Modèles corticaux géométriques et simulation de la perception

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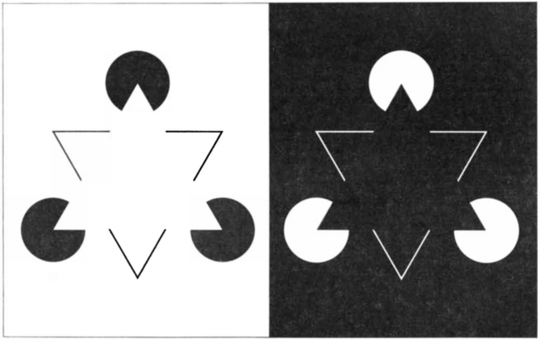
Orientation sensitivity and illusory contours

G. Kanizsa[1]: *Certain combination of incomplete figures give rise to clearly visible contours even when the contours do not actually exist. It appears that such contours are supplied by the visual system.*



[1]Kanizsa, G. *Subjective Contours*. Scientific American 234, no. 4 (1976): 48-53.

Outline of the talk

Geometric structure of V1

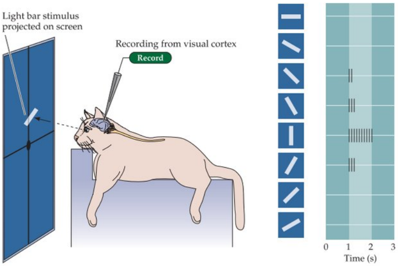
A sub-Riemannian structure for V1

A geometric model of the auditory cortex

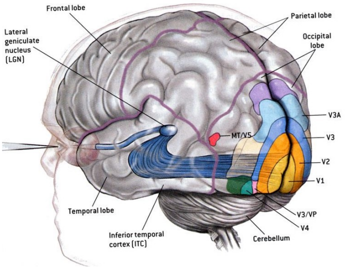
Geometric structure of the primary visual cortex V1

Orientation sensitivity

Hubel and Wiesel's 1959 experiment (Nobel prize in 1981):



Experimental setup



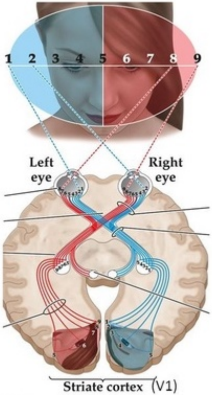
Visual cortex

Heightened visual sensitivity heightened by orientation information

- In V1: presence of neurons sensitive to orientation[2] in $\mathbb{P}^1$

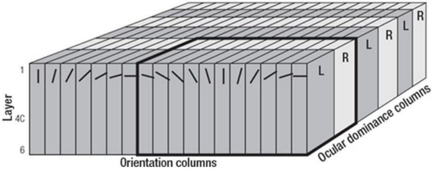
[2]Set of orientation: $\mathbb{P}^1 = [0, \pi]/\sim$ where $0 \sim \pi$.

V1 is linked to the field of view: an image is mapped one-to-one on V1



Retinotopic map

Sensitivities to orientation are then distributed in V1 according to a columnar structure

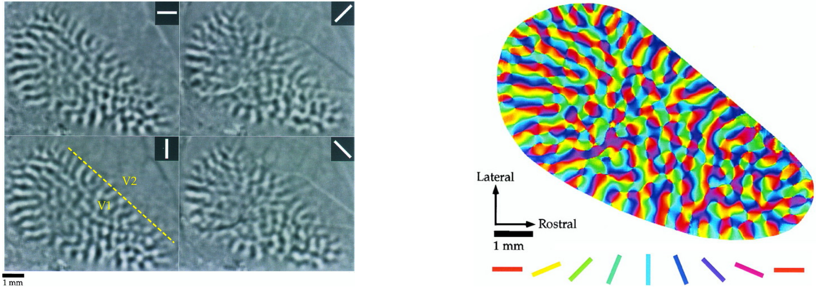


Columns and hypercolumns

We represent sensitivities in V1 as belonging to $\mathbb{R}^2 \times \mathbb{P}^1$

