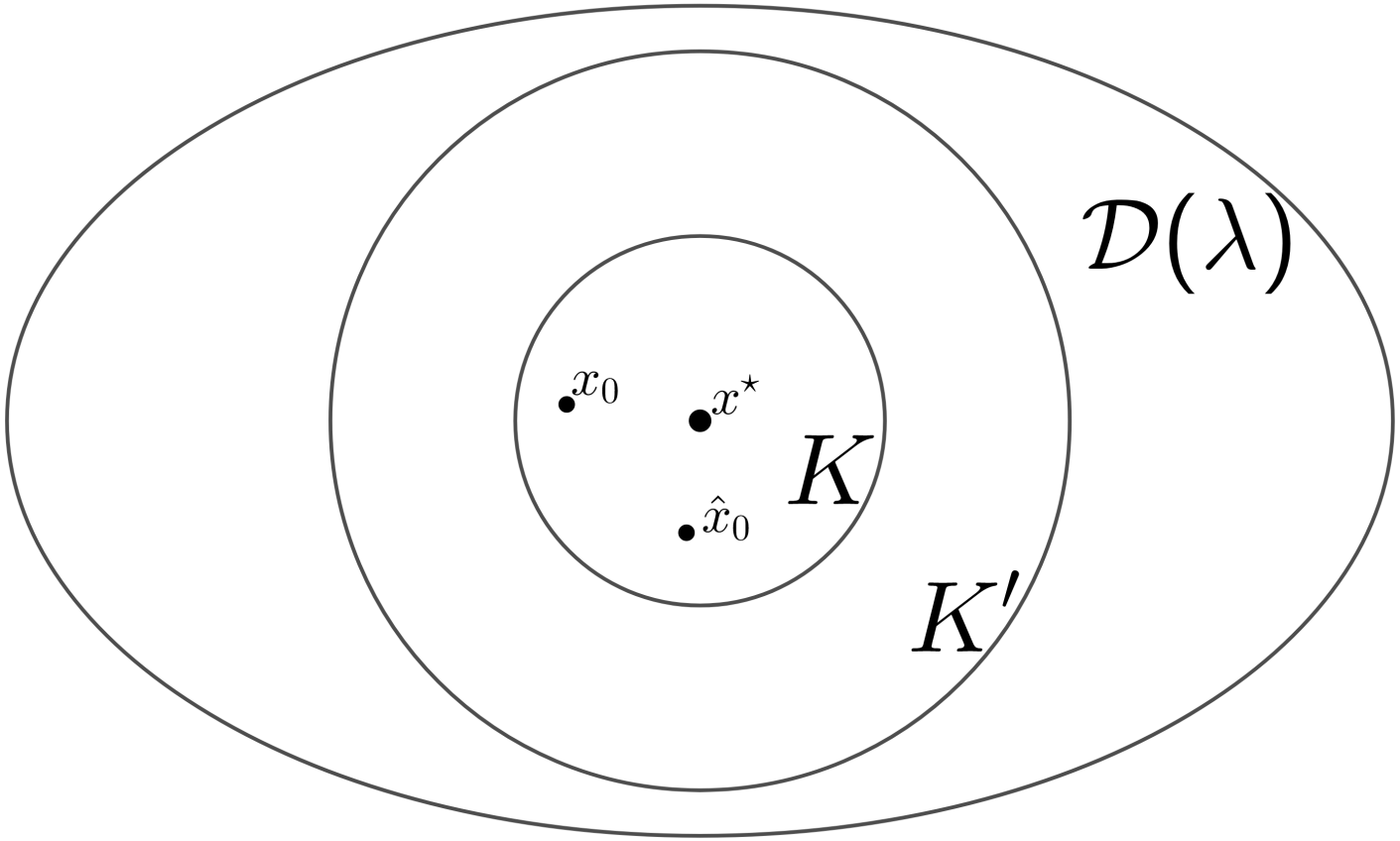

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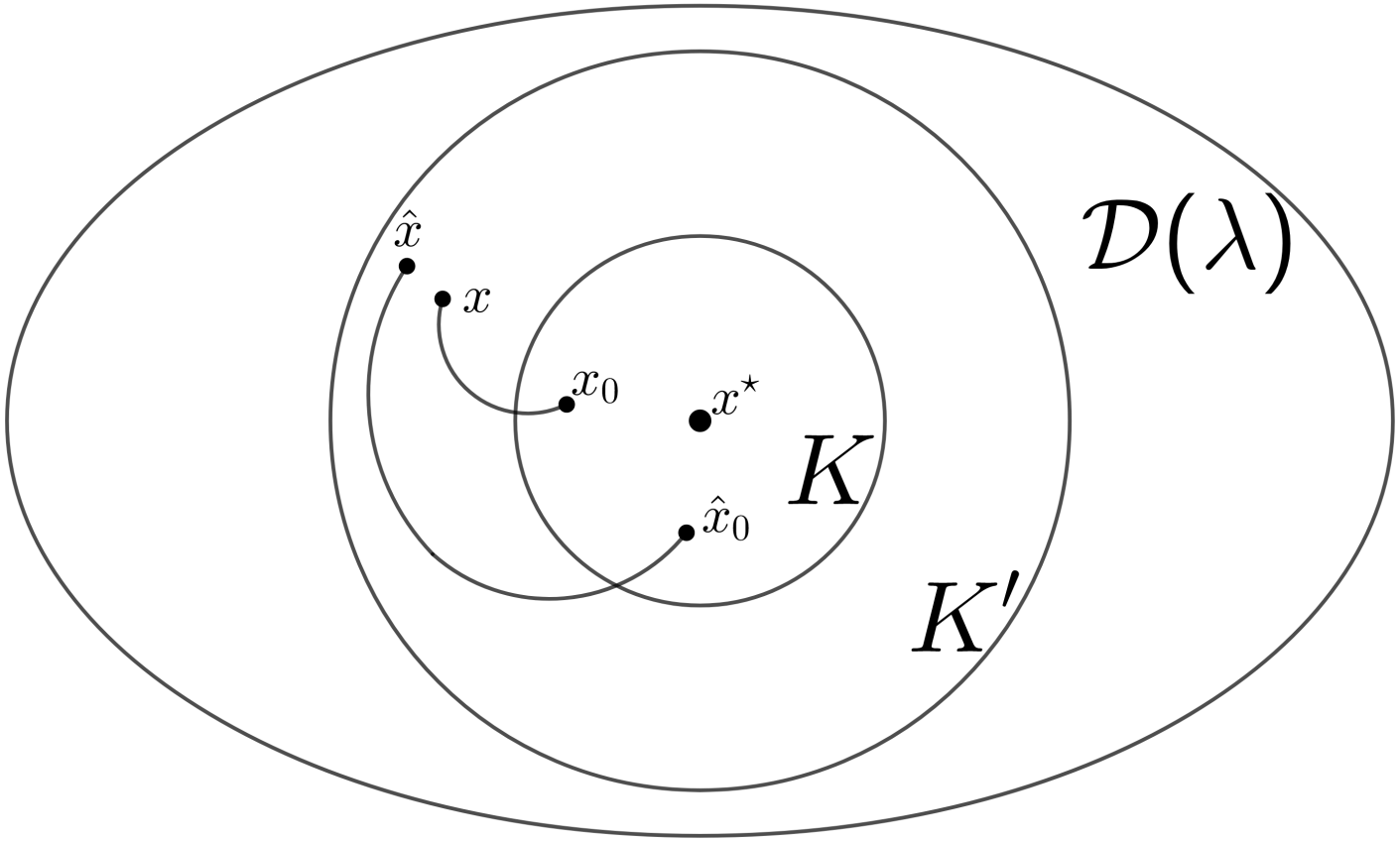

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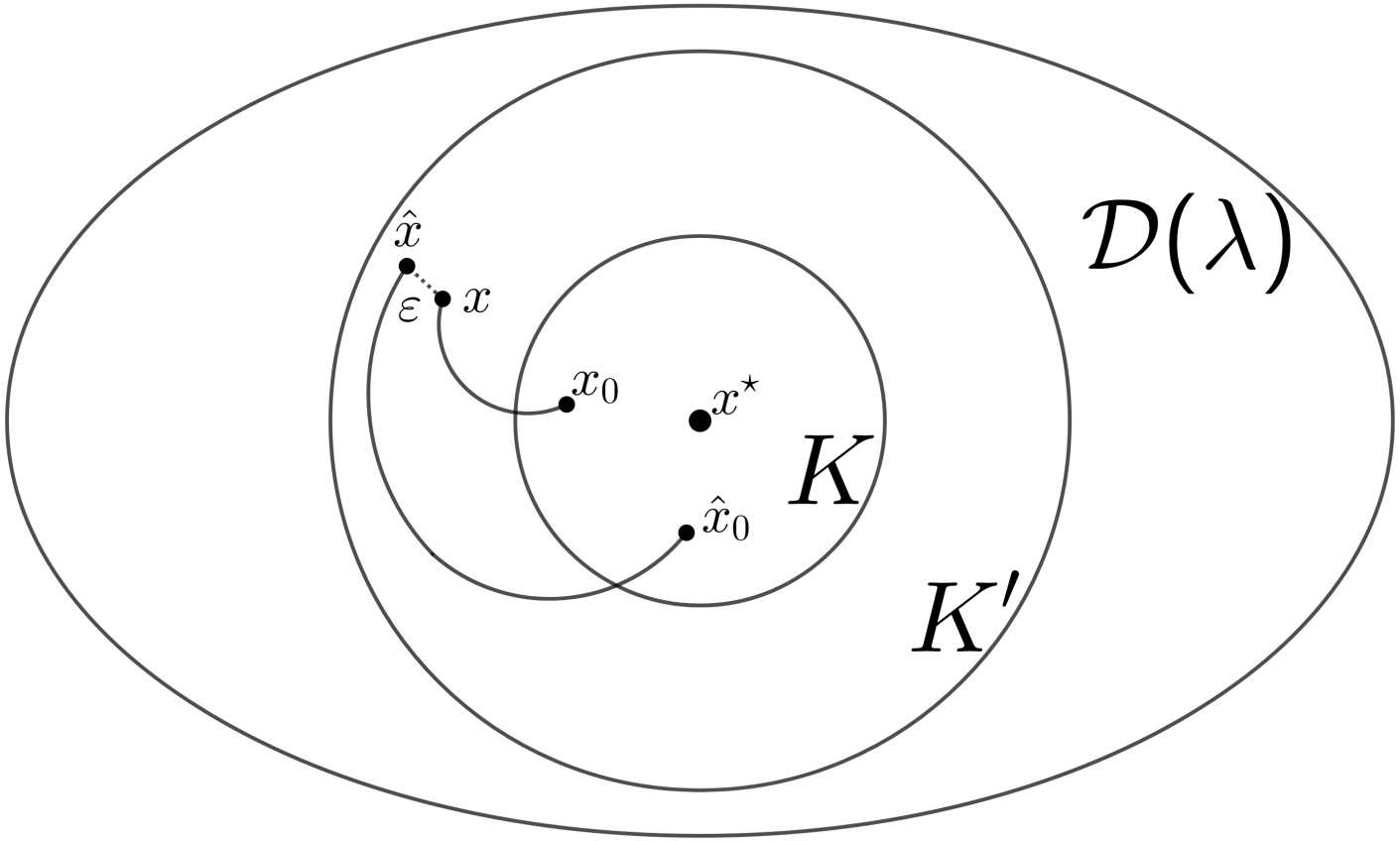

