# On deciding stability of high frequency amplifiers

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December 10, 2019
*Journées McTAO-CAGE*

# Summary

1. Stability Conditions for Time-varying Networks of Telegrapher's Equations or Difference Delay Equations

2. Stability of high frequency amplifiers

# Example time-varying Networks of Telegrapher's equations

**Telegrapher's equations** :

$$\begin{cases} C_k \dfrac{\partial v_k(t,x)}{\partial t} = -\dfrac{\partial i_k(t,x)}{\partial x}, \\ L_k \dfrac{\partial i_k(t,x)}{\partial t} = -\dfrac{\partial v_k(t,x)}{\partial x}, \end{cases} \qquad (t,x) \in \Omega\,,$$

$$\Omega \;=\; \{(t,x) \in^2,\; 0 < x < 1 \text{ and } 0 < t < +\infty\},$$

With $k \in \{1, \cdots, N\}$ where $N$ positive integer denoting the number of Telegrapher's equations.

# General time-varying Networks of Telegrapher's equations

$$v(x,t) = \begin{pmatrix} v_1(x,t) \\ \vdots \\ v_N(x,t) \end{pmatrix}, \quad i(x,t) = \begin{pmatrix} i_1(x,t) \\ \vdots \\ i_N(x,t) \end{pmatrix}$$

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**Assumption (Dissipativity)**

*The map $t \mapsto \mathbf{A}(t)$ is continuous and bounded.*

$$\mathbf{A}(t) + \mathbf{A}^*(t) \geq \alpha\, Id, \qquad \alpha > 0 \qquad t \in \mathbb{R}.$$

# Equivalence with linear time-varying delay system

- $$\begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \\ y_1(t) \\ \vdots \\ y_N(t) \end{pmatrix} = (I + \mathbf{A}(t)\,\mathbf{K})^{-1}\,(I - \mathbf{A}(t)\,\mathbf{K})\,P_2 \begin{pmatrix} x_1(t-\tau_1) \\ \vdots \\ x_N(t-\tau_N) \\ y_1(t-\tau_1) \\ \vdots \\ y_N(t-\tau_N) \end{pmatrix}.$$
- K diagonal positive matrix and $P_2$ permutation matrix.
- One to one linear relation between $i_k, v_k$ and $x_k, y_k$.

# Generalities on the linear time-varying delay systems

$$z(t) = \sum_{i=1}^{N} D_i(t)\, z(t - \eta_i) \quad t \geq 0. \tag{1}$$

## Theorem

*Let* $\phi$ *be an element of* $L^2([-\eta_N, 0], \mathbb{R}^d)$.

(i) *There is a unique solution* $z$ *to* (1) *in* $L^2_{loc}([-\eta_N, +\infty), \mathbb{R}^d)$ *meeting the initial condition* $z_{|[-\eta_N,0]} = \phi$.

(ii) *Moreover if* $\phi$ *is continuous on* $[-\eta_N, 0]$ *with* $\phi(0) = \sum_{i=1}^{N} D_i(0)\, \phi(-\eta_i)$ , *then* $z \in C^0([-\eta_N, +\infty), \mathbb{R}^d))$.

# Definition Stability

$$z(t) = \sum_{i=1}^{N} D_i(t)\, z(t-\eta_i) \tag{2}$$

## Definition

The system (2) is $L^2$ (resp. $C^0$) exponentially stable if there exists $\gamma, K > 0$ such that :

$$\|z(t+\cdot)\|_{L^2([-\eta_N,0],\mathbb{R}^d)} \leq K e^{-\gamma t} \|z(\cdot)\|_{L^2([-\eta_N,0],\mathbb{R}^d)}, \;\; t \geq 0$$

$$\text{(resp. } \|z(t+\cdot)\|_{C^0([-\eta_N,0],\mathbb{R}^d)} \leq K e^{-\gamma t} \|z(\cdot)\|_{C^0([-\eta_N,0],\mathbb{R}^d)}, \;\; t \geq 0 \text{ ).}$$

