Stabilization of non-uniformly observable control systems

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Joint work with Jean-Paul Gauthier, Ludovic Sacchelli and Ulysse Serres
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Mini-workshop: Défauts d’observabilité des systèmes non-linéaires
Introduction
Consider a nonlinear observation-control system:

\[
\begin{aligned}
\dot{x} &= f(x, u) \\
y &= h(x)
\end{aligned}
\]

\(x \in \mathbb{R}^n\) is the state, \(u \in \mathbb{R}^p\) is the input and \(y \in \mathbb{R}^m\) is the output.

**Semi-global dynamic output feedback stabilization:**

For each compact set \(K \subset \mathbb{R}^n\), find a dynamic output feedback

\[
\begin{aligned}
\dot{x} &= \nu(\hat{x}, u, y) \\
u &= \omega(\hat{x}, y)
\end{aligned}
\]

and a compact set \(\hat{K}\) such that \((x^*, \hat{x}^*)\) is an asymptotically stable equilibrium with basin of attraction containing \(K \times \hat{K}\) in closed-loop.
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**Semi-global dynamic output feedback stabilization:**

For each compact set \(K \subset \mathbb{R}^n\), find a dynamic output feedback

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\dot{\hat{x}} &= \nu(\hat{x}, u, y) \\
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and a compact set \(\hat{K}\) such that \((0, 0)\) is an asymptotically stable equilibrium with basin of attraction containing \(K \times \hat{K}\) in closed-loop.
Dynamic output feedback stabilization

**State feedback stabilization:** Find a feedback $\lambda$ such that the origin is a globally asymptotically stable equilibrium point of the vector field $x \mapsto f(x, \lambda(x))$.

**Observer-based strategy:** Design an observer system

$$\dot{\hat{x}} = \nu(\hat{x}, u, y)$$

such that $\hat{x} - x \to 0$ for all initial conditions in $K \times \hat{K}$ and use the control $u = \lambda(\hat{x})$ with $\lambda$ globally stabilizing.

**Closed-loop:**

$$\begin{cases} 
\dot{x} = f(x, \lambda(\hat{x})) \\
y = h(x) 
\end{cases}, \quad \begin{cases} 
\dot{\hat{x}} = \nu(\hat{x}, u, y) \\
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Dynamic output feedback stabilization

Dynamic output feedback stabilization diagram

$D(\lambda)$

$x^*$
Dynamic output feedback stabilization

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Dynamic output feedback stabilization

Definition (Observability and Uniform observability). A system is said to be *observable* in time $T$ for an input $u$ if and only if, for all initial conditions $x_a \neq x_b \in \mathbb{R}^n$,

$$\text{measure} \left\{ t \in [0, T] \mid y(t; x_a) \neq y(t; x_b) \right\} > 0$$

If it is observable in any time $T > 0$ for all inputs $u$, then it is said to be *uniformly observable* in small time.
Uniform observability

Theorem (Teel and Praly, 1994). If the system is

- globally state feedback stabilizable
- completely uniformly observable

then it is also semi-globally stabilizable by dynamic output feedback.

Problem: It is not generic to be uniformly observable when \( m \leq p \) (Gauthier and Kupka, 2001).

What if there is no uniform observability?
Uniform observability

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Non-uniformly observable systems

Definition (Control value at the target).

If \((x, \hat{x}) \to (0, 0)\), then \(u = \varpi(\hat{x}, h(x)) \to u^*\).

Does the constant input \(u \equiv 0\) make the system observable?

\[
\begin{aligned}
\dot{x} &= f(x, 0) \\
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\]

- Yes: Observable target
- No: Unobservable target
Non-uniformly observable systems

Definition (Control value at the target).

If \((x, \hat{x}) \to (0, 0)\), then \(u = \varpi(\hat{x}, h(x)) \to u^* = 0\).

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Observable target

No

Unobservable target
Main obstacles and tools

Main problems to be tackled:

- Observable target: avoid observability singularities when possible
- Unobservable target: observation and stabilization are antagonistic

Main techniques to deal with these problems:

- Feedback perturbation
- Dissipative systems
- Embeddings
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- Dissipative systems
- Embeddings
Observable target
Systems under consideration: SISO bilinear systems with linear output

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\dot{x} &= (A + uB)x + bu \\
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\(x \in \mathbb{R}^n, \ u \in \mathbb{R}, \ y \in \mathbb{R}, \ A, B \in \mathbb{R}^{n \times n}, \ C \in \mathbb{R}^{1 \times n}, \ b \in \mathbb{R}^n.\)

State feedback: \(u = \lambda(x)\) is globally asymptotically stabilizing.