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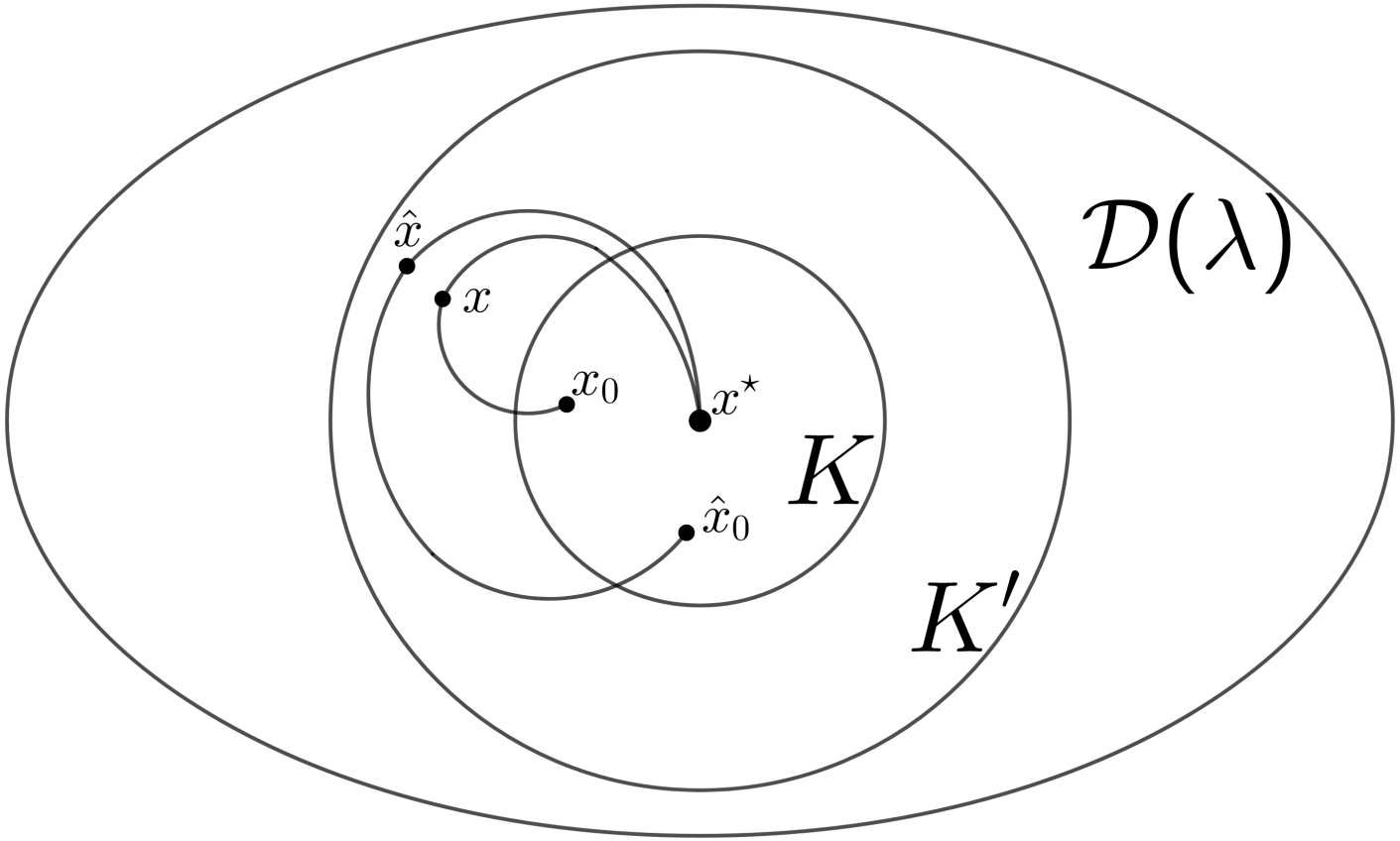


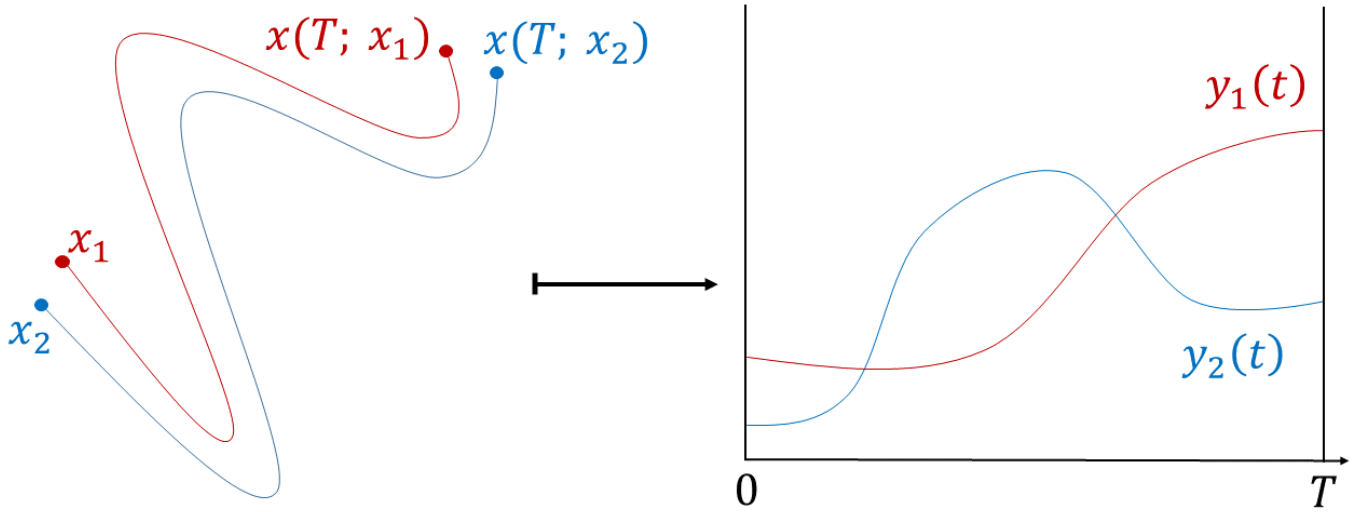
Dynamic output feedback stabilization diagram

Dynamic output feedback stabilization

Definition (Observability and Uniform observability). A system is said to be *observable* in time T for an input u if and only if, for all initial conditions $x_a \neq x_b \in \mathbb{R}^n$,

$$\text{measure}\left\{t \in [0, T] \mid y(t; x_a) \neq y(t; x_b)\right\} > 0$$

If it is observable in any time $T > 0$ **for all inputs** u, then it is said to be **uniformly observable** in small time.