V1 cartography

With modern imaging techniques, precise images of V1's organization are obtained



Orientation sensitivity of V1 neurons in a tree shrew.[3]

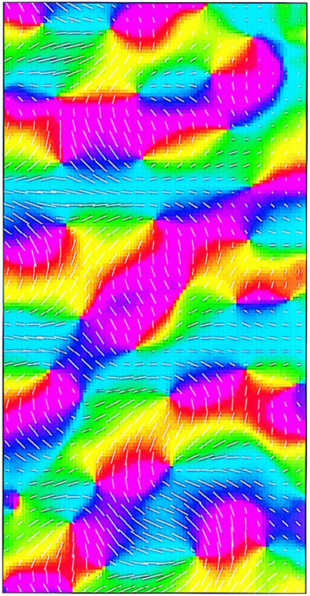
- Mapping the 3D space $\mathbb{R}^2 \times \mathbb{P}^1$ into $\mathbb{R}^2$ introduces singularities

[3] Bosking, W.H., et al. *Orientation Selectivity and the Arrangement of Horizontal Connections in Tree Shrew Striate Cortex* Journal of Neuroscience 15 March 1997, 17 (6) 2112-2127

Line fields singularities

▶ *Generic singularities of line fields on 2D manifolds* . U. Boscain, **L. Sacchelli**, M. Sigalotti. Diff. G. & App. 2016.

▶ *A bisector line field approach to interpolation of orientation fields* . N. Boizot, **L. Sacchelli**. JMIV 2021.



Map $x \in \mathbb{R}^2 \mapsto \theta \in \mathbb{P}^1$ is similar to vector fields.

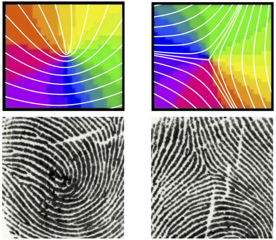
Sections of $PT\mathbb{R}^2 \simeq \mathbb{R}^2 \times \mathbb{P}^1$ are ill equipped for the challenge of modelling V1.

Hence the introduction of the bisector line field

Patterns in V1 match the ones appearing in

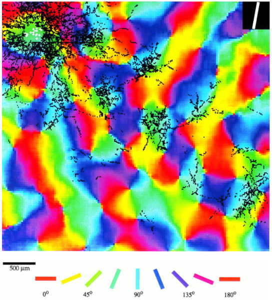
– liquid crystals in nematic phase

– finger ridges

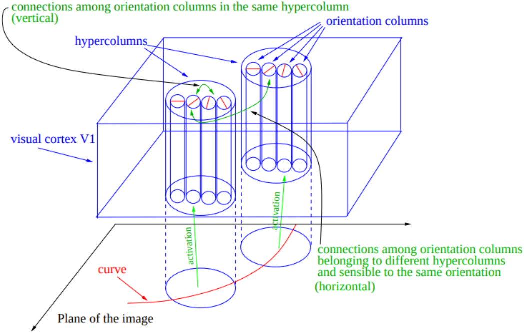


The Citti-Petitot-Sarti model: a sub-Riemannian structure for V1

Meta-structure: connections $\mathbb{R}^2 \times \mathbb{P}^1$



Connections in V1 are highly anisotropic



Model of V1

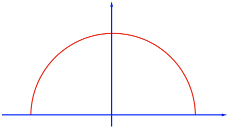
- The meta-structure of V1 matches $\mathbb{R}^2 \times \mathbb{P}^1$

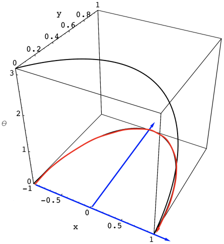
How is it translated to perceived curves ?

Curve perception and lift to the contact space

A smooth curve $t \mapsto (x(t), y(t))$ in $\mathbb{R}^2$ is detected by V1 as a curve $t \mapsto (x(t), y(t), \theta(t))$ in $\mathbb{R}^2 \times \mathbb{P}^1$ such that

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} /\!\!/ \begin{pmatrix} \cos\theta(t) \\ \sin\theta(t) \end{pmatrix}$$





This can be rewritten in the frame (X_1, X_2) such that

$$X_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

Then the curve follows the dynamical system

$$\frac{d}{dt} \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} = u(t) X_1 + v(t) X_2$$

Then length of a curve $\gamma : [0,1] \to \mathbb{R}^2 \times \mathbb{P}^1$ such that $\dot{\gamma} = uX_1(\gamma) + vX_2(\gamma)$ is

$$\ell(\gamma) = \int_0^1 |\dot{\gamma}| dt = \int_0^1 \sqrt{u^2 + v^2} dt.$$

In order to respect the lift structure, this induces a distance between neurons of V1 by a sub-Riemannian optimal control problem

The Citti-Petitot-Sarti[4] model of V1

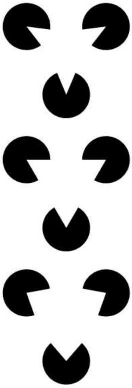
Neural connections follow the metric over $\mathbb{R}^2 \times \mathbb{P}^1$

$$\gamma(0) = p, \quad \gamma(1) = q, \qquad \dot{\gamma} = uX_1(\gamma) + vX_2(\gamma),$$

$$\int_0^1 \sqrt{u^2 + v^2} dt \to \min$$

[4] Petitot, J., Tondut, Y. *Vers une neurogéométrie. Fibrations corticales, structures de contact et contours subjectifs modaux*. Math. Sci. Hum. 145, 5-101 (1999)
Citti, G., Sarti, A. *A cortical based model of perceptual completion in the rototranslation space*. J. Math. Imag. Vis. 24(3), 307-326 (2006)

Illusory contours

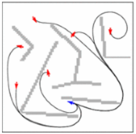


The optimal control problem $\gamma(0) = p, \quad \gamma(1) = q$

$$\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} = u(t) \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} + v(t) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \int_0^1 \sqrt{u^2 + v^2} dt \to \min$$

- This is a classical problem of length/curvature trade-off[5] with

$$\int_0^1 \sqrt{u^2 + v^2} dt = \int_0^1 \sqrt{\dot{x}^2 + \dot{y}^2} \sqrt{1 + \kappa^2} dt$$



[5] The Citti-Petitot-Sarti model focuses on $\mathbb{R}^2 \times \mathbb{P}^1$ however similar models on $\mathbb{R}^2 \times \mathbb{S}^1$ had already been proposed (see Mumford 94).

Diffusions and image inpainting

We have a model for neuronal connection in an ideal continuous visual cortex.

- Neuronal activity follows random walks in this structure.

Resulting diffusion process: $\partial_t \phi = \Delta \phi := X_1^2 \phi + X_2^2 \phi$

This allows to consider images perceived by V1 as whole for inpainting[6]

Original image

Corrupted image

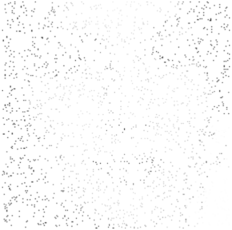
V1 diffusion

Full algorithm

[6] Boscain, U., et al.Image Reconstruction Via Non-Isotropic Diffusion in Dubins/Reed-Shepp-Like Control Systems. 53th IEEE Conference on Decision and Control, 2014

State of the art from this school

Algorithm based on the structure of the primary visual cortex (V1)



Reconstruction of an image with 97% of pixels missing
From Boscain, Prandi, *et al.* (2018).