Observer system:

\[
\dot{x} = (A + uB)\hat{x} + bu - PC^*C(\hat{x} - x)
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- \(\dot{P} = 0\) (Luenberger observer)
- \(\dot{P} = (A + uB)P + P(A + uB)^* + Q - PC^*CP\) (Kalman observer)
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Feedback perturbation

**Issue:** What happens if the chosen control $u = \lambda(\hat{x})$ makes the system unobservable?

**General idea:** Feedback modification

- Periodic additive perturbation
  
  $$u(t) = \lambda(\hat{x}(t)) \rightarrow u(t) = \lambda(\hat{x}(t)) + d(t)$$

  - [Combes et al., 2016]: virtual measurements

- Switching strategy
  
  - Excite the system to estimate the state, then control to stabilize, etc.
  - [Coron, 1994]: local stabilization
  - [Shim and Teel, 2002]: practical stabilization

- Smooth autonomous perturbation
  
  $$u(t) = \lambda(\hat{x}(t)) \rightarrow u(t) = \lambda(\hat{x}(t)) + \delta(\hat{x}(t))$$

  - [Lagache, Serres and Gauthier, 2017]: additive perturbation
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  $$u(t) = \lambda(\hat{x}(t)) \rightarrow u(t) = (\lambda + \delta)(\hat{x}(t))$$

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**Strategy** of [Lagache, Serres and Gauthier, 2017]:

1. Show that there exists a (smooth) small perturbation $\delta$ such that 
   $$u = (\lambda + \delta) \circ \hat{x}$$ makes the system observable.
2. Show that for this input, the observer converges to the state (and remains in a fixed compact set).
3. Show asymptotic stability.

**Remarks:**

- Example of quantum control;
- **Unobservable** target;
- **Practical** stabilization and **exact** stabilization.

**Towards a generalization?**
Feedback perturbation

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**Towards a generalization?**
Avoiding observability singularities

**Theorem (Brivadis, Gauthier, Sacchelli and Serres, 2021).**

\[
\begin{aligned}
\dot{x} &= (A + uB)x + bu \\
y &= Cx, \\
\dot{\hat{x}} &= (A + uB)\hat{x} + bu - PC^*C(\hat{x} - x)
\end{aligned}
\]

If
- \(\lambda\) is a smooth globally stabilizing feedback
- \((C, A)\) and \((C, B)\) are observable

then, for any compact \(K\), there exists \(\mathcal{N} \subset C^\infty(\mathbb{R}^n, \mathbb{R})\),

\[
\mathcal{N} = \left\{ \delta \mid \sup_{K_x} |\partial^\sigma \delta| < \varepsilon \text{ for } |\sigma| < k \text{ and } \delta = 0 \text{ in a nbh. of 0} \right\}
\]

such that
- \(\delta \in \mathcal{N} \implies \lambda + \delta\) is a semi-globally stabilizing state feedback
- \(\exists \Delta\) open and dense (Whitney \(C^\infty\)) in \(\mathcal{N}\) such that:
  \(\forall \delta \in \Delta, \forall (x_0, \hat{x}_0, P_0) \in K\), the system is observable in small time
  for the control \(u = (\lambda + \delta) \circ \hat{x}\).
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  for the control \(u = (\lambda + \delta) \circ \hat{x}\).
Avoiding observability singularities

We can prove the convergence of bounded trajectories.

If $x$ and $\hat{x}$ remain in $K'$, then $\hat{x} - x$ converges to zero (thanks to $\delta$), and stabilization is achieved.

Otherwise, one must adapt the observer gain. How does it affect $\delta$?
Avoiding observability singularities

We can prove the convergence of **bounded trajectories**.

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Dissipative systems

Definition (Dissipativity).

\[
\begin{align*}
\dot{x} &= A(u)x + B(u) \\
y &=Cx
\end{align*}
\]

with

\[
x'A(u)x \leq 0, \quad \forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^p.
\]

Luenberger observer:

\[
\begin{align*}
\dot{x} &= A(u)x - \alpha C' C \varepsilon \\
\dot{\varepsilon} &= (A(u) - \alpha C' C) \varepsilon
\end{align*}
\]

\(\alpha > 0\)

The norm of \(\varepsilon\) is non-increasing:

\[
\frac{1}{2} \frac{d}{dt} |\varepsilon|^2 = \varepsilon' A(u) \varepsilon - \alpha \varepsilon' C' C \varepsilon \leq -\alpha |C \varepsilon|^2 \leq 0
\]
Dissipative systems