# Stability linear time-invariant delay system

$$z(t) = \sum_{i=1}^{N} D_i \, z(t - \eta_i) \tag{3}$$

**Theorem (Henry-Hale Theorem, [Hen74, HVL93])**

*The following properties are equivalent :*

1. *System (3) is $L^2$ exponentially stable.*
2. *System (3) is $C^0$ exponentially stable.*
3. *There exists $\beta < 0$ for which*

$$Id - \sum_{i=1}^{N} D_i \, e^{-\lambda \eta_i} \text{ is invertible for all } \lambda \in \mathbb{C} \text{ such that } \Re(\lambda) > \beta.$$

# Stability time-invariant network of Telegrapher's equation

**Theorem**

*Under the dissipativity assumption, the delay difference system associated to the network of Telegrapher's equations with time-invariant boundaries is $C^0$ exponentially stable.*

*Proof :*

1. $$\sum_{i=1}^{N} D_i\, e^{-\lambda\tau_i} = [Id + \mathbf{AK}]^{-1}[Id - \mathbf{AK}]\, P_2 \begin{bmatrix} e^{-\lambda\eta_1} & & \\ & \ddots & \\ & & e^{-\lambda\eta_N} \end{bmatrix}.$$

2. There exists $|||\cdot|||_{\mathbf{K}}$, $\left|\left|\left| \sum_{i=1}^{N} D_i\, e^{-\lambda\tau_i} \right|\right|\right|_{\mathbf{K}} \leq e^{-\Re(\lambda)\eta_1}$ .

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Few references :

- Statibility criteria through Perron-Froebenius theorem
  Ngoc and Huy [NH15]
  $\rightarrow$ Not suitable for our case
- Stability criteria through joint spectral radius
  Chitour, Mazanti and Sigalotti [CMS16]
  $\rightarrow$ Combinatorics too complicated

Under the dissipativity assumption, we want to prove the $C^0$ exponential stability of the time-varying telegrapher's equations.

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Our Strategy :

$\rightarrow$ Prove $L^2$ stability through Lyapunov functional for system of PDEs
$\rightarrow$ Prove that stability $L^2$ is equivalent to the stability $C^0$ for difference delay system ([CMS16] or [BFLP19])
$\rightarrow$ Conclude that we have the $C^0$ stability for our system of PDEs.

# L2 stability for PDE

**Theorem**

*Under the dissipativity, the delay system associated to the time-varying network of telegrapher's equation is $L^2$ exponentially stable.*

1. Lyapunov function :

$$E_k(t) = \frac{1}{2}\int_0^1 \left[C_k v_k^2(t,x) + L_k i_k^2(t,x)\right] dx\,, \quad E(t) = \sum_{k=1}^{N} E_k(t)\,.$$

2. $\frac{d}{dt}E(t) \leq -\alpha \sum_{k=1}^{N}\left[i_k^2(t,0) + i_k^2(t,1)\right].$
3. We can prove : $E(t) \leq Ke^{-\gamma t}E(0)$, $\gamma, K > 0$.

# Equivalence $C^0$ and $L^2$ stability for difference delay system

**Proposition ([CMS16], BFLP 2019)**

*A periodic delay system is $L^2$ exponentially stable if and only if it is $C^0$ exponentially stable.*

*Sketch of proof :*

1. $$z(t) = \sum_{q=1}^{+\infty} M_q(t)\,\phi(t-\sigma_q), \qquad t \geq 0,$$

   for at most $Ct^N$ non zero terms.
2. Using suitable test initial data, $L^2$ or $C^0$ stability implies :

$$|||M_q(s)||| \leq C_0\, e^{-\gamma s}, \qquad \gamma, C_0 > 0.$$

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# Stability for a subclass of delay system

$$z(t) = \sum_{i=1}^{N} D_i(t)\, z(t-\eta_i),\ \eta_i \text{ distinct}$$

**Theorem**

*If we have :*

1. *Disjoint column properties,*
2. *The sum of the matrices $D_i(t)$ is uniformly contractive:*
$$\left|\left|\left|\sum_{i=1}^{N} D_i(t)\right|\right|\right|_2 \leq \nu,\ \nu < 1,$$