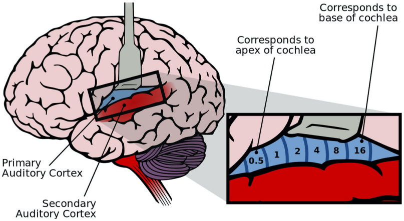
Main features

- Invariant under natural operations on images (roto-translations)
- Naturally highly parallelizable
- Respects crossings

A V1-inspired geometric model of the auditory cortex

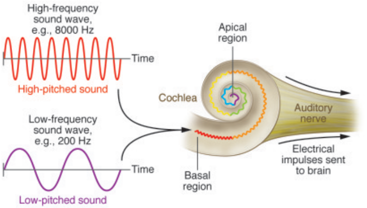
▶ *A bio-inspired geometric model for sound reconstruction.* U. Boscain, D. Prandi, **L. Sacchelli**, G. Turco. Journal of mathematical neuroscience, 2021.

Goals: obtain a auditory cortex model that is V1-inspired. Use it to design a bio-inspired sound reconstruction algorithm.



- Sound signal come equipped with *features* that we want to preserve
- Perceived sounds are carried to the time frequency domain before being processed
- The time frequency domain can be expanded with a 3D space with structure
- With neuronal activity simulations, input signals are extended with their features preserved

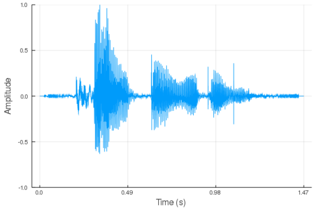
The auditory system



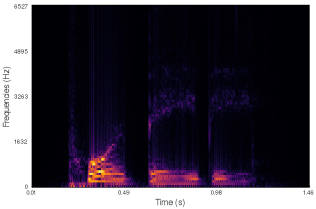
- Hair cells in the cochlea are *tonotopically* organized.
- The Auditory Cortex (AC) receives frequency-wise information.

Mathematical interpretation

AC receives the short-time Fourier transform (STFT) of the sound.



Sound $s : [0, T] \to \mathbb{R}$.



STFT $S : [0, T] \times \mathbb{R} \to \mathbb{C}$.

Structure of the Auditory Cortex (AC)

Neurophysiological facts

- V1 and AC share many structural features
- Sensitivity of V1 neurons to orientation (first order jet of the image)

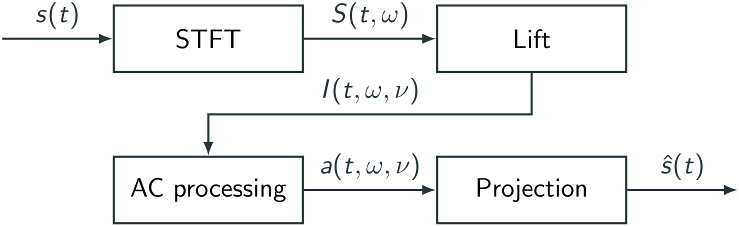
Mathematical fact

- AC is fed the STFT $S(t, \omega)$ of the sound signal at time t
- Its first order jet is the *instantaneous chirpiness* $\nu = \frac{d\omega}{dt}$.

Assumption

AC is modeled as the set $\{(\omega, \nu) \in \mathbb{R} \times \mathbb{R}\}$.

Sound processing algorithm



We need to explain:

- Lift procedure
- Signal processing in the Auditory Cortex
- Projection of the processed signal to a sound

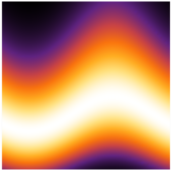
Lift to AC

Lift of curves

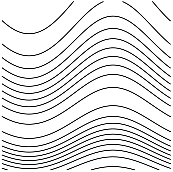
A curve $t \mapsto \omega(t)$ in the time-frequency domain is lifted to AC as $t \mapsto \left(\omega(t), \frac{d\omega(t)}{dt}\right)$

The lift of S to AC is defined by

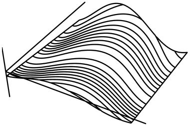
$$I(t,\omega,\nu) = \begin{cases} S(t,\omega) & \text{if } \nu\partial_\omega|S|(t,\omega) + \partial_t|S|(\tau,\omega) = 0 \\ 0 & \text{otherwise.} \end{cases}$$



