A bound on the iteration order of Fuller singularities for generic control-affine time-optimal systems

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Time-optimal trajectories of control-affine systems

$$\dot{q} = f_0(q) + \sum_{i=1}^{m} u_i f_i(q), \qquad \|u\| \leq 1, \quad q \in M$$

M smooth n-dimensional manifold, $u = (u_1, \ldots, u_m)$
$f_0, \ldots, f_m \in \mathrm{Vec}(M) \longrightarrow$ smooth (i.e., C^∞) vector fields on M

Time optimal problem: $q(0) = q_0, \quad q(T) = q_1, \quad T \to \min$

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Regularity of a time-optimal trajectory $q : [0, T] \to M$ measured in terms of maximal domain of smoothness

$$O = \bigcup_{\omega \text{ open}, \, q|_\omega \text{ smooth}} \omega, \qquad \Sigma = [0, T] \setminus O$$

Is Σ empty? finite? countable? of finite measure? of empty interior?

In general: No regularity

(SUSSMANN - 1986) for ANY $t \mapsto u(t) \in [-1, 1]$ measurable and any M, q_0, there exist $f_0, f_1 \in \mathrm{Vec}(M)$ such that the admissible trajectory driven by u and starting at q_0 is time-optimal for $\dot{q} = f_0(q) + u f_1(q)$

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Natural question: generic properties, i.e., properties that hold for all time-optimal trajectories of the control-affine system, provided that $(f_0, \ldots, f_m)$ belongs to an open and dense subset of $\mathrm{Vec}(M)^{m+1}$ for the C^∞ Whitney topology

Bang, singular, and Fuller

Definition (Bang and singular arcs)

An arc is a connected component of O. An arc ω is said to be bang if $\|u\| \equiv 1$ (a.e.) on ω, and singular otherwise Two arcs are concatenated if they are separated by a single point (an isolated point of Σ) which is called a switching time

Definition (Fuller times)

Let Σ_0 be the set of isolated points in Σ (switching times) The elements of $\Sigma \setminus \Sigma_0$ are Fuller times
By recurrence, let

$$\Sigma_k \text{ set of isolated points of } \Sigma \setminus (\cup_{j=0}^{k-1} \Sigma_j) \qquad k \in \mathbb{N} \cup \{\infty\}$$

If $t \in \Sigma_k$ then t is a Fuller time of order k

Σ_k countable for all $k \geq 0$.

Single-input case: previous results

- For $n = 2$, Σ is finite generically and for all analytic systems Lobry 1970, Sussmann 1982, 1987
- Finiteness of Σ close to points at which some suitable non-dependence condition between Lie brackets holds ($n = 3, 4$) Agrachev, Bressan, Gamkrelidze, Krener, Schättler, S., Sussmann,. . . .
- For n large enough time-extremal trajectories of generic systems might exhibit Fuller phenomenon ($\#\Sigma = \infty$) Kupka 1990, Zelikin–Borisov 1994, 2004, Borisov 1998, 2004
- Generically, for every extremal trajectory $q : [0, T] \to M$, the set O is dense in $[0, T]$ Agrachev 1995
- Generically, for any extremal triple $(q(\cdot), u(\cdot), \lambda(\cdot))$ such that $\langle \lambda(t), f_1(q(t)) \rangle \equiv 0$ on $[0, T]$, O is dense in $[0, T]$ Bonnard–Kupka 1997 and also of full measure Chitour–Jean–Trélat 2008

Main result in the single-input case

Theorem (Boarotto–Sigalotti, ANIHP, 2019)

There exists $K(n) \in \mathbb{N}$ such that, for a generic pair (f_0, f_1), every time-optimal trajectory of

$$\dot{q} = f_0(q) + u f_1(q), \quad q \in M,\ u \in [-1, 1],$$

has at most Fuller times of order $K(n)$, i.e.,

$$\Sigma = \Sigma_0 \cup \cdots \cup \Sigma_{K(n)}.$$

- In particular, u can be taken smooth out of a finite union of discrete sets (hence, out of a countable set)
- as a byproduct of the proof, for (f_0, f_1) generic, for a trajectory s. t. $\langle \lambda(t), f_1(q(t)) \rangle \equiv 0$, $[0, T] \setminus O$ countable.