*then the difference delay system is $C^0$ stable.*

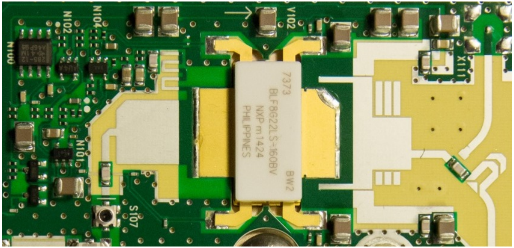
# Summary

# Inside

An amplifier is made of interconnected

1. resistors, inductors, capacitors,
2. diodes/transistors,
3. lossless transmission lines wich cannot be neglected at high frequency inducing delays.

Forcing periodic signal ▶▶ periodic solution in the amplifier ▶▶ amplified signal.



# Motivation

- Amplifiers at high frequency are ubiquitous (Cell phones, relays...). They need to be quick to design and produced in large quantites.
- Computer-assisted design (CAD) and simulation before production.
- Powerful "frequency simulation" tools give a reliable prediction of the response, but that response might be unstable.
- **Need** for a tool to predict stability/unstability in the frequency domain.

# Harmonic Balance

The Harmonic Balance method, through Fourier development, Laplace transform and fixed point methods permits to :

- approximate the periodic solution of the circuit,
- linearize the circuit around the periodic solution,
- give a frequency response to a periodic signal wich disturbs the linearized circuit.

Harmonic balance method : Numerically implemented.

Our focus : Structure of the harmonic transfer function, its singularities, links with stability.

General system, $T$-periodic :

$$\begin{cases} \frac{dx(t)}{dt} = A_1(t)x(t) + \sum_{i=0}^{N} B_{1,i}(t)y(t-\tau_i) \\ y(t) = \sum_{i=1}^{N} B_{2,i}(t)y(t-\tau_i) + A_2(t)x(t), \;\; t \geq s, \end{cases}$$

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- Solution operator $U(t,s) : L^2 \to L^2$
- Monodromy operator $U(T,0)$
- $$\left. \begin{array}{c} L^2 \text{ exponential} \\ \textit{stability} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} Sp(U(T,0)) \text{ included in} \\ \text{disc of radius } r < 1 \end{array} \right.$$

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# Behaviour at high frequency

High frequency system :

$$\begin{cases} x(t) = 0 \\ y(t) = \sum_{i=1}^{N} B_{2,i}(t) y(t-\tau_i), \ \ t \geq s, \end{cases}$$

- $\tilde{L}^2 := \{0_n\} \times L^2([-\tau_N, 0], \mathbb{R}^k)$.
- Solution operator $V(t,s) : \tilde{L}^2 \to \tilde{L}^2$.
- Monodromy operator $V(T,0)$.
- 

$L^2$ or $C^0$ exponential *stability* $\Bigg\} \Leftrightarrow \Big\{$ High Frequency dissipativity

# Compact perturbation

**Lemma**

*We have :*

$$U(t,s) = V(t,s)P + K(t,s), \;\; t \geq s$$

*with $K(t,s)$ compact operator $L^2 \to L^2$ for all $t,s$ and $P$ the canonical projection $L^2 \to \tilde{L}^2$.*

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**Theorem**

*The monodromy operator $U(T,0)$ possesses at most a finite number of eigenvalues $\zeta_1, \cdots, \zeta_n$ outside a disk of a radius strictly less than 1.*

# $L^2$ stability equivalent to $C^0$ stability

**Theorem**

*The general system is $L^2$ exponentially stable if and only if it is $C^0$ exponentially stable.*

The stability of high frequency amplifiers depends on a finite number of unstable eigenvalues.

# