Dwell-time control sets and applications to the stability analysis of linear switched systems

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Outline

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2. Invariant control sets of general nonlinear systems
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5. Dwell-time invariant control sets and linear switched systems with dwell-time

Linear switched systems and their Lyapunov exponents

Linear switched systems

Consider a switched system on $\mathbb{R}^d$ of the type

$$\dot{x} = A(t)x, \qquad A(\cdot) \in \mathcal{S} \tag{Σ}$$

$\mathcal{S}$ class of signals from $\mathbb{R}$ to some set $S \subset M_d(\mathbb{R})$.
Examples:

- $\mathcal{S}_0 = \{A : \mathbb{R} \to S \mid A \text{ piecewise constant}\} \longrightarrow$ arbitrary switching
- $\mathcal{S}_\tau = \{A : \mathbb{R} \to S \mid$ A piecewise constant, discontinuities at distance $\geq \tau\} \longrightarrow$ switching with (guaranteed) dwell-time τ
- other classes can be introduced in terms of average dwell-time constraints, persistence of excitation, Lipschitz constraints, ...

Crucial property of switched systems: uniform asymptotic stability with respect to $A \in \mathcal{S}$

Measures of stability of linear switched systems

$A \in \mathcal{S} \longrightarrow$ fundamental matrix $\Phi_A(\cdot)$ solution to

$$\frac{d}{dt}\Phi_A(t) = A(t)\Phi_A(t), \quad \Phi_A(0) = \mathrm{Id}_d$$

- $\mathcal{S}$-attractive: $\Phi_A(t) \to 0$ for every $A \in \mathcal{S}$
- $\mathcal{S}$-uniform exponential stability: $\exists C, \lambda > 0$ s.t. $\forall A \in \mathcal{S}$

$$\|\Phi_A(t)\| \leq Ce^{-\lambda t}, \quad \forall t \geq 0 \qquad (\star)$$

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- uniform exponential rate:

$$\lambda(\mathcal{S}) = \limsup_{t \to +\infty} \sup_{A \in \mathcal{S}} \frac{\log(\|\Phi_A(t)\|)}{t} = \inf\{\lambda \mid \exists C \text{ s.t. } (\star)\, \forall A \in \mathcal{S}\}$$

- $\mathcal{S}$-uniform exponential stability $\Leftrightarrow \lambda(\mathcal{S}) < 0$

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- $\mathcal{S}$-uniform exponential stability $\Leftrightarrow$ $\lambda(\mathcal{S}) < 0$
- maximal Lyapunov exponent:

$$\widehat{\lambda}(\mathcal{S}) = \sup_{A \in \mathcal{S}} \limsup_{t \to +\infty} \frac{\log(\|\Phi_A(t)\|)}{t} = \inf\{\lambda \mid \forall A \in \mathcal{S},\ \exists C \text{ s.t. } (\star)\,\}$$

- $\widehat{\lambda}(\mathcal{S}) \leq \lambda(\mathcal{S})$

Equality between $\lambda(\mathcal{S}_\tau)$ and $\widehat{\lambda}(\mathcal{S}_\tau)$

Lemma (Fenichel)

Let $\mathcal{S} = \mathcal{S}_\tau$ for $\tau \geq 0$. Then (Σ) is $\mathcal{S}$-attractive if and only if it is $\mathcal{S}$-uniformly exponentially stable

Corollary

For every $\tau \geq 0$, $\lambda(\mathcal{S}_\tau) = \widehat{\lambda}(\mathcal{S}_\tau)$

From now on

$$\lambda_\tau(S) := \lambda(\mathcal{S}_\tau)$$

Our aim: give a useful characterization of $\lambda_\tau(S)$

Probabilistic Lyapunov exponents for Piecewise Deterministic Markov Processes (PDMP)

Let $S = \{1, \ldots, N\}$ and $Q = (q_{ij})_{i,j=1}^{N}$ be Markov transition matrix ($q_{ij} \geq 0$, $\sum_{j=1}^{N} q_{ij} = 1$)
A trajectory is a random variable, as well as its switching law $(i_k, t_k)_{k \in \mathbb{N}}$:

- the initial index i_1 in S is a random variable
- transition $A_{i_k} \to A_{i_{k+1}}$ at time t_k with probability $q_{i_k i_{k+1}}$
- we can introduce a dwell-time:

$$P(\{t_{k+1} - t_k \leq \theta\}) = \begin{cases} 0 & \text{if } \theta < \tau \\ \nu \int_{\tau}^{\theta} e^{-\nu(t-\tau)} dt & \text{if } \theta \geq \tau \end{cases}$$

- duration of each interval between switching times:

$$P(\{t_{k+1} - t_k \leq \theta\}) = \nu \int_0^{\theta} e^{-\nu t} dt$$

Furstenberg–Kesten theorem: if Q is strongly connected, then, with probability one $\exists \lim_{t \to \infty} \frac{1}{t} \|\Phi_A(t)\| = \chi_\tau(S, \nu, Q)$

Invariant control sets of general nonlinear systems

Invariant control sets

M manifold, F family of smooth complete vector fields on M
$f(\cdot) \in \mathcal{F}_0$ piecewise constant with values in F, q initial condition
$\longrightarrow$ solution $t \mapsto \phi(t, q, f)$
Attainable set from $q \in M$: $A(q) = \{\phi(t, q, f) \mid t \geq 0,\ f \in \mathcal{F}_0\}$

Definition

$\emptyset \neq D \subset M$ *invariant control set (ICS)* if $D = \overline{A(q)}$ for every $q \in D$

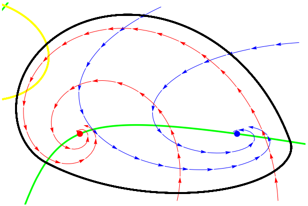
EXAMPLE: $A_1, \ldots, A_m \in M_d(\mathbb{R})$, $x_1, \ldots, x_m \in \mathbb{R}^d$.
If

- $\dot{x} = A_{i(t)}x$ is asymptotically stable (with arbitrary switching)

then

$$\overline{A(x_1)} = \bigcap_{\Omega \neq \emptyset \text{ compact invariant}} \Omega$$

is a ICS for $\dot{x} = A_{i(t)}(x - x_{i(t)})$



Existence of invariant control sets

Theorem (see, e.g., Colonius–Kliemann, 2000)

Let M be compact. For each $q \in M$ there exists a nonempty ICS D_q contained in $\overline{A(q)}$. Assume, moreover, that F has the Lie algebra rank condition (LARC). Then

- *D_q has nonempty interior*
- *there exists $\mathfrak{C}_q \subset D_q$ open and dense in D_q such that $A(q') = \mathfrak{C}_q$ for every $q' \in \mathfrak{C}_q$*
- *there exist finitely many distinct ICS*

- existence by Zorn lemma: $\overline{A(q')} \subset \overline{A(q)}$ if $q' \in \overline{A(q)}$
- nonempty interior by Krener theorem ($D_q = \overline{A(q')}$ for $q' \in D_q$)

Invariant control sets for projected linear switched systems and periodization

Some link between linear switched systems and invariant control sets

Interesting properties on the behavior of a linear switched system can be deduced from its angular component:

$x(t) \longrightarrow (\|x(t)\|, [x(t)]) =: (r(t), s(t)) \in (0, +\infty) \times \mathbb{RP}^{d-1}$

Using local identification $[x] = \frac{x}{\|x\|}$, $\dot{x} = Ax$ can be rewritten as

$$\frac{\dot{r}}{r} = \langle s, As \rangle, \qquad \dot{s} = (A - \langle s, As \rangle \mathrm{Id}_d)s =: (\pi_* A)s$$

$(\pi\Sigma)$ projected linear system on $\mathbb{RP}^{d-1}$ associated with $F := \pi_* S$

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$(\pi\Sigma)$ projected linear system on $\mathbb{RP}^{d-1}$ associated with $F := \pi_* S$

- [Arnold, Kliemann, Oeljeklaus, 1986] $\longrightarrow$ if F LARC on $\mathbb{RP}^{d-1}$, then $(\pi\Sigma)$ has a unique ICS D and $\mathrm{int}(D) \neq \emptyset$
- [Colonius, Kliemann, 1993] $\longrightarrow$ if F LARC on $\mathbb{RP}^{d-1}$, then $\lambda_0(S)$ is equal to

$$\lambda_0^{\mathrm{per}}(S) := \sup \left\{ \limsup_{t \to +\infty} \frac{\log(\|\Phi_A(t)x_0\|)}{t} \mid A \in \mathcal{S}_0,\ (A(\cdot), \pi\Phi_A(\cdot)x_0) \text{ periodic} \right\}$$

- For PDMP ICSs characterize support of invariant measures [Benaïm, Colonius, Lettau, 2017]

Interest of periodization

The identity $\lambda_0(S) = \lambda_0^{\text{per}}(S)$:

- provides a monotone finite horizon approximation scheme
- proves the Gelfand-like formula

$$\lambda_0(S) = \limsup_{t \to +\infty} \sup_{A \in \mathcal{S}_0,\ x_0 \neq 0} \frac{\log(\rho(\Phi_A(t)))}{t}$$

  with ρ spectral radius
- can be used to show continuity of $S \mapsto \lambda_0(S)$
- first introduced to bound large deviations for Piecewise Deterministic Markov Processes [Arnold, Kliemann, 1987]

Periodization (proof by Colonius and Kliemann)

$$\lambda_0(S) = \sup\left\{\limsup_{t\to+\infty} \frac{\log(\|\Phi_A(t)x_0\|)}{t} \mid A \in \mathcal{S}_0,\ x_0 \neq 0\right\}$$

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Let $x(t) = \Phi_A(t)x_0$ be (quasi-)maximizing for $\lambda_0(S)$
In order to prove that $\lambda_0^{\mathrm{per}}(S) \geq \lambda_0(S) - \varepsilon$ we should be able to close the loop and, for t large, use $(\pi\Sigma)$ to go from $[x(t)]$ to $[x_0]$

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- Step 1: Choose x_0 appropriately. Take D the unique ICS for $(\pi\Sigma)$, fix $v_1, \dots, v_d$ linearly independent in $\mathrm{int} D$. Since $\|M\| = \max_{i=1}^d \|Mv_i\|$ is a norm on $M_d(\mathbb{R})$, we can take as x_0 one of the v_i
- Step 2: guarantee that there exist a uniform controllability time T for driving $(\pi\Sigma)$ from any point in D to any of the v_i within time T

Dwell-time invariant control sets for general nonlinear systems

Goal: extend control sets analysis to the dwell-time case

- The definition of invariant control sets does not suit the dwell-time case (invariance fails to see dwell-time)
- Mathematically, the difficulty come from non-concatenability of the class of admissible signals
- Equivalently, the family of admissible flows is not a semigroup
- Idea: recover main geometric properties by looking not at attainable sets (built with entire trajectories issuing from a point) but only at points which are attainable in a *concatenable* manner

Dwell-time attainable set: $A_\tau(q) = \{\phi(T, q, f) \mid f|_{[0,T]} \in \mathcal{F}_\tau\}$ with

$$\mathcal{F}_\tau = \{f_1 * \cdots * f_m \mid m \in \mathbb{N},\ f_i \text{ constant on a interval of length } \geq \tau\}$$

Note: $\mathcal{F}_\tau$ not shift invariant!

Construction

Semigroups of concatenable flows

$$\mathfrak{S}_\tau = \{\phi(T, \cdot, f) \mid f|_{[0,T]} \in \mathcal{F}_\tau\}$$

Then $A_\tau(q) = \mathfrak{S}_\tau(q)$.

Definition

D is a *dwell-time invariant control set* (τ-ICS) if $D = \overline{A_\tau(q)}$ for every $q \in D$.

Remark

[San Martin, 1993] *already studied control sets for orbits of not necessarily connected semigroups, in a setting which does not directly applies here (semigroup with nonempty interior in a Lie group G and action on some X/G)*

Basic properties of dwell-time attainable and control sets

Theorem

Let M be compact, $\tau \geq 0$. For each $q \in M$ there exists a τ-ICS D_q contained in $\overline{A_\tau(q)}$. If, moreover, F has the LARC, then $\mathrm{int}\, D_q \neq \emptyset$

Remark: if there exists $\bar{q}$ such that $\bar{q} \in D_q$ for every $q \in M$, then there exists a unique τ-ICS $(= \overline{A_\tau(\bar{q})})$