Single-input time-extremal trajectories and the switching function

By the Pontryagin maximum principle, if $q : [0, T] \to M$ is time-optimal for

$$\dot{q} = f_0(q) + u f_1(q) \qquad u \in [-1, 1]$$

then $\exists$ extremal lift $\lambda : [0, T] \to T^*M \setminus \{0\}$ of $q(\cdot)$ such that

$$\dot{\lambda}(t) = \vec{h_0}(\lambda(t)) + u(t)\, \vec{h_1}(\lambda(t)) \quad \text{a.e. } t \in [0, T]$$

where $h_i(\lambda) = \langle \lambda, f_i(q) \rangle$, $i = 0, 1$, and

$$u(t)\langle \lambda(t), f_1(q(t)) \rangle = |\langle \lambda(t), f_1(q(t)) \rangle| \quad \text{a.e. } t \in [0, T]$$

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Equivalently, $\forall X \in \mathrm{Vec} M$,

$$\frac{d}{dt}\langle \lambda(t), X(q(t)) \rangle = \langle \lambda(t), [f_0 + u(t) f_1, X](q(t)) \rangle \quad \text{a.e. } t \in [0, T]$$

and

$(*)$ $u(t) = \mathrm{sgn}(\langle \lambda(t), f_1(q(t)) \rangle)$ a.e. $t \in [0, T]$ s.t. $\langle \lambda(t), f_1(q(t)) \rangle \neq 0$

(*) = First feedback expression of $u(\cdot)$ in terms of (f_0, f_1).

Switching function

Let $f_\pm = f_0 \pm f_1$ and, for $I = (i_1 \cdots i_k)$ word with letters in $\{+, -, 0, 1\}$

$$f_I = [f_{i_1}, \ldots, [f_{i_{k-1}}, f_{i_k}] \cdots], \quad h_I(t) = \langle \lambda(t), f_I(q(t)) \rangle$$

The PMP implies

$$\dot{h}_I(t) = h_{0I}(t) + u(t) h_{1I}(t) \qquad \text{a.e. } t \in [0, T]$$

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In particular, for

$$h_1(t) = \langle \lambda(t), f_1(q(t)) \rangle \qquad \text{switching function}$$

we have

$$\dot{h}_1(t) = h_{01}(t) \qquad \forall t \in [0, T]$$

$$\ddot{h}_1(t) = h_{001}(t) + u(t) h_{101}(t) \qquad \text{a.e. } t \in [0, T]$$

Bonnard–Kupka 1997: generically, if $h_1 \equiv 0$, then $h_{101}(t) \neq 0$ and

$$(**)\ u(t) = -h_{001}(t)/h_{101}(t) \text{ a.e. } t \in O.$$

(**) = Second feedback expression of $u(\cdot)$ in terms of (f_0, f_1).

Strategy of the proof

- At switching times $h_1 = 0$ and more generally $h_1|_\Sigma \equiv 0$

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- Higher order Fuller times are accumulations of accumulations and new relations between $\lambda(t)$ and the brackets of $f_0(q(t))$ and $f_1(q(t))$ can be derived
  Initialization on $\Sigma \setminus \Sigma_0$ ($0 = h_1 = h_{01} = \dots$)
  Recursion $\Sigma \setminus \cup_{j=0}^{k} \Sigma_j \longrightarrow \Sigma \setminus \cup_{j=0}^{k+1} \Sigma_j$

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- At high order Fuller times $(\lambda(t), j^N(f_0, f_1)(q(t)))$ belongs to a set of large codimension in $T^*_{q(t)}M \times J^N_{q(t)}M \times J^N_{q(t)}M$, where N large and $J^N M$ is the bundle of N-jets of vector fields on M