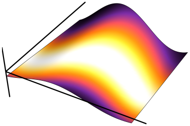
Heatmap of $|S|$



Level sets of $|S|$



Lift of level sets



Lift of S

AC processing

Neuronal dynamics are modeled via neural mean field equations:

$$\partial_t a(t,\omega,\nu) = -\alpha a(t,\omega,\nu) + \beta I(t,\omega,\nu) \\ + \gamma \int_{\mathbb{R}^2} k_\tau(\omega,\nu \| \omega',\nu') \sigma(a(t-\tau,\omega',\nu'))\, d\omega'\, d\nu'$$

First introduced by Wilson & Cowan. Some features are:

- For $\gamma = 0$ reduce to a low-pass filter: $\partial_t a = -\alpha a + \beta I$;
- Integral interaction term with delay τ;
- Non-linearity σ (sigmoid and/or saturation)

What about the interaction kernel k_τ?

Interaction kernel in AC

Observation:

jet-space structure $\rightsquigarrow$ control system structure

Proposed model for AC

Neural connections follow the following control-affine system:

$$\begin{cases} \dot{\omega} = \nu & (\omega, \nu) \in \mathbb{R}^2 \\ \dot{\nu} = \mathrm{u} & \mathrm{u} : [0, T] \to \mathbb{R} \end{cases}$$

$\rightsquigarrow$ k_τ is the kernel of the associated Kolmogorov equation

$$\partial_t \varphi = \mathcal{L}\varphi, \quad \text{where} \quad \mathcal{L} = \nu\partial_\omega + \partial_\nu^2,$$

Projection

Recovering a sound $\hat{s}$ from the processed sound signal a:

1. Apply the left inverse of the lift operation:

$$\int_{-\infty}^{+\infty} a(t, \omega, \nu)\, d\nu.$$

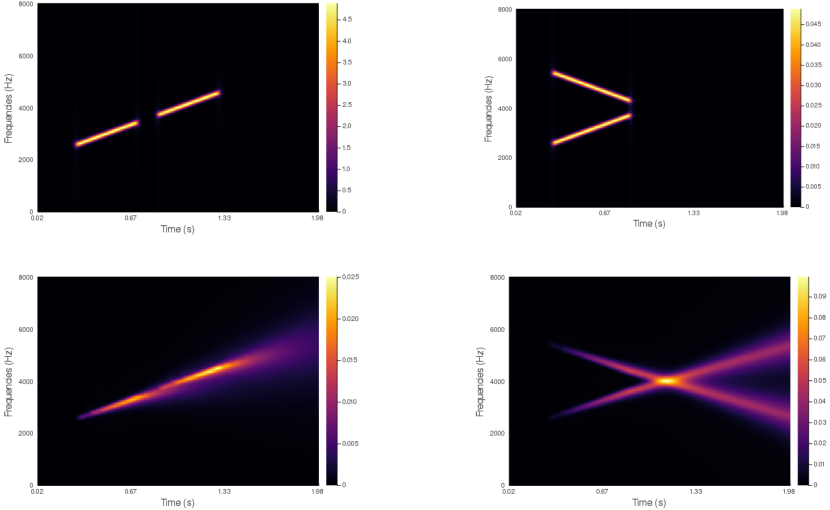
2. Apply the inverse of the STFT operator:

$$\hat{s} = \mathrm{STFT}^{-1}\left(\int_{-\infty}^{+\infty} a(t, \omega, \nu) d\nu\right)$$

Theorem

The resulting signal $\hat{s}$ is real-valued and thus correctly represents a sound signal.

Numerical experiments



Conclusion

Let's recap the V1-inspired framework

- Variational information is captured from the input signal
- This translates into a lift to a higher dimensional structured space
- Sub-Riemannian/anisotropic diffusions according to the structure enhance the variational information

These results on sound will be studied further for validation and possible applications to the problem of degraded speech.

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Thank you for your attention!