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Université Nice - Sophia Antipolis

Activity Report 2014

Project-Team MCTAO

Mathematics for Control, Transport and Applications

IN COLLABORATION WITH: Laboratoire Jean-Alexandre Dieudonné (JAD)

RESEARCH CENTER
Sophia Antipolis - Méditerranée

THEME
Optimization and control of dynamic systems

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Project-Team MCTAO

Keywords: Control Theory, Nonlinear Control, Optimal Control, Optimal Transport

McTAO is a common team with University of Nice, and it also has permanent members in Dijon; an agreement has been signed with University of Bourgogne, that officially makes it another location of the team.

Creation of the Team: 2012 January 01*, updated into Project-Team:* 2013 January 01.

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2. Overall Objectives

2.1. Overall Objectives

The core endeavor of this team is to develop methods in control theory for finite-dimensional nonlinear systems, as well as in optimal transport, and to be involved in applications of these techniques. Some mathematical fields like dynamical systems and optimal transport may benefit from control theory techniques. Our primary domain of industrial applications will be space engineering, namely designing trajectories in space mechanics using optimal control and stabilization techniques: transfer of a satellite between two Keplerian orbits, rendez-vous problem, transfer of a satellite from the Earth to the Moon or more complicated space missions. A second field of applications is quantum control with applications to Nuclear Magnetic Resonance and medical image processing.

3. Research Program

3.1. Control Systems

Our effort is directed toward efficient methods for the *control* of real (physical) systems, based on a *model* of the system to be controlled. *System* refers to the physical plant or device, whereas *model* refers to a mathematical representation of it.

We mostly investigate nonlinear systems whose nonlinearities admit a strong structure derived from physics; the equations governing their behavior are then well known, and the modeling part consists in choosing what phenomena are to be retained in the model used for control design, the other phenomena being treated as perturbations; a more complete model may be used for simulations, for instance. We focus on systems that admit a reliable finite-dimensional model, in continuous time; this means that models are controlled ordinary differential equations, often nonlinear.

Choosing accurate models yet simple enough to allow control design is in itself a key issue; however, modeling or identification as a theory is not per se in the scope of our project.

The extreme generality and versatility of linear control do not contradict the often heard sentence "most real life systems are nonlinear". Indeed, for many control problems, a linear model is sufficient to capture the important features for control. The reason is that most control objectives are local, first order variations around an operating point or a trajectory are governed by a linear control model, and except in degenerate situations (non-controllability of this linear model), the local behavior of a nonlinear dynamic phenomenon is dictated by the behavior of first order variations. Linear control is the hard core of control theory and practice; it has been pushed to a high degree of achievement –see for instance some classics: [45], [35]– that leads to big successes in industrial applications (PID, Kalman filtering, frequency domain design, H^∞ robust control, etc...). It must be taught to future engineers, and it is still a topic of ongoing research.

Linear control by itself however reaches its limits in some important situations:

1. **Non local control objectives.** For instance, steering the system from a region to a reasonably remote other one (path planning and optimal control); in this case, local linear approximation cannot be sufficient.
   It is also the case when some domain of validity (e.g. stability) is prescribed and is larger than the region where the linear approximation is dominant.
2. **Local control at degenerate equilibria.** Linear control yields local stabilization of an equilibrium point based on the tangent linear approximation if the latter is controllable. When it is *not*, and this occurs in some physical systems at interesting operating points, linear control is irrelevant and specific nonlinear techniques have to be designed.
   This is in a sense an extreme case of the second paragraph in point 1: the region where the linear approximation is dominant vanishes.
3. **Small controls.** In some situations, actuators only allow a very small magnitude of the effect of control compared to the effect of other phenomena. Then the behavior of the system without control plays a major role and we are again outside the scope of linear control methods.
4. **Local control around a trajectory.** Sometimes a trajectory has been selected (this appeals to point 1), and local regulation around this reference is to be performed. Linearization in general yields, when the trajectory is not a single equilibrium point, a *time-varying* linear system. Even if it is controllable, time-varying linear systems are not in the scope of most classical linear control methods, and it is better to incorporate this local regulation in the nonlinear design, all the more so as the linear approximation along optimal trajectories is, by nature, often non controllable.

Let us discuss in more details some specific problems that we are studying or plan to study: classification and structure of control systems in section 3.2, optimal control, and its links with feedback, in section 3.3, the problem of optimal transport in section 3.4, and finally problems relevent to a specific class of systems where the control is "small" in section 3.5.

3.2. Structure of nonlinear control systems

In most problems, choosing the proper coordinates, or the right quantities that describe a phenomenon, sheds light on a path to the solution. In control systems, it is often crucial to analyze the structure of the model, deduced from physical principles, of the plant to be controlled; this may lead to putting it via some transformations in a simpler form, or a form that is most suitable for control design. For instance, equivalence to a linear system may allow to use linear control; also, the so-called "flatness" property drastically simplifies path planning [40], [51].

A better understanding of the "set of nonlinear models", partly classifying them, has another motivation than facilitating control design for a given system and its model: it may also be a necessary step towards a theory of "nonlinear identification" and modeling. Linear identification is a mature area of control science; its success is mostly due to a very fine knowledge of the structure of the class of linear models: similarly, any progress in the understanding of the structure of the class of nonlinear models would be a contribution to a possible theory of nonlinear identification.

These topics are central in control theory, but raise very difficult mathematical questions: static feedback classification is a geometric problem which is feasible in principle, although describing invariants explicitly is technically very difficult; and conditions for dynamic feedback equivalence and linearization raise unsolved mathematical problems, that make one wonder about decidability [1].

3.3. Optimal control and feedback control, stabilization

3.3.1. Optimal control.

Mathematically speaking, optimal control is the modern branch of the calculus of variations, rather well established and mature [18], [49], [26], [58]. Relying on Hamiltonian dynamics is now prevalent, instead of the standard Lagrangian formalism of the calculus of variations. Also, coming from control engineering, constraints on the control (for instance the control is a force or a torque, which are naturally bounded) or the state (for example in the shuttle atmospheric re-entry problem there is a constraint on the thermal flux) are imposed; the ones on the state are usual but these on the state yield more complicated necessary optimality conditions and an increased intrinsic complexity of the optimal solutions. Also, in the modern treatment, ad-hoc numerical schemes have to be derived for effective computations of the optimal solutions.

What makes optimal control an applied field is the necessity of computing these optimal trajectories, or rather the controls that produce these trajectories (or, of course, close-by trajectories). Computing a given optimal trajectory and its control as a function of time is a demanding task, with non trivial numerical difficulties: roughly speaking, the Pontryagin Maximum Principle gives candidate optimal trajectories as solutions of a two point boundary value problem (for an ODE) which can be analyzed using mathematical tools from geometric control theory or solved numerically using shooting methods. Obtaining the *optimal synthesis* –the optimal control as a function of the state– is of course a more intricate problem [26], [31].

These questions are not only academic for minimizing a cost is *very* relevant in many control engineering problems. However, modern engineering textbooks in nonlinear control systems like the "best-seller" [42] hardly mention optimal control, and rather put the emphasis on designing a feedback control, as regular and explicit as possible, satisfying some qualitative (and extremely important!) objectives: disturbance attenuation, decoupling, output regulation or stabilization. Optimal control is sometimes viewed as disconnected from automatic control... we shall come back to this unfortunate point.

[1]Consider the simple system with state $(x, y, z) \in I\!R^3$ and two controls that reads $\dot{z} = (\dot{y} - z\dot{x})^2\dot{x}$ after elimination of the controls; it is not known whether it is equivalent to a linear system, or flat; this is because the property amounts to existence of a formula giving the general solution as a function of two arbitrary functions of time and their derivatives up to a certain order, but no bound on this order is known a priori, even for this very particular example.

3.3.2. Feedback, control Lyapunov functions, stabilization.

A control Lyapunov function **(CLF)** is a function that can be made a Lyapunov function (roughly speaking, a function that decreases along all trajectories, some call this an "artificial potential") for the closed-loop system corresponding to *some* feedback law. This can be translated into a partial differential relation sometimes called "Artstein's (in)equation" [21]. There is a definite parallel between a CLF for stabilization, solution of this differential inequation on the one hand, and the value function of an optimal control problem for the system, solution of a HJB equation on the other hand. Now, optimal control is a quantitative objective while stabilization is a qualitative objective; it is not surprising that Artstein (in)equation is very under-determined and has many more solutions than HJB equation, and that it may (although not always) even have smooth ones.

We have, in the team, a longstanding research record on the topic of construction of CLFs and stabilizing feedback controls. This is all the more interesting as our line of research has been pointing in almost opposite directions. [36], [55], [57] insist on the construction of continuous feedback, hence smooth CLFs whereas, on the contrary, [34], [59], [60] proceed with a very fine study of non-smooth CLFs, yet good enough (semi-concave) that they can produce a reasonable discontinuous feedback with reasonable properties.

3.4. Optimal Transport

We believe that matching optimal transport with geometric control theory is one originality of our team. We expect interactions in both ways.

The study of optimal mass transport problems in the Euclidean or Riemannian setting has a long history which goes from the pioneer works of Monge [53] and Kantorovitch [46] to the recent revival initiated by fundamental contributions due to Brenier [32] and McCann [52].

Th same transportation problems in the presence of differential constraints on the set of paths —like being an admissible trajectory for a control system— is quite new. The first contributors were Ambrosio and Rigot [19] who proved the existence and uniqueness of an optimal transport map for the Monge problem associated with the squared canonical sub-Riemannian distance on the Heisenberg groups. This result was extended later by Agrachev and Lee [16], then by Figalli and Rifford [37] who showed that the Ambrosio-Rigot theorem holds indeed true on many sub-Riemannian manifolds satisfying reasonable assumptions. The problem of existence and uniqueness of an optimal transport map for the squared sub-Riemannian distance on a general complete sub-Riemannian manifold remains open; it is strictly related to the regularity of the sub-Riemannian distance in the product space, and remains a formidable challenge. Generalized notions of Ricci curvatures (bounded from below) in metric spaces have been developed recently by Lott and Villani [50] and Sturm [63], [64]. A pioneer work by Juillet [43] captured the right notion of curvature for subriemannian metric in the Heisenberg group; Agrachev and Lee [17] have elaborated on this work to define new notions of curvatures in three dimensional sub-Riemannian structures. The optimal transport approach happened to be very fruitful in this context. Many things remain to do in a more general context.

3.5. Small controls and conservative systems, averaging

Using averaging techniques to study small perturbations of integrable Hamiltonian systems dates back to H. Poincaré or earlier; it gives an approximation of the (slow) evolution of quantities that are preserved in the non-perturbed system. It is very subtle in the case of multiple periods but more elementary in the single period case, here it boils down to taking the average of the perturbation along each periodic orbit; see for instance [20], [62].

When the "perturbation" is a control, these techniques may be used after deciding how the control will depend on time and state and other quantities, for instance it may be used after applying the Pontryagin Maximum Principle as in [23], [24], [33], [41]. Without deciding the control a priori, an "average control system" may be defined as in [22].

The focus is then on studying into details this simpler "averaged" problem, that can often be described by a Riemannian metric for quadratic costs or by a Finsler metric for costs like minimum time.

This line of research stemmed out of applications to space engineering, see section 4.1. For orbit transfer in the two-body problem, an important contribution was made by B. Bonnard, J.-B. Caillau and J. Gergaud [24] in explicitly computing the solutions of the average system obtained after applying Pontryagin Maximum Principle to minimizing a quadratic integral cost; this yields an explicit calculation of the optimal control law itself. Studying the Finsler metric issued form the time-minimal case is in progress.

4. Application Domains

4.1. Space engineering, satellites, low thrust control

Space engineering is very demanding in terms of safe and high-performance control laws (for instance optimal in terms of fuel consumption, because only a finite amount of fuel is onborad a sattelite for all its "life"). It is therefore prone to real industrial collaborations.

We are especially interested in trajectory control of space vehicles using their own propulsion devices, outside the atmosphere. Here we discuss "non-local" control problems (in the sense of section 3.1 point 1): orbit transfer rather than station keeping; also we do not discuss attitude control.

In the geocentric case, a space vehicle is subject to
- gravitational forces, from one or more central bodies (the corresponding acceleration is denoted by $F_{\text{grav.}}$ below),
- a thrust, the control, produced by a propelling device; it is the $G\,u$ term below; assume for simplicity that control in all directions is allowed, *i.e.* G is an invertible matrix
- other "perturbating" forces (the corresponding acceleration is denoted by F_2 below).

In position-velocity coordinates, its dynamics can be written as

$$\ddot{x} \;=\; F_{\text{grav.}}(x,t)\;\left[+\;F_2(x,\dot{x},t)\right]\;+\;G(x,\dot{x})\,u\;,\qquad \|u\|\leq u_{\max}. \tag{1}$$

In the case of a single attracting central body (the earth) and in a geocentric frame, $F_{\text{grav.}}$ does not depend on time, or consists of a main term that does not depend on time and smaller terms reflecting the action of the moon or the sun, that depend on time. The second term is often neglected in the design of the control at first sight; it contains terms like athmospheric drag or solar pressure. G could also bear an explicit dependence on time (here we omit the variation of the mass, that decreases proportionnally to $\|u\|$.

4.1.1. Low thrust

Low thrust means that $u_{\max}$ is small, or more precisely that the maximum magnitude of $G\,u$ is small with respect to the one of $F_{\text{grav.}}$ (but in genral not compared to F_2). Hence the influence of the control is very weak instantaneously, and trajectories can only be significantly modified by accumulating the effect of this low thrust on a long time. Obviously this is possible only because the free system is somehow conservative. This was "abstracted" in section 3.5.

Why low thrust ? The common principle to all propulsion devices is to eject particles, with some relative speed with respect to the vehicle; conservation of momentum then induces, from the point of view of the vehicle alone, an external force, the "thrust" (and a mass decrease). Ejecting the same mass of particles with a higher relative speed results in a proportionally higher thrust; this relative speed (specific impulse, I_{sp}) is a characteristic of the engine; the higher the I_{sp}, the smaller the mass of particles needed for the same change in the vehicle momentum. Engines with a higher I_{sp} are highly desirable because, for the same maneuvers, they reduce the mass of "fuel" to be taken on-board the satellite, hence leaving more room (mass) for the payload. "Classical" chemical engines use combustion to eject particles, at a somehow limited speed even with very efficient fuel; the more recent electric engines use a magnetic field to accelerate particles and eject them at a considerably higher speed; however electrical power is limited (solar cells), and only a small amount of particles can be accelerated per unit of time, inducing the limitation on thrust magnitude.

Electric engines theoretically allow many more maneuvers with the same amount of particles, with the drawback that the instant force is very small; sophisticated control design is necessary to circumvent this drawback. High thrust engines allow simpler control procedures because they almost allow instant maneuvers (strategies consist in a few burns at precise instants).

4.1.2. Typical problems

Let us mention two.

- *Orbit transfer or rendez-vous*. It is the classical problem of bringing a satellite to its operating position from the orbit where it is delivered by the launcher; for instance from a GTO orbit to the geostationary orbit at a prescribed longitude (one says rendez-vous when the longitude, or the position on the orbit, is prescribed, and transfer if it is free). In equation (1) for the dynamics, $F_{\mathrm{grav.}}$ is the Newtonian gravitation force of the earth (it then does not depend on time); F_2 contains all the terms coming either from the perturbations to the Newtonian potential or from external forces like radiation pressure, and the control is usually allowed in all directions, or with some restrictions to be made precise.
- *Three body problem*. This is about missions in the solar system leaving the region where the attraction of the earth, or another single body, is preponderant. We are then no longer in the situation of a single central body, $F_{\mathrm{grav.}}$ contains the attraction of different planets and the sun. In regions where two central bodies have an influence, say the earth and the moon, or the sun and a planet, the term $F_{\mathrm{grav.}}$ in (1) is the one of the restricted three body problem and dependence on time reflects the movement of the two "big" attracting bodies.

  An issue for future experimental missions in the solar system is interplanetary flight planning with gravitational assistance. Tackling this global problem, that even contains some combinatorial problems (itinerary), goes beyond the methodology developed here, but the above considerations are a brick in this puzzle.

4.1.3. Properties of the control system.

If there are no restrictions on the thrust direction, i.e., in equation (1), if the control u has dimension 3 with an invertible matrix G, then the control system is "static feedback linearizable", and a fortiori flat, see section 3.2. However, implementing the static feedback transformation would consist in using the control to "cancel" the gravitation; this is obviously impossible since the available thrust is very small. As mentioned in section 3.1, point 3, the problem remains fully nonlinear in spite of this "linearizable" structure [2].