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- The projection $\pi(A)$ on $J^N_{q(t)}M \times J^N_{q(t)}M$ of a set A of large codimension has itself large codimension
- Fuller times of too large order can be ruled out by Thom's transversality theorem

$$(f_0, f_1) \text{ generic} \Longrightarrow j^N(f_0, f_1)(q) \notin \pi(A) \ \forall q \in M$$

Initialization: dependence conditions on $\Sigma \setminus \Sigma_0$

Proposition (No genericity assumption here)

Let $t \in \Sigma \setminus \Sigma_0$.
Then $h_1(t) = h_{01}(t) = 0$ *and, in addition, either* $h_{+01}(t) = 0$ *or* $h_{-01}(t) = 0$.

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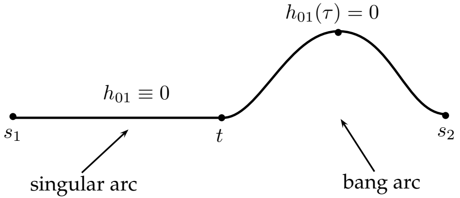
- if a bang arc B is a concatenated to a singular arc S then h_{+01} or h_{-01} vanish on $B \cup S$



- By this kind of reasoning we can reduce the problem to the case where t is the limit of an infinite sequence of concatenated bang arcs $\longrightarrow$ next slide

Infinitely many concatenated bang arcs

Lemma

Assume $\exists$ *infinite sequence of concatenated bang arcs converging to* $\tau \in [0, T]$. *Then either* $h_{+01}(\tau) = 0$ *or* $h_{-01}(\tau) = 0$.

Lengths of subsequent bang arcs $\{\tau_i\}_{i \in \mathbb{N}}$.
Proof by contradiction: if $h_{+01}(\tau)h_{-01}(\tau) \neq 0$, then $|\ddot{h}_1| \geq C_0 > 0$ on bang arcs because $\ddot{h}_1 = \alpha h_{+01} + (1 - \alpha)h_{-01}$

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Then

- if $(t_i, t_i + \tau_i)$ is a bang arc then

$$\dot{h}_1(t_i) = O(\tau_i), \qquad \dot{h}_1(t_i + \tau_i) = -\dot{h}_1(t_i) + O(\tau_i^2)$$

- $\tau_{i+1} = O(\tau_i)$

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- $\tau_{i+1} = O(\tau_i)$
- $\tau_{i+2} = \tau_i + O(\tau_i^2)$
- $T \geq \sum_{i=1}^{\infty} \tau_i = +\infty$, contradiction.

Recursion $\Sigma \setminus \cup_{j=0}^{k}\Sigma_j \longrightarrow \Sigma \setminus \cup_{j=0}^{k+1}\Sigma_j$

$\Sigma \setminus \cup_{j=0}^{k}\Sigma_j \ni t_n \to t \in \Sigma \setminus \cup_{j=0}^{k+1}\Sigma_j$
I_1, I_2 words with letters in $\{+,-,0,1\}$, length$(I_1) \leq$length(I_2),

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Up to subsequences

$$\frac{1}{t - t_n} \int_{t_n}^{t} u(s) ds \to \bar{u} \in [-1, 1].$$

Then, for $j = 1, 2$,

$$0 = \frac{h_{I_j}(t) - h_{I_j}(t_n)}{t - t_n} = \frac{\int_{t_n}^{t} (h_{0I_j}(s) + u(s) h_{1I_j}(s)) ds}{t - t_n} \to h_{0I_j}(t) + \bar{u} h_{1I_j}(t).$$

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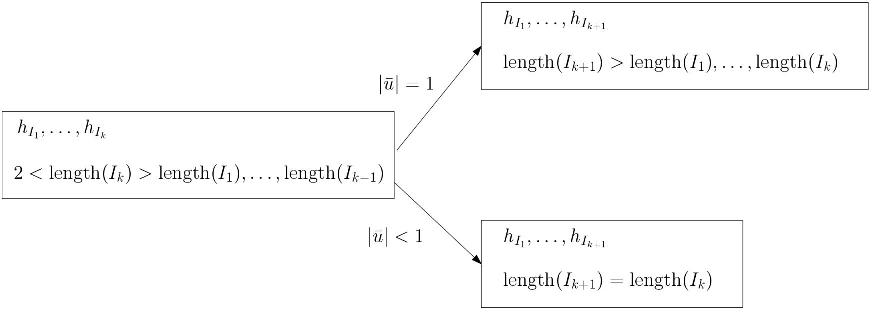
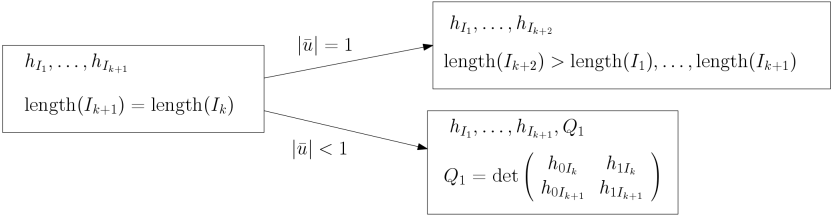
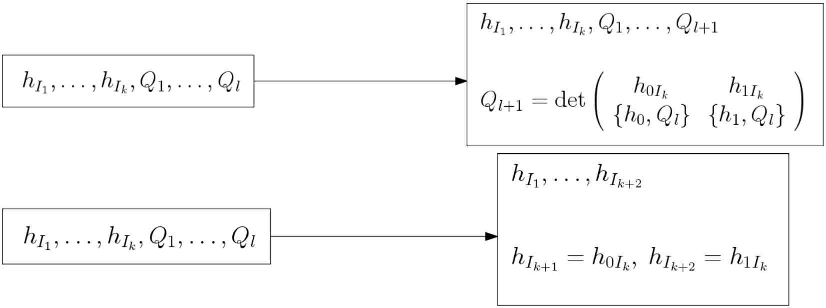
- $|\bar{u}| = 1 \longrightarrow$ new word $J = \pm I_2$ of longer length with $h_J(t) = 0$
- $|\bar{u}| < 1$ & $I_2 = \star I_1$, $\star \in \{+,-\}$ $\longrightarrow$ can replace $h_{I_2} = 0$ with 2 indep. conditions $h_{0I_1} = 0 = h_{1I_1}$ (as in initialization step)
- $|\bar{u}| < 1$ & length(I_1)=length(I_2) $\longrightarrow$ $\det\begin{pmatrix} h_{0I_1}(t) & h_{1I_1}(t) \\ h_{0I_2}(t) & h_{1I_2}(t)\end{pmatrix} = 0$

Generating new conditions





Generating more conditions



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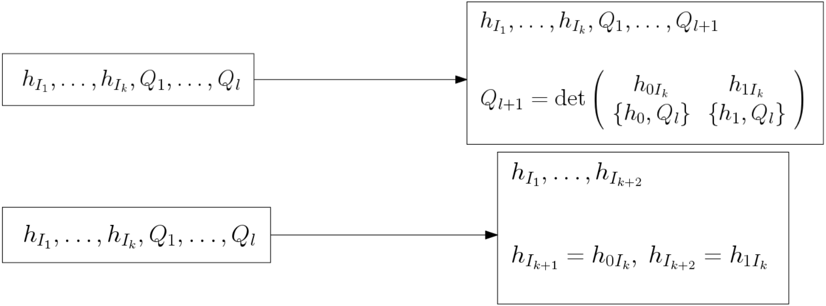


Two reasons for losing dependence:

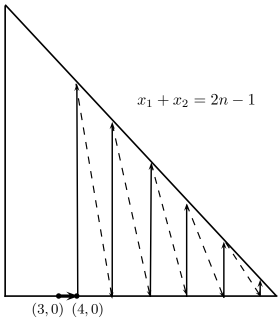
- $h_{0I_k} = 0 = h_{1I_k}$: all determinants $Q_1, \dots, Q_{l+1}$ vanish
- $f_0 \wedge f_1 = 0$: conditions on lower order jets
  e.g., if $f_0(q) = 0$ then $[f_0, [f_0, \dots, [f_0, f_1] \cdots](q)$ only depends on $f_1(q)$ and $Df_0(q)$

Hoarding dependence conditions in $\{f_0 \wedge f_1 \neq 0\}$

If $h_{I_1} = 0, \ldots, h_{I_k} = 0, Q_1 = 0, \ldots, Q_l = 0$ are independent conditions at $\lambda(t)$ then $(\lambda(t), j^N(f_0, f_1)(q(t)))$ is in a codimension $k + l$ set (N large enough).
For $k + l$ larger than $2n - 1$ we get a condition on $j^N(f_0, f_1)$ that is generically nowhere satisfied on M.
Any higher Fuller order gives rise to one of the following moves

- $(k, 0) \longrightarrow (k + 1, 0)$
- $(k, l) \longrightarrow (k, l + 1)$
- $(k, l) \dashrightarrow (k + 2, 0)$

To find $K(n)$ we compute the longest sequence of moves staying in $\{k + l \leq 2n - 1\}$



Iterations along $\mathcal{C} = \{f_0 \wedge f_1 = 0\}$

Lemma

Let $q : [0, T] \to M$ *trajectory associated with control* u*. Take* $t_\infty \in [0, T]$ *such that* $f_1(q(t_\infty)) \wedge [f_0, f_1](q(t_\infty)) \neq 0$*. Let* $\{t_i\}_{i\in\mathbb{N}}$ *converge to* t_∞ *and* $q(t_i) \in \mathcal{C}$ *for every* $i \in \mathbb{N}$*. Then there exists* $\bar{u} = \lim_{i\to\infty} \frac{1}{t_\infty - t_i} \int_{t_i}^{t_\infty} u(\tau) d\tau$ *and* $f_0(q(t_\infty)) + \bar{u} f_1(q(t_\infty)) = 0$*.*

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Let $q : [0, T] \to M$ trajectory associated with control u. Take $t_\infty \in [0, T]$ such that $f_1(q(t_\infty)) \wedge [f_0, f_1](q(t_\infty)) \neq 0$. Let $\{t_i\}_{i\in\mathbb{N}}$ converge to t_∞ and $q(t_i) \in \mathcal{C}$ for every $i \in \mathbb{N}$. Then there exists $\bar{u} = \lim_{i\to\infty} \frac{1}{t_\infty - t_i} \int_{t_i}^{t_\infty} u(\tau)d\tau$ and $f_0(q(t_\infty)) + \bar{u} f_1(q(t_\infty)) = 0$.

$\Omega = \{t \in [0, T] \mid q(t) \in \mathcal{C},\ h_1(t) = 0\}$.
Ω_0 isolated points of Ω, Ω_k isolated points of $\Omega \setminus (\bigcup_{j=0}^{k-1} \Omega_j)$.

Lemma

If $\{t_i\}_{i\in\mathbb{N}} \subset \Omega \setminus (\bigcup_{j=0}^{k} \Omega_j)$ are such that $t_\infty = \lim_{i\to\infty} t_i$ and $f_1(q(t_\infty)) \wedge [f_0, f_1](q(t_\infty)) \neq 0$, then $\mathrm{ad}^j_{h_{\bar{u}}}(h_1)(\lambda(t_\infty)) = 0$, for every $0 \leq j \leq k+2$. In particular, generically with respect to (f_0, f_1), $\Omega = \Omega_0 \cup \cdots \cup \Omega_{n-2}$.