Lemma

Let F satisfy LARC and assume that $D \subset M$ is a τ-ICS. Then

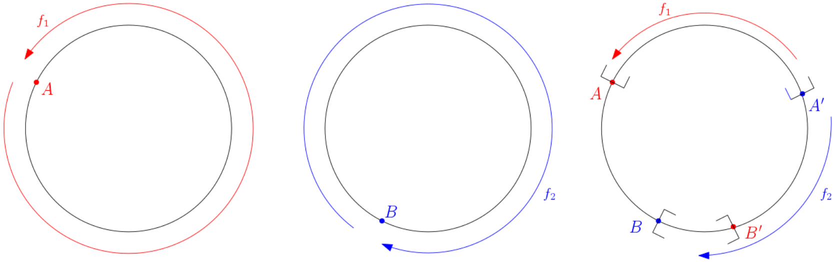
(i) $\overline{\mathrm{int}(D)} = D$

(ii) $\Phi(\mathrm{int}(D)) \subset \mathrm{int}(D)$ *for every* $\Phi \in \mathfrak{S}_\tau$

(iii) *There exists an open and dense subset* $\mathfrak{C}$ *of* D *such that* $\mathfrak{C} = \mathfrak{S}_\tau(q)$ *for all* $q \in \mathfrak{C}$

Dwell-time invariant control sets and linear switched systems with dwell-time

Example of dwell-time control set for projected linear switched system



f_1, f_2 vector fields on $\mathbb{RP}^1$, conjugate to $\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x$

$$A' := e^{\tau f_1}(B), \quad B' := e^{\tau f_2}(A)$$

- $D = \widehat{AA'} \cup \widehat{B'B}$ unique τ-ICS
- if τ_* is such that $e^{\tau_* f_1}(B) = e^{\tau_* f_2}(A)$ then

  $\tau \leq \tau_* \longrightarrow D = \widehat{AB}; \qquad \tau > \tau_* \longrightarrow D$ disconnected
- $\mathfrak{C} = \mathrm{int}(\widehat{AA'}) \cup \mathrm{int}(\widehat{B'B})$ ($\neq \mathrm{int} D$ for $\tau = \tau_*$)

Periodization without LARC condition

Theorem (F. Boarotto, M.S., JDE, to appear)

Let $S \subset M_d(\mathbb{R})$ *and* $\tau \geq 0$. *Then* $\lambda_\tau(S) = \lambda_\tau^{\mathrm{per}}(S)$

Idea: restrict the projected system to some orbit for the family $\pi_* S$
$O([x_0]) = \{[x(t)] \mid \dot{x} = A(t)x,\ x(0) = x_0,\ A(t) \in S \cup -S\}$
$(M, F) = (O([x_0]), \pi_* S)$

Advantages

- $O([x_0])$ has the structure of smooth manifold (Orbit theorem)
- LARC of the system restricted to M is for free

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Difficulties

- we should guarantee that the orbit carries all informations about asymptotic behavior $\longrightarrow$ reduction to irreducible case
- existence of τ-ICS requires compactness of orbits

Existence of a closed orbit

Theorem

Let B be the group generated by $\{e^{tA_j} \mid t \in \mathbb{R},\ j = 1, \ldots, m\}$ (any connected Lie subgroup of $\mathrm{GL}(\mathbb{R}, d)$). Then the action

$$\varphi : B \times \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}, \quad \varphi(b, x) = \frac{bx}{\|bx\|},$$

induced by B on the $(d-1)$-dimensional unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ admits at least one closed orbit in $\mathbb{S}^{d-1}$ (and the same is true for $\mathbb{RP}^{d-1}$)

Existence (and even uniqueness) of τ-ICS is obtained and Colonius–Kliemann's periodization argument can be performed, proving $\lambda_\tau(S) = \lambda_\tau^{\mathrm{per}}(S)$

Conclusions and perspectives

- We proved that the maximal Lyapunov exponent of linear switched systems with dwell-time can be characterized using only trajectories with periodic angular component (new also in the case $\tau = 0$ when the LARC does not hold)

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- existence of a compact orbit for a projected linear system could be useful for other control problems
- τ-ICS to characterizes support of the invariant measure for piecewise deterministic random process with dwell time
- Ongoing work: adapt our technique to a more abstract setting applying to other non-concatenable classes of switching signals