4.1.4. Context for these applications

The geographic proximity of Thales Alenia Space, in conjunction with the "Pole de compétitivité" PEGASE in PACA region is an asset for a long term collaboration between Inria - Sophia Antipolis and Thales Alenia Space (Thales Alenia Space site located in Cannes hosts one of the very few European facilities for assembly, integration and tests of satellites).

[2] However, the linear approximation around *any* feasible trajectory is controllable (a periodic time-varying linear system); optimal control problems will have no singular or abnormal trajectories.

B. Bonnard and J.-B. Caillau in Dijon have had a strong activity in optimal control for space, in collaboration with the APO Team from IRIT at ENSEEIHT (Toulouse), and sometimes with EADS, for development of geometric methods in numerical algorithms.

4.2. Quantum Control

These applications started by a collaboration between B. Bonnard and D. Sugny (a physicist from ICB) in the ANR project Comoc, localized mainly at the University of Dijon. The problem was the control of the orientation of a molecule using a laser field, with a model that does take into account the dissipation due to the interaction with the environment, molecular collisions for instance. The model is a dissipative generalization of the finite dimensional Schrödinger equation, known as Lindblad equation. It is a 3-dimensional system depending upon 3 parameters, yielding a very complicated optimal control problem that we have solved for prescribed boundary conditions. In particular we have computed the minimum time control and the minimum energy control for the orientation or a two-level system, using geometric optimal control and appropriate numerical methods (shooting and numerical continuation) [29], [28].

More recently, based on this project, we have reoriented our control activity towards Nuclear Magnetic Resonance (MNR). In MNR medical imaging, the contrast problem is the one of designing a variation of the magnetic field with respect to time that maximizes the difference, on the resulting image, between two different chemical species; this is the "contrast". This research is conducted with Prof. S. Glaser (TU-München), whose group is performing both in vivo and in vitro experiments; experiments using our techniques have successfully measured the improvement in contrast between materials chemical species that have an importance in medicine, like oxygenated and de-oxygenated blood, see [27]; this is however still to be investigated and improved. The model is the Bloch equation for spin $\frac{1}{2}$ particles, that can be interpreted as a sub-case of Lindblad equation for a two-level system; the control problem to solve amounts to driving in minimum time the magnetization vector of the spin to zero (for parameters of the system corresponding to one of the species), and generalizations where such spin $\frac{1}{2}$ particles are coupled: double spin inversion for instance.

Note that a reference book by B. Bonnard and D. Sugny has been published on the topic [30].

4.3. Applications of optimal transport

Optimal Transportation in general has many applications. Image processing, biology, fluid mechanics, mathematical physics, game theory, traffic planning, financial mathematics, economics are among the most popular fields of application of the general theory of optimal transport. Many developments have been made in all these fields recently. Two more specific fields:
- In image processing, since a grey-scale image may be viewed as a measure, optimal transportation has been used because it gives a distance between measures corresponding to the optimal cost of moving densities from one to the other, see e.g. the work of J.-M. Morel and co-workers [54].
- In representation and approximation of geometric shapes, say by point-cloud sampling, it is also interesting to associate a measure, rather than just a geometric locus, to a distribution of points (this gives a small importance to exceptional "outlier" mistaken points); this was developed in Q. Mérigot's PhD [56] in the GEOMETRICA project-team. The relevant distance between measures is again the one coming from optimal transportation.
- A collaboration between Ludovic Rifford and Robert McCann from the University of Toronto aims at applications of optimal transportation to the modeling of markets in economy; it was to subject of Alice Erlinger's PhD, unfortunately interrupted.

Applications *specific to the type of costs that we consider*, i.e. these coming from optimal control, are concerned with evolutions of densities under state or velocity constraints. A fluid motion or a crowd movement can be seen as the evolution of a density in a given space. If constraints are given on the directions in which these densities can evolve, we are in the framework of non-holonomic transport problems.

4.4. Applications to some domains of mathematics

Control theory (in particular thinking in terms of inputs and reachable set) has brought novel ideas and progresses to mathematics. For instance, some problems from classical calculus of variations have been revisited in terms of optimal control and Pontryagin's Maximum Principle [44]; also, closed geodesics for perturbed Riemannian metrics where constructed in [47], [48] using control techniques.

Inside McTAO, a work like [39], [38] is definitely in this line, applying techniques from control to construct some perturbations under constraints of Hamiltonian systems to solve longstanding open questions in the field of dynamical systems. Also, in [61], geometric control is applied successfully to obtain genericity properties for Hamiltonian systems.

5. New Software and Platforms

5.1. Hampath

Participants: Jean-Baptiste Caillau, Olivier Cots [corresponding participant], Joseph Gergaud.

Hampath is a software developped to solve optimal control problems but also to study Hamiltonian flow. It has been developped since 2009 by members of the APO team from Institut de Recherche en Informatique de Toulouse, jointly with colleagues from the Université de Bourgogne. It is now updated with McTAO team members. See more on http://cots.perso.math.cnrs.fr/hampath/.

6. New Results

6.1. Optimal control for quantum systems and NMR

Participants: Bernard Bonnard, Mathieu Claeys [Imperial College, UK], Olivier Cots, Thierry Combot, Pierre Martinon [project team COMMANDS], Alain Jacquemard [Université de Bourgogne, IMB].

- The contrast imaging problem in nuclear magnetic resonance can be modeled as a Mayer problem, in the terminology of optimal control. The candidates as minimizers are selected among a set of extremals, solutions of a Hamiltonian system given by the Pontryagin Maximum Principle; sufficient second order conditions are known; they form the geometric foundations of the HAMPATH code which combines shooting and continuation methods.

  In [4], based on these theoretical studies, a thorough analysis of the case of deoxygenated/oxygenated blood samples is pursued, based on many numerical experiments.
- We initiated more than a year ago a program to compare and study the complementarities between these methods based on the Pontryagin Maximum Principle are known as indirect methods,
  - with the so-called direct methods where optimal control is seem as a generic optimization problem, as implemented in the Bocop software, developed in the COMMANDS project-team,
  - and with LMI techniques used to obtain global bounds on the extremum;
  this was naturally done in collaboration with Pierre Martinon, an important contributor to Bocop and with Mathieu Claeys (LAAS CNRS, a PhD student supervised by J.-B. Lasserre, now with Imperial College). The results are very promising, and there is a gain, numerically, in using both direct and indirect methods while working towards global optimality (in the contrast problem there are many local optima and the global optimality is a complicated issue). This is presented in [3].

This also led to use algebraic techniques to further analyse the equations and their dependance of the materials to be discriminated [10].

- For time minimal control of a linear spin system with Ising coupling (more complex than the model above), we also analysed *integrability* properties of extremal solutions of the Pontryagin Maximum Principle, in relation with conjugate and cut loci computations. Restricting to the case of three spins, as in [11], the problem is equivalent to analyze a family of almost-Riemannian metrics on the sphere S^2 , with Grushin equatorial singularity. The problem can be lifted into a SR-invariant problem on $SO(3)$, this leads to a complete understanding of the geometry of the problem and to an explicit parametrization of the extremals using an appropriate chart as well as elliptic functions. This approach is compared with the direct analysis of the Liouville metrics on the sphere where the parametrization of the extremals is obtained by computing a Liouville normal form. This is backed by an algebraic approach applying differential Galois theory to integrability.

6.2. Conjugate and cut loci computations and applications

Participants: Bernard Bonnard, Olivier Cots, Jean-Baptiste Caillau, Alessio Figalli [Univ. of Texas at Austin, USA], Thomas Gallouët [MEPHYSTO project-team], Ludovic Rifford.

- Many optimal control problems from mechanics or quantum systems (see [11] and the last paragraph of section 6.1) lead to studying some king of singular metrics, sometimes known as almost-Riemannian. This led us to consider, in [2], metrics on the two-sphere of revolution of the following find: they are Riemannian on each open hemisphere whereas one term of the corresponding tensor becomes infinite on the equator. Length minimizing curves can be computed and structure results on the cut and conjugate loci given, extending those in [25]. These results rely on monotonicity and convexity properties of the quasi-period of the geodesics; such properties are studied on an example with elliptic transcendency. A suitable deformation of the round sphere allows to reinterpretate the equatorial singularity in terms of concentration of curvature and collapsing of the sphere.
- It is known that convexity of the injectivity domain (the boundary of which is sent by the exponential map to the first cut locus) and the “Ma–Trudinger–Wang condition” (an positivity condition on the Ma–Trudinger–Wang tensor) both play a very important role in the continuity of solutions of optimal transport problems. This led to study these properties on their own, and it is still an open question to decide under which conditions the latter implies the former. In [13], it is proved that the MTW condition implies the convexity of injectivity domains on a smooth nonfocal compact Riemannian manifold. This improves a previous result by Loeper and Villani.

6.3. Averaging in control and application to space mechanics

Participants: Bernard Bonnard, Helen-Clare Henninger, Jana Němcová [Institute of Chemical Tech, Prague, CZ], Jean-Baptiste Pomet, Jeremy Rouot.

As explained in sections 3.5 and 4.1, control problems where the non controlled system is conservative and the control effect is small compared to the free dynamics lead to computing an average system. This computation may be explicit or numerical.

Even though it will not be always the case that an explicit expression is available, it is interesting to study that case thoroughly.

- In [23], [24], a smooth Riemannian metric was introduced to describe the energy minimizing orbital transfer with low propulsion. We have pursued a study of its deformation due to the standard perturbations in space mechanics, e.g. oblate spheroid shape of the Earth and lunar attraction. In [12], using Hamiltonian formalism, we describe the effects of the perturbations on the orbital transfers and the deformation of the conjugate and cut loci of the original metric. This is done using averaging with respect to both the proper frequency of the space vehicle and the moon frequency.
- The average system has the advantage of being more controllable (it has new virtual controls), but often displays singularities that were not present in the original system. It is the case when minimum time is considered instead of the quadratic energy criterium. We are conducted an analysis of this average minimum time Hamiltonian flow.

In [6], we compare the two problems for planar transfers. While the energy case leads to analyze a 2D Riemannian metric using the standard tools of Riemannian geometry (curvature computations, geodesic convexity), the time minimal case is associated to a Finsler metric which is not smooth. Nevertheless a qualitative analysis of the geodesic flow is given in this article to describe the optimal transfers. In particular we prove geodesic convexity of the elliptic domain.

6.4. Applications of control methods to dynamical systems

Participants: Gonzalo Contreras, Alessio Figalli, Ayadi Lazrag, Ludovic Rifford, Raffael Ruggiero.

Ludovic Rifford and collaborators have been applying with success, techniques from geometric control theory to open problems in dynamical systems, mostly on genericity properties and using controllability methods to build suitable perturbations.

This has been applied to closing geodesics and weak-KAM theory [39], [38].

Ayadi Lazrag's PhD also deals with such problems; applying techniques close to these in [61], he established a version of Francks' lemma for geodesic flows; one goal is to apply this to persistence problems. The approach relies on control theory results, with order 2 conditions. See [14] and [15], where a non trivial conjecture on generic hyperbolicity of the so-called Aubry set of a Hamiltonian is solved on compact surfaces and in the C^2 topology (for genericity).

7. Bilateral Contracts and Grants with Industry

7.1. Thales Alenia Space - Inria

"Transfert orbital dans le problème des deux et trois corps avec la technique de propulsion faible".

This contract started October, 2012 for 3 years. It partially supports Helen Heninger's PhD.

The goal is to improve transfer strategies for guidance of a spacecraft in the gravitation field of one central body (the two-body problem) or two celestial bodies (three-body problem).

7.2. CNES - Inria - UMB

This three year contract will formally started in 2014. It involves CNES and McTAO both through Inria and through Université de Bourgogne. It concerns averaging techniques in orbit transfers around the earth while taking into acount many perturbation of the main force (gravity for the earth considered as circular). The objective is to validate numerically and theoretically the approximations made by using averaging, and to propose methods that refine the approximation.

8. Partnerships and Cooperations

8.1. Regional Initiatives

- The "région" *Provence Alpes Côte d'Azur* (PACA) partially supports Helen Heninger's PhD . The other part comes from Thales Alenia space, see section 7.1.
- The "région" *Provence Alpes Côte d'Azur* (PACA) partially supports Jérémy Rouot's PhD.

8.2. National Initiatives

8.2.1. IMB - Université de Bourgogne, Dijon

The team is officially a common team with University of Nice, but also has very strong links with Université de Bourgogne and IMB (Institute of Mathematics in Burgundy). Bernard Bonnard is currently on leave from Université de Bourgogne; Jean-Baptiste Caillau collaborates actively with us; there is also an active common seminar http://math.unice.fr/~rifford/publis/Journee_McTAO/J_McTAO.html . A formal convention between Inria and Université de Bourgogne has been signed in 2014. It makes the IMB control team a part of McTAO as of January, 2015.