This concludes the proof of the single-input result
$(K(n) = (n-1)^2)$

Multi-input case

$$\dot{q} = f_0(q) + \sum_{i=1}^{m} u_i f_i(q) \qquad u = (u_1, \ldots, u_m) \in U$$

- Switching studied for U ball in Agrachev–Biolo 2018 and, in some particular cases, Caillau–Daoud 2012, Orieux–Roussarie 2019
- Fuller phenomenon for U polytope structurally stable for extremal trajectories Zelikin–Lokutsievskiy–Hildebrand 2012, 2015

We focus on the case $U = B_1 = \{u \mid \|u\| \leq 1\}$

PMP multi-input

For $q : [0, T] \to M$ time-optimal there exists extremal lift $\lambda : [0, T] \to T^*M$ such that $\sum_{i=1}^{m} u_i(t)\langle\lambda(t), f_i(q(t))\rangle = \max_{u\in B_1} \sum_{i=1}^{m} u_i\langle\lambda(t), f_i(q(t))\rangle$ and $\forall X \in \mathrm{Vec}(M)$

$$\frac{d}{dt}\langle\lambda(t), X(q(t))\rangle = \langle\lambda(t), [f_0, X](q(t))\rangle + \sum_{i=1}^{m} u_i(t)\langle\lambda(t), [f_i, X](q(t))\rangle$$

Define

- $h_I(t) = \langle\lambda(t), f_I(q(t))\rangle$ for I word in $\{0, \ldots, m\}$
- $h_{\mathbf{m}} = (h_1, \ldots, h_m)^T \quad h_{0\mathbf{m}} = (h_{01}, \ldots, h_{0m})^T$
- $H_{\mathbf{mm}} = (h_{ij})_{i,j=1}^{m}$ Goh matrix

PMP implies $u(t) = h_{\mathbf{m}}(t)/\|h_{\mathbf{m}}(t)\|$ when $h_{\mathbf{m}}(t) \neq 0$ and

$$\dot{h}_{\mathbf{m}}(t) = h_{0\mathbf{m}}(t) - H_{\mathbf{mm}}(t)u(t) \qquad \text{a.e. } t \in [0, T]$$

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Corollary

$t \in \Sigma \setminus \Sigma_0$, $\det(H_{\mathbf{mm}}(t)) \neq 0 \Longrightarrow h_{0\mathbf{m}}(t) \in H_{\mathbf{mm}}(t)\partial B_1$

REMARK: is m is odd then the antisymmetric matrix $H_{\mathbf{mm}}$ is never invertible, hence $\det(H_{\mathbf{mm}}(t)) = 0$

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Assumption. m even

Main result in the multi-input case

Theorem (Boarotto–C.–Sigalotti, SICON 2020)

M smooth manifold of dimension n, m even, $m+1 \leq n$. There exist $K = K(n) \in \mathbb{N}$ and an open and dense set $\mathcal{U}$ of

$$\{(f_0, \dots, f_m) \in \mathrm{Vec}(M)^{m+1} \mid f_0(q), \dots, f_m(q) \text{ linearly independent } \forall q \in M\}$$

such that, if $(f_0, \dots, f_m) \in \mathcal{U}$, then every time-optimal trajectory of

$$\dot{q} = f_0(q) + \sum_{i=1}^{m} u_i f_i(q) \qquad \|u\| \leq 1$$

has at most Fuller times of order K, i.e.,

$$\Sigma = \Sigma_0 \cup \cdots \cup \Sigma_K$$

Comments on the proof

- Role played by the condition $h_{+01} = 0$ or $h_{-01} = 0$ on $\Sigma \setminus \Sigma_0$ now played by

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- use lower-semicontinuity of the rank to put together the Fuller order estimates

Perspectives and open problems

Single-input

- What is the minimal $K(n)$ (for time-extremal and time-optimal trajectories)? We can show $K(n) \leq (n-1)^2$. Can we at least bound $K(n)$ by a sub-quadratic function in n?
- Is optimality of (iterated) Fuller extremals structurally stable?
- What can be said for M, f_0, f_1 analytic?

Multi-input

- case m even, $u \in B_1 \subset \mathbb{R}^m$: remove linearly independent condition
- case m odd, $u \in B_1 \subset \mathbb{R}^m$
- case $u \in [-1, 1]^m \subset \mathbb{R}^m$ or general polytopes