Uniform observability

Theorem (Teel and Praly, 1994). If the system is

- globally state feedback stabilizable
- completely uniformly observable

then it is also semi-globally stabilizable by dynamic output feedback.

Problem: It is not generic to be uniformly observable when $m \leqslant p$ (Gauthier and Kupka, 2001).

What if there is no uniform observability?

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Non-uniformly observable systems

Definition (Control value at the target).

$$\text{If } (x, \hat{x}) \to (0,0), \text{ then } u = \varpi(\hat{x}, h(x)) \to u^\star.$$

Does the constant input $u \equiv 0$ make the system observable?

$$\begin{cases} \dot{x} = f(x, 0) \\ y = h(x) \end{cases}$$

Yes → Observable target

No → Unobservable target

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Main obstacles and tools

Main problems to be tackled:

- Observable target: avoid observability singularities when possible
- Unobservable target: observation and stabilization are antagonistic

Main techniques to deal with these problems:

- Feedback perturbation
- Dissipative systems
- Embeddings

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Observable target

Context

Systems under consideration: SISO bilinear systems with linear output

$$\begin{cases} \dot{x} = (A + uB)x + bu \\ y = Cx \end{cases}$$

$x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$, $A, B \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{1 \times n}$, $b \in \mathbb{R}^n$.

State feedback: $u = \lambda(x)$ is globally asymptotically stabilizing.

Observer system:

$$\dot{\hat{x}} = (A + uB)\hat{x} + bu - PC^*C(\hat{x} - x)$$

- $\dot{P} = 0$ (Luenberger observer)
- $\dot{P} = (A + uB)P + P(A + uB)^* + Q - PC^*CP$ (Kalman observer)

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Feedback perturbation

Issue: What happens if the chosen control $u = \lambda(\hat{x})$ makes the system unobservable?

General idea: Feedback modification

- Periodic additive perturbation

$$u(t) = \lambda(\hat{x}(t)) \longrightarrow u(t) = \lambda(\hat{x}(t)) + d(t)$$

  - [Combes et al., 2016]: virtual measurements
- Switching strategy
  - Excite the system to estimate the state, then control to stabilize, etc.
  - [Coron, 1994]: local stabilization
  - [Shim and Teel, 2002]: practical stabilization
- Smooth autonomous perturbation

$$u(t) = \lambda(\hat{x}(t)) \longrightarrow u(t) = \lambda(\hat{x}(t)) + \delta(\hat{x}(t))$$

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$$u(t) = \lambda(\hat{x}(t)) \longrightarrow u(t) = (\lambda + \delta)(\hat{x}(t))$$

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Feedback perturbation

Strategy of [Lagache, Serres and Gauthier, 2017]:

1. Show that there exists a (smooth) small perturbation δ such that $u = (\lambda + \delta) \circ \hat{x}$ makes the system observable.
2. Show that for this input, the observer converges to the state (and remains in a fixed compact set).
3. Show asymptotic stability.

Remarks:

- Example of quantum control;
- **Unobservable** target;
- **Practical** stabilization and **exact** stabilization.

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Avoiding observability singularities

Theorem (Brivadis, Gauthier, Sacchelli and Serres, 2021).

$$\begin{cases} \dot{x} = (A + uB)x + bu \\ y = Cx, \end{cases}, \quad \dot{\hat{x}} = (A + uB)\hat{x} + bu - PC^*C(\hat{x} - x)$$

If

- λ is a smooth globally stabilizing feedback
- (C, A) and (C, B) are observable

then, for any compact K, there exists $\mathcal{N} \subset C^\infty(\mathbb{R}^n, \mathbb{R})$,

$$\mathcal{N} = \left\{ \delta \;\middle|\; \sup_{K_x} |\partial^\sigma \delta| < \varepsilon \text{ for } |\sigma| < k \text{ and } \delta = 0 \text{ in a nbh. of } 0 \right\}$$

such that

- $\delta \in \mathcal{N} \implies \lambda + \delta$ is a **semi-globally stabilizing state feedback**
- $\exists \Delta$ **open and dense** (Whitney C^∞) in $\mathcal{N}$ such that:
  $\forall \delta \in \Delta$, $\forall (x_0, \hat{x}_0, P_0) \in K$, the system is **observable** in small time for the control $u = (\lambda + \delta) \circ \hat{x}$.

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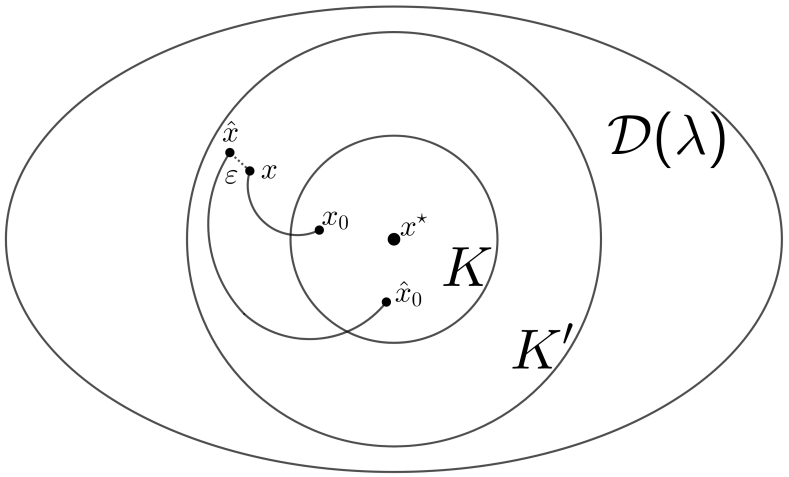
Avoiding observability singularities

We can prove the convergence of **bounded trajectories**.



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Otherwise, one must adapt the observer gain. How does it affect δ?

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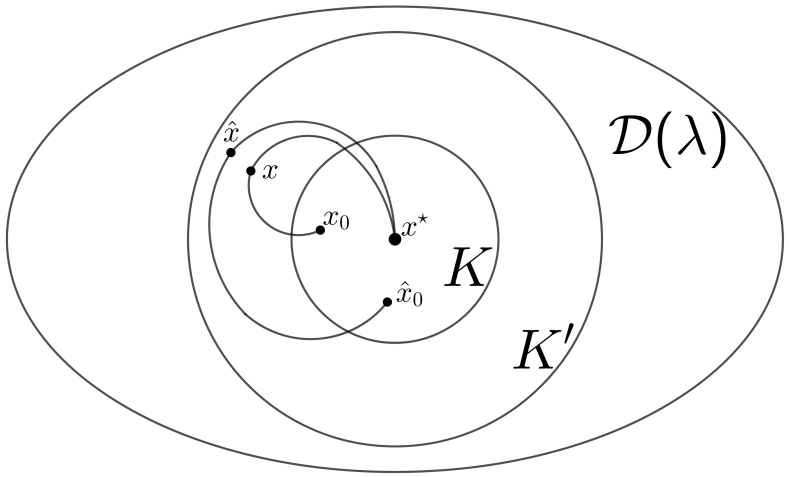

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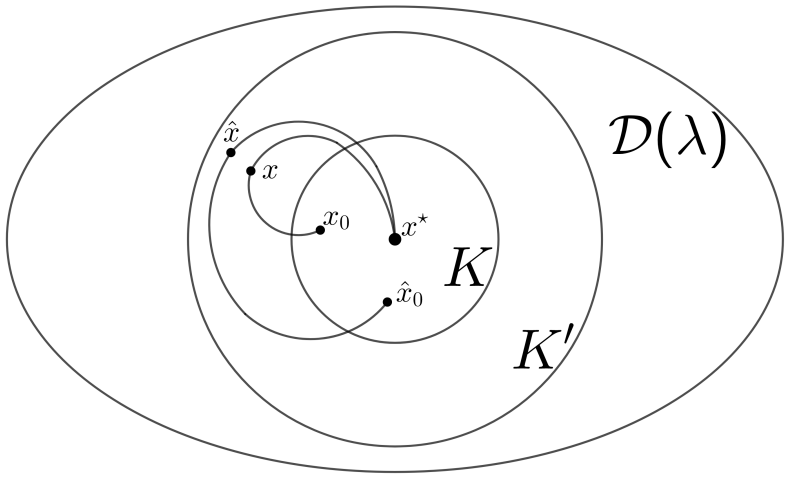

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Dissipative systems

Definition (Dissipativity).

$$\begin{cases} \dot{x} = A(u)x + B(u) \\ y = Cx \end{cases}$$

with

$$x'A(u)x \leqslant 0, \qquad \forall x \in \mathbb{R}^n,\ \forall u \in \mathbb{R}^p.$$

Luenberger observer:

$$\begin{cases} \dot{\hat{x}} = A(u)\hat{x} - \alpha C'C\varepsilon \\ \dot{\varepsilon} = (A(u) - \alpha C'C)\,\varepsilon \end{cases} \qquad \alpha > 0$$

The norm of ε is **non-increasing**:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\varepsilon|^2 = \underbrace{\varepsilon' A(u)\varepsilon}_{\leqslant 0} - \alpha\varepsilon' C'C\varepsilon \leqslant -\alpha|C\varepsilon|^2 \leqslant 0$$

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Dissipative systems

Theorem (Sacchelli, Brivadis, Andrieu, Serres and Gauthier, 2020). If

- λ is a globally stabilizing feedback
- $A(u)$ is dissipative

then the system is **semi-globally asymptotically stabilizable** if and only if $(C, A(0))$ is **detectable**. The dynamic output feedback is given by

$$\begin{cases} \dot{\hat{x}} = A(u)\hat{x} + B(u) - \alpha C'(C\hat{x} - y) \\ u = \lambda(\hat{x}) \end{cases}$$

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Dissipative systems

Theorem (Sacchelli, Brivadis, Andrieu, Serres and Gauthier, 2020). If

- λ is a globally stabilizing feedback
- $A(u)$ is dissipative

then the system is **globally asymptotically stabilizable** if and only if $(C, A(0))$ is **detectable**. The dynamic output feedback is given by

$$\begin{cases} \dot{\hat{x}} = A(u)\hat{x} + B(u) - \alpha(\hat{x})C'(C\hat{x} - y) \\ u = \lambda(\hat{x}) \end{cases}$$

Strategy of the proof:

- $|\varepsilon|$ is non-increasing
- choose α sufficiently small

Unobservable target

An illustrative example

Example inspired by [Coron, 1994]:

$$\begin{cases} \dot{x} = Jx + bu \\ y = h(|x|) \end{cases}, \qquad J' = -J, \quad (J, b) \text{ controllable}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$.

Unobservability at $u \equiv 0$:

$$|x_a| = |x_b| \implies y(t; x_a) \equiv y(t; x_b)$$

Theorem (Brivadis, Gauthier, Sacchelli and Serres, 2021).
If J is **not** invertible, then the system **not** stabilizable by means of dynamic output feedback.

This is due to a **topological constraint**.

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Embedding into a dissipative system

Theorem (Brivadis, Gauthier, Sacchelli and Serres, 2021).
If J is **invertible**, (J, b) is **controllable** and $h(|x|) = |x|^2/2$, then the system is stabilizable by means of dynamic output feedback.

Idea: Embedding into a bilinear dissipative system with linear output.

$$z = \tau(x) = (x, \frac{1}{2}|x|^2).$$

$$\begin{cases} \dot{x} = Jx + bu \\ y = \frac{1}{2}|x|^2 \\ x(0) \in \mathbb{R}^n \end{cases} \xrightarrow[\tau]{} \begin{cases} \dot{z} = A(u)z + Bu \\ y = Cz \\ z(0) \in \tau(\mathbb{R}^n) \subset \mathbb{R}^{n+1} \end{cases}$$

$$A(u) = \begin{pmatrix} J & 0 \\ ub' & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}.$$

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Embedding into a dissipative system

Observer system:

$$\begin{cases} \dot{\varepsilon} = (A(u) - L(u)C)\,\varepsilon \\ \dot{\hat{z}} = A(u)\hat{z} + Bu - L(u)C\varepsilon. \end{cases}$$

with $L(u) = \begin{pmatrix} bu \\ \alpha \end{pmatrix}$ and $\alpha > 0$. Then,

$$A(u) - L(u)C = \underbrace{\begin{pmatrix} J & -bu \\ ub' & 0 \end{pmatrix}}_{\text{skew-symmetric}} - \underbrace{\alpha C'C.}_{\substack{\text{Luenberger} \\ \text{correction term}}}$$

Feedback law:

$$u = \begin{pmatrix} K & 0 \end{pmatrix} \hat{z} + \delta \hat{z}_{n+1} \text{ with } J + bK \text{ Hurwitz and } \delta > 0$$

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Sketch of the proof:

- Trajectories are **bounded**: choose α large enough
- **Observability analysis:** If $(\varepsilon_0, \hat{z}_0) \neq (0,0)$, then the perturbed feedback makes the system observable in small time. Hence, $\varepsilon \to 0$.
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Theorem (Brivadis, Sacchelli, 2021).
If (J, b) is controllable and $h(|x|) = |x|^2/2$, Then the system is **globally stabilizable** by means of a **periodic time-varying** dynamic output feedback.

Proof by **switching strategy**:

- **Observation phase**: On $[t_k, t_k + T)$, apply a control making the system observable except at the target point. Use the embedded observer.
- **Stabilization phase**: On $[t_k + T, t_{k+1})$, apply a stabilizing control, based on a stabilizing feedback and the observer.

Open question: How to embed a nonlinear system into a system for which an observer with dissipative error can be designed?

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Embedding into an infinite-dimensional dissipative system

Definition (Unitary representation). A unitary representation of a group $\mathbb{G}$ is a group morphism $\rho : \mathbb{G} \to \mathcal{L}(H)$, where $\mathcal{L}(H)$ are the bounded endomorphism of the Hilbert space H and $\rho(g)$ is a unitary operator for all $g \in \mathbb{G}$.

When considering control systems, the group of interest is the Lie **group of diffeomorphisms** generated by the dynamics:

For all $u \in \mathbb{R}$ and all $T > 0$, the flow over $[0, T]$ of

$$\dot{x} = f(x, u)$$

defines a diffeomorphism of $\mathbb{R}^n$.

We can use a unitary representation ρ to define an **embedding into a dissipative system**.

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The dynamical system

$$\dot{x} = Jx + bu, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

generates the group $SE(2) = \mathbb{R}^2 \rtimes SO(2)$ (motions of the plane).

Unitary representations: $\rho : SE(2) \to \mathcal{L}(L^2(\mathbb{S}^1; \mathbb{C}))$

$$\rho((x_1, x_2), \beta) \cdot \xi(\theta) = e^{i\mu(x_1 \cos\theta + x_2 \sin\theta)} \xi(\theta - \beta)$$

This defines an **embedding**: $\tau : \mathbb{R}^2 \to L^2(\mathbb{S}^1; \mathbb{C})$

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$$\langle \tau(x), 1 \rangle_{L^2} = J_0(\mu|x|).$$

J_0: Bessel function of the first kind of order 0.

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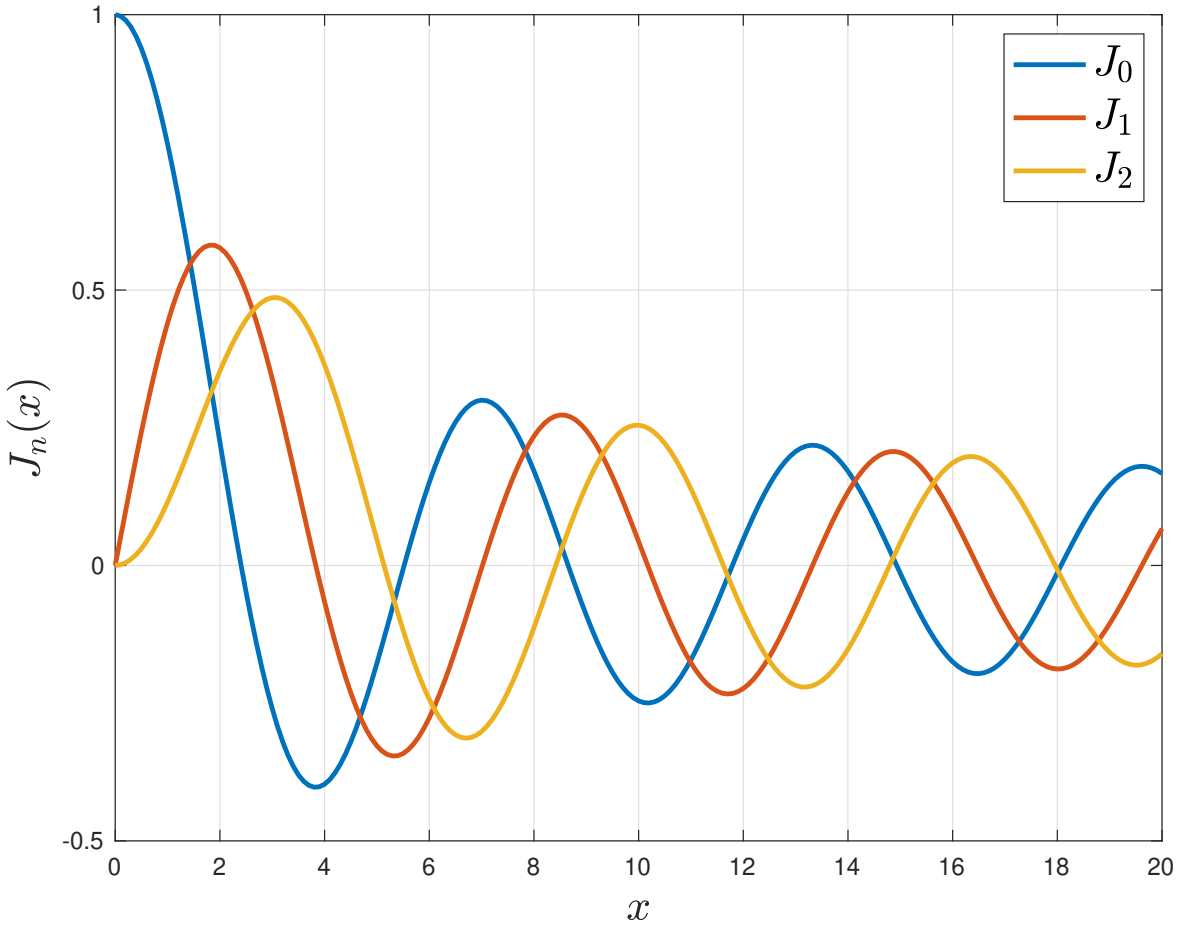
Embedding into an infinite-dimensional dissipative system



Bessel functions

Embedding into an infinite-dimensional dissipative system

The system is now embedded via

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$$A(u) = -\frac{\partial}{\partial \theta} + i\mu u \sin(\theta), \qquad Cz = \langle z, 1 \rangle_{L^2}.$$

How to build an observer for this infinite-dimensional system?

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Infinite-dimensional observer

Luenberger observer:

$$\begin{cases} \dot{z} = A(u(t))z, \quad z(0) = z_0 \in H \\ y = Cz \end{cases}$$

$$\begin{cases} \dot{\hat{z}} = A(u(t))\hat{z} - \alpha C^* C\varepsilon, \quad \hat{z}(0) = \hat{z}_0 \in H \\ \dot{\varepsilon} = (A(u(t)) - \alpha C^* C)\varepsilon, \quad \varepsilon(0) = \varepsilon_0 = \hat{z}_0 - z_0 \end{cases}$$

- H and Y are Hilbert spaces, $C \in \mathscr{L}(H, Y)$
- $A(u(t)) : \mathcal{D}(A) \to H$ generates an evolution system $(\mathbb{T}(t,s))_{t \geqslant s \geqslant 0}$
- $z, \hat{z}, \varepsilon \in C^0(\mathbb{R}_+; H)$

Example: $H = L^2(\mathbb{S}^1; \mathbb{C})$, $Y = \mathbb{C}$, $\mathcal{D}(A) = H^1(\mathbb{S}^1; \mathbb{C})$

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Recall that $A(u) = -\frac{\partial}{\partial\theta} + i\mu u \sin(\theta)$ is **skew-adjoint**. Hence

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon(t)\|_H^2 &= \langle \varepsilon(t),\ \dot{\varepsilon}(t)\rangle_H \\ &= \langle \varepsilon(t),\ A(u(t))\varepsilon(t)\rangle_H - \alpha\,\langle \varepsilon(t),\ C^*C\varepsilon(t)\rangle_H \\ &\leqslant -\alpha\,\|C\varepsilon(t)\|_Y^2 \\ &\leqslant 0.\end{aligned}$$

More generally, if

$$\langle A(u(t))z,\ z\rangle_H \leqslant p\,\|Cz\|_Y^2\,, \quad \forall z \in H,\ \forall t \geqslant 0 \quad (\textbf{weak detectability})$$

for some $p > 0$, then

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Infinite-dimensional observer

Definition (Exact observability).
The system is **exactly** observable on $[t_0, t_0 + T]$ if for some $k > 0$,

$$\int_{t_0}^{t_0+T} \|C\mathbb{T}(t, t_0)z_0\|_Y^2 \mathrm{d}t \geqslant k\|z_0\|_H^2, \quad \forall z_0 \in H$$

Definition (Approximate observability).

$$\mathcal{O} = \left\{ z_0 \in H \mid \int_{t_0}^{t_0+T} \|C\mathbb{T}(t, t_0)z_0\|_Y^2 \mathrm{d}t = 0 \right\}^{\perp}$$

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Infinite-dimensional observer

Theorem (Brivadis, Andrieu, Serres and Gauthier, 2021).
Assumptions:

- $((A(u(t)))_{t\geqslant 0}, C)$ is weakly detectable, $\alpha > p$
- $\exists (t_n)_{n\geqslant 0} \to +\infty$ and an evolution system $(\mathbb{T}_\infty(t,s))_{0\leqslant s\leqslant t}$ s.t.

$$\|\mathbb{T}(t_n+t,t_n) - \mathbb{T}_\infty(t,0)\|_{\mathscr{L}(H)} \underset{n\to+\infty}{\longrightarrow} 0 \text{ uniformly in } t\in[0,\tau], \forall \tau > 0.$$

Let $\mathcal{O}$ be the observable subspace of $(\mathbb{T}_\infty, C)$ in infinite-time. Then

$$\langle \varepsilon(t_n),\ \psi\rangle_H \underset{n\to+\infty}{\longrightarrow} 0, \qquad \forall \psi\in\mathcal{O},\ \forall \varepsilon_0 \in H.$$

Moreover, if $(t_{n+1}-t_n)_{n\geqslant 0}$ is bounded and $\mathcal{O} = H$, then $\varepsilon(t) \underset{t\to+\infty}{\overset{weak}{\longrightarrow}} 0$.

Infinite-dimensional observer

Theorem (Brivadis, Andrieu, Serres and Gauthier, 2021).
Assumptions:

- $((A(u(t)))_{t\geqslant 0}, C)$ is weakly detectable, $\alpha > p$
- $\exists (t_n)_{n\geqslant 0} \to +\infty$ and an evolution system $(\mathbb{T}_\infty(t,s))_{0\leqslant s\leqslant t}$ s.t.

$$\|\mathbb{T}(t_n+t,t_n) - \mathbb{T}_\infty(t,0)\|_{\mathscr{L}(H)} \underset{n\to+\infty}{\longrightarrow} 0 \text{ uniformly in } t\in[0,\tau], \forall \tau > 0.$$

Let $\mathcal{O}$ be the observable subspace of $(\mathbb{T}_\infty, C)$ in infinite-time. Then

$$\langle \varepsilon(t_n),\ \psi\rangle_H \underset{n\to+\infty}{\longrightarrow} 0, \qquad \forall \psi\in\mathcal{O},\ \forall \varepsilon_0 \in H.$$

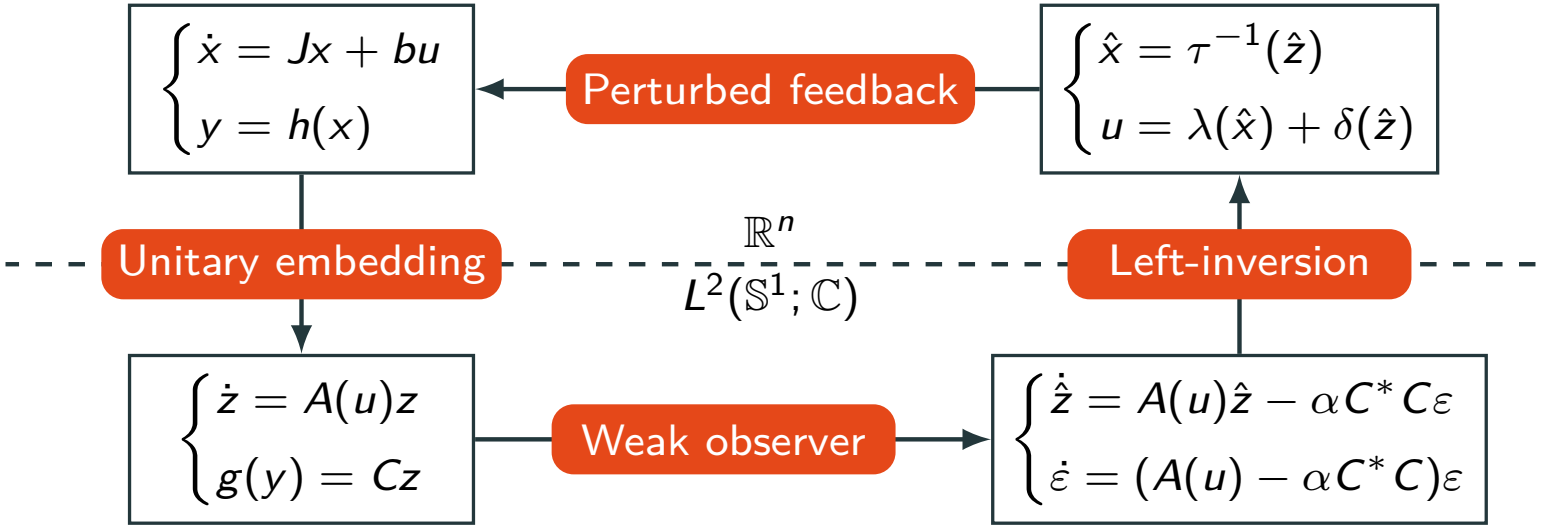
Moreover, if $(t_{n+1}-t_n)_{n\geqslant 0}$ is bounded and $\mathcal{O} = H$, then $\varepsilon(t) \underset{t\to+\infty}{\overset{weak}{\longrightarrow}} 0$.

Infinite-dimensional observer

Theorem (Brivadis, Gauthier, Sacchelli and Serres, 2021). If

- $\exists g \mid g(h(r\cos(s), r\sin(s))) = \sum_{\text{finite}} c_k J_k(\mu r) e^{-iks}$
- $u \equiv 0$ is an isolated observability singularity

then there exist a **left-inverse** τ^{-1} and a **feedback** $u = \lambda(\hat{x}) + \delta(\hat{z})$ such that the closed-loop system is asymptotically stable.



Unobservable target

Main differences with the finite dimensional embedding:

- The embedded system is unitary.
- ε is convergent in the weak topology.
- Depending on the isolation of $u \equiv 0$, a discretization of the control may be required.

Towards a generalization:

- The existence of an embedding of a dynamical system into a unitary one, either finite or infinite-dimensional, has been studied for observer design in [Celle, Gauthier, Kazakos and Sallet, 1989]
- If $\varepsilon \overset{weak}{\to} 0$, we can prove the existence of pseudo-inverse τ^{-1} such that $\tau^{-1}(\hat{z}) - x \to 0$.

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Conclusion and Perspectives

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Guidelines for output feedback stabilization:

- Uniform observability
- **Dissipative** systems
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  - **Feedback perturbation** to get observability
  - Convergence of **bounded trajectories**

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Our references on this subject:

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Thank you for your attention