8.2.2. Explosys (franco-german ANR project)

Bernard Bonnard is a memebr of this project, accepted in October, 2014. The coordinators are Dominique Sugny (Dijon) and Stefen Glaser (Munich). The budget is approximately 500 K€.

8.2.3. Others

Bernard Bonnard and Ludovic Rifford participate in the GDR MOA, a CNRS network on Mathematics of Optimization and Applications. http://gdrmoa.univ-perp.fr/.

Jean-Baptiste Caillau is in the board of governors of the group SMAI-MODE (http://smai.emath.fr/spip.php?article338).

Jean-Baptiste Caillau is a member of the Centre de Compétences Techniques (CCT) Mécanique orbitale du CNES

Jean-Baptiste Caillau is the corresponding member in Dijon for the Labex AMIES (http://www.agence-maths-entreprises.fr/).

8.3. International Initiatives

There is a strong collaboration with the control group in the University of Hawaii around M. Chyba. The purpose of the collaboration is to study the aspects of the contrast problem in Nuclear Magnetic Resonance.

9. Dissemination

9.1. Promoting Scientific Activities

9.1.1. Scientific events organisation

Ludovic Rifford was in the scientific comitee of the "Conference of Calculus of Variations: Geometry, Inequalities, and Design" within the Thematic Program on Variational Problems in Physics, Economics and Geometry, Fields INstitute, Toronto.

9.1.2. Journals

9.1.2.1. Member of the editorial board

L. Rifford is a member of the editorial board of *Discrete and Continuous Dynamical Systems - Series A* (AIMS Journal).

9.1.2.2. Reviewer

The members of the team reviewed numerous papers for international journals including: SIAM J. on Control and Optimisation, International J. of Control, IEEE Trans. Automatic Control, Acta Applicandae Mathematicae, Journal of Dynamical and Control Systems.

9.2. Teaching - Supervision - Juries

9.2.1. Teaching

B. Bonnard and L. Rifford did their teaching duty at Univ. Nice and Univ. Bourgogne (Esirem).

9.2.2. Supervision

Ph: Ayadi Lazrag, *Théorie de contrôle et systèmes dynamiques* (Control theory and dynamical systems), defended September 25, 2014, University of Nice, advisor: Ludovic Rifford.

Ph: Lionel Jassionnesse, *Contrôle optimal et métriques de Clairaut-Liouville avec applications*, Université de Bourgogne, started october, 2010, advisor: Bernard Bonnard.

PhD in progress: Helen Heninger, subject: *Étude des solutions du transfert orbital avec poussée faible dans le probleme des deux ou trois corps*, Université de Nice Sophia Antipolis, started october, 2012, advisors: Bernard Bonnard and Jean-Baptiste Pomet.

PhD in progress: Jérémy Rouot, subject: *Moyennisation en contrôle et en contrôle optimal, effet des perturbations non périodiques*, Université de Nice Sophia Antipolis, started october, 2013, advisors: Bernard Bonnard and Jean-Baptiste Pomet.

PhD in progress: Zeinab Badredine, subject: *Techniques d'intégrabilité en dynamique des spins et applications au contrôle optimal*, Université de Bourgogne, started october, 2014, advisors: Bernard Bonnard and Ludovic Rifford.

MSc: Sofya Maslovskaya, *Finsler metric associated with average minimum time problems*, Ensta ParisTech, supervisors: Jean-Baptiste Pomet.

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Articles in International Peer-Reviewed Journals

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International Conferences with Proceedings

[7] B. BONNARD, H. HENNINGER, J.-B. POMET. *Time minimization versus energy minimization in the one-input controlled Kepler problem with weak propulsion*, in "21st International Symposium on Mathematical Theory of Networks and Systems", Groningen, Netherlands, July 2014, pp. 686-688, https://hal.inria.fr/hal-01112429

Scientific Books (or Scientific Book chapters)

[8] B. BONNARD, M. CHYBA. *Singular trajectories in optimal control*, in "Encyclopedia of Systems and Control", J. BAILLIEUL, T. SAMAD (editors), Springer, February 2015, https://hal.inria.fr/hal-00939089

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