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Dissipative systems

Theorem (Sacchelli, Brivadis, Andrieu, Serres and Gauthier, 2020). If

- $\lambda$ is a globally stabilizing feedback
- $A(u)$ is dissipative

then the system is semi-globally asymptotically stabilizable if and only if $(C, A(0))$ is detectable. The dynamic output feedback is given by

$$
\begin{cases}
\dot{x} = A(u)\hat{x} + B(u) - \alpha C'(C\hat{x} - y) \\
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Strategy of the proof:

- $|\epsilon|$ is non-increasing
- choose $\alpha$ sufficiently small
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**Strategy of the proof:**

- \(|\varepsilon|\) is non-increasing
- choose \( \alpha \) sufficiently small
Unobservable target
An illustrative example

Example inspired by [Coron, 1994]:

\[
\begin{cases}
\dot{x} = Jx + bu \\
y = h(|x|)
\end{cases}, \quad J' = -J, \quad (J, b) \text{ controllable}
\]

with \( x \in \mathbb{R}^n, u \in \mathbb{R}, y \in \mathbb{R} \).

**Unobservability** at \( u \equiv 0 \):

\[ |x_a| = |x_b| \implies y(t; x_a) \equiv y(t; x_b) \]

**Theorem** (Brivadis, Gauthier, Sacchelli and Serres, 2021).

If \( J \) is not invertible, then the system **not** stabilizable by means of dynamic output feedback.

This is due to a **topological constraint**.
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Embedding into a dissipative system

Theorem (Brivadis, Gauthier, Sacchelli and Serres, 2021). If $J$ is invertible, $(J, b)$ is controllable and $h(|x|) = |x|^2/2$, then the system is stabilizable by means of dynamic output feedback.

**Idea:** Embedding into a bilinear dissipative system with linear output.

$$z = \tau(x) = (x, \frac{1}{2}|x|^2).$$

\[
\begin{align*}
\dot{x} &= Jx + bu \\
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x(0) &\in \mathbb{R}^n
\end{align*} \quad \rightarrow \quad \begin{align*}
\dot{z} &= A(u)z + Bu \\
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z(0) &\in \tau(\mathbb{R}^n) \subset \mathbb{R}^{n+1}
\end{align*}
\]

$$A(u) = \begin{pmatrix} J & 0 \\ ub' & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}.$$
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Embedding into a dissipative system

Observer system:

\[
\begin{align*}
\dot{\varepsilon} &= (A(u) - L(u)C)\varepsilon \\
\dot{\hat{z}} &= A(u)\hat{z} + Bu - L(u)C\varepsilon.
\end{align*}
\]

with \( L(u) = \begin{pmatrix} bu \\ \alpha \end{pmatrix} \) and \( \alpha > 0 \). Then,

\[ A(u) - L(u)C = \begin{pmatrix} J & -bu \\ ub' & 0 \end{pmatrix} - \underbrace{\alpha C' C}_{\text{Luenberger correction term}} \]

Feedback law:

\[ u = \begin{pmatrix} K & 0 \end{pmatrix}\hat{z} + \delta \hat{z}_{n+1} \] with \( J + bK \) Hurwitz and \( \delta > 0 \)
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Sketch of the proof:

- Trajectories are **bounded**: choose $\alpha$ large enough
- **Observability analysis**: If $(\varepsilon_0, \hat{z}_0) \neq (0, 0)$, then the perturbed feedback makes the system observable in small time. Hence, $\varepsilon \to 0$.
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Theorem (Brivadis, Sacchelli, 2021).

If \((J, b)\) is controllable and \(h(|x|) = |x|^2/2\), Then the system is globally stabilizable by means of a periodic time-varying dynamic output feedback.

Proof by switching strategy:

- **Observation phase:** On \([t_k, t_k + T]\), apply a control making the system observable except at the target point. Use the embedded observer.

- **Stabilization phase:** On \([t_k + T, t_{k+1}]\), apply a stabilizing control, based on a stabilizing feedback and the observer.

Open question: How to embed a nonlinear system into a system for which an observer with dissipative error can be designed?
Embedding into a dissipative system

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Definition (Unitary representation). A unitary representation of a group $G$ is a group morphism $\rho : G \rightarrow \mathcal{L}(H)$, where $\mathcal{L}(H)$ are the bounded endomorphism of the Hilbert space $H$ and $\rho(g)$ is a unitary operator for all $g \in G$.

When considering control systems, the group of interest is the Lie group of diffeomorphisms generated by the dynamics:

For all $u \in \mathbb{R}$ and all $T > 0$, the flow over $[0, T]$ of

$$\dot{x} = f(x, u)$$

defines a diffeomorphism of $\mathbb{R}^n$.

We can use a unitary representation $\rho$ to define an embedding into a dissipative system.
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Embedding into an infinite-dimensional dissipative system

The dynamical system

\[ \dot{x} = Jx + bu, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

generates the group \( SE(2) = \mathbb{R}^2 \rtimes SO(2) \) (motions of the plane).

**Unitary representations:** \( \rho : SE(2) \rightarrow \mathcal{L}(L^2(S^1; \mathbb{C})) \)

\[ \rho((x_1, x_2), \beta) \cdot \xi(\theta) = e^{i\mu(x_1 \cos \theta + x_2 \sin \theta)} \xi(\theta - \beta) \]

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\[ \langle \tau(x), 1 \rangle_{L^2} = J_0(\mu|x|). \]

\( J_0 \): Bessel function of the first kind of order 0.
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\[ \langle \tau(r \cos(s), r \sin(s)), e^{ik\theta} \rangle_{L^2} = i^k J_k(\mu r) e^{-iks}. \]

\( J_k \): Bessel function of the first kind of order \( k \).
Embedding into an infinite-dimensional dissipative system

Bessel functions

$J_0(x)$
$J_1(x)$
$J_2(x)$
Embedding into an infinite-dimensional dissipative system

The system is now embedded via

\[ z = \tau(x) : \theta \in S^1 \mapsto e^{i\mu(x_1 \cos(\theta) + x_2 \sin(\theta))}. \]

\[
\begin{align*}
\dot{x} &= Jx + bu \\
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\]

\[
\begin{align*}
\dot{z} &= A(u)z \\
J_0(\mu y) &= Cz \\
z_0 &\in \text{Im } \tau \subset L^2(S^1; \mathbb{C})
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\]

\[ A(u) = -\frac{\partial}{\partial \theta} + i\mu u \sin(\theta), \quad Cz = \langle z, 1 \rangle_{L^2}. \]

How to build an observer for this infinite-dimensional system?
Embedding into an infinite-dimensional dissipative system

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\[
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\end{cases}
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\[ A(u) = -\frac{\partial}{\partial \theta} + i\mu u \sin(\theta), \quad Cz = \langle z, \sum_{\text{finite}} c_k e^{ik\theta} \rangle_{L^2}. \]

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How to build an observer for this infinite-dimensional system?
**Infinite-dimensional observer**

**Luenberger observer:**

\[
\begin{aligned}
\dot{z} &= A(u(t))z, \quad z(0) = z_0 \in H \\
y &= Cz \\
\dot{\hat{z}} &= A(u(t))\hat{z} - \alpha C^* C \varepsilon, \quad \hat{z}(0) = \hat{z}_0 \in H \\
\dot{\varepsilon} &= (A(u(t)) - \alpha C^* C)\varepsilon, \quad \varepsilon(0) = \varepsilon_0 = \hat{z}_0 - z_0
\end{aligned}
\]

- $H$ and $Y$ are Hilbert spaces, $C \in \mathcal{L}(H, Y)$
- $A(u(t)) : \mathcal{D}(A) \to H$ generates an evolution system $(T(t, s))_{t \geq s \geq 0}$
- $z, \hat{z}, \varepsilon \in C^0(\mathbb{R}_+; H)$

**Example:** $H = L^2(S^1; \mathbb{C}), \ Y = \mathbb{C}, \ \mathcal{D}(A) = H^1(S^1; \mathbb{C})$
**Infinite-dimensional observer**

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**Example:** $H = L^2(\mathbb{S}^1; \mathbb{C}), \ Y = \mathbb{C}, \ \mathcal{D}(A) = H^1(\mathbb{S}^1; \mathbb{C})$
Infinite-dimensional observer

Recall that $A(u) = -\frac{\partial}{\partial \theta} + i\mu u \sin(\theta)$ is **skew-adjoint**. Hence

\[
\frac{1}{2} \frac{d}{dt} \|\epsilon(t)\|_H^2 = \langle \epsilon(t), \dot{\epsilon}(t) \rangle_H
\]

\[
= \langle \epsilon(t), A(u(t))\epsilon(t) \rangle_H - \alpha \langle \epsilon(t), C^* C \epsilon(t) \rangle_H
\]

\[
\leq -\alpha \| C \epsilon(t) \|_Y^2
\]

\[
\leq 0.
\]

More generally, if

\[
\langle A(u(t))z, z \rangle_H \leq p \| Cz \|_Y^2, \quad \forall z \in H, \forall t \geq 0 \quad (**weak\ detectability**)
\]

for some $p > 0$, then

\[
\frac{1}{2} \frac{d}{dt} \|\epsilon(t)\|_H^2 \leq -(\alpha - p) \| C \epsilon(t) \|_Y^2 \leq 0 \quad \text{if } \alpha > p.
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Infinite-dimensional observer

Definition (Exact observability).
The system is **exactly** observable on \([t_0, t_0 + T]\) if for some \(k > 0\),

\[
\int_{t_0}^{t_0+T} \| C \mathbb{T}(t, t_0) z_0 \|^2_Y dt \geq k \| z_0 \|^2_H, \quad \forall z_0 \in H
\]

Definition (Approximate observability).

\[
\emptyset = \left\{ z_0 \in H \mid \int_{t_0}^{t_0+T} \| C \mathbb{T}(t, t_0) z_0 \|^2_Y dt = 0 \right\} \perp
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The system is **approximately** observable on \([t_0, t_0 + T]\) if \(\emptyset = H\).
Infinite-dimensional observer

Definition (Exact observability).
The system is **exactly** observable on \([t_0, t_0 + T]\) if for some \(k > 0\),

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\int_{t_0}^{t_0 + T} \| C_T(t, t_0)z_0 \|_Y^2 \, dt \geq k \| z_0 \|_H^2, \quad \forall z_0 \in H
\]

Definition (Approximate observability).

\[
\mathcal{O} = \left\{ z_0 \in H \mid \int_{t_0}^{t_0 + T} \| C_T(t, t_0)z_0 \|_Y^2 \, dt = 0 \right\} \perp
\]

The system is **approximately** observable on \([t_0, t_0 + T]\) if \(\mathcal{O} = H\).
Theorem (Brivadis, Andrieu, Serres and Gauthier, 2021).
Assumptions:

- \(((A(u(t))))_{t \geq 0}, C)\) is weakly detectable, \(\alpha > p\)
- \(\exists (t_n)_{n \geq 0} \to +\infty\) and an evolution system \((T_\infty(t, s))_{0 \leq s \leq t}\) s.t.

\[
\|T(t_n + t, t_n) - T_\infty(t, 0)\|_{\mathcal{L}(H)} \xrightarrow{n \to +\infty} 0
\]
uniformly in \(t \in [0, \tau], \forall \tau > 0\).

Let \(\mathcal{O}\) be the observable subspace of \((T_\infty, C)\) in infinite-time. Then

\[
\langle \varepsilon(t_n), \psi \rangle_H \xrightarrow{n \to +\infty} 0, \quad \forall \psi \in \mathcal{O}, \forall \varepsilon_0 \in H.
\]

Moreover, if \((t_{n+1} - t_n)_{n \geq 0}\) is bounded and \(\mathcal{O} = H\), then \(\varepsilon(t) \xrightarrow{t \to +\infty} 0\).
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Infinite-dimensional observer

Theorem (Brivadis, Gauthier, Sacchelli and Serres, 2021). If

- $\exists g \mid g(h(r \cos(s), r \sin(s))) = \sum_{\text{finite}} c_k J_k(\mu r)e^{-iks}$
- $u \equiv 0$ is an isolated observability singularity

then there exist a **left-inverse** $\tau^{-1}$ and a **feedback** $u = \lambda(\hat{x}) + \delta(\hat{z})$ such that the closed-loop system is asymptotically stable.
Main differences with the finite dimensional embedding:

- The embedded system is unitary.
- $\varepsilon$ is convergent in the weak topology.
- Depending on the isolation of $u \equiv 0$, a discretization of the control may be required.

Towards a generalization:

- The existence of an embedding of a dynamical system into a unitary one, either finite or infinite-dimensional, has been studied for observer design in [Celle, Gauthier, Kazakos and Sallet, 1989]
- If $\varepsilon \xrightarrow{\text{weak}} 0$, we can prove the existence of pseudo-inverse $\tau^{-1}$ such that $\tau^{-1}(\hat{z}) - x \to 0$. 
Unobservable target

**Main differences** with the finite dimensional embedding:

- The embedded system is unitary.
- $\varepsilon$ is convergent in the weak topology.
- Depending on the isolation of $u \equiv 0$, a discretization of the control may be required.

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- **Dissipative** systems
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  - Feedback perturbation to get observability
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  → Asymptotic stability?
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Our references on this subject:


Thank you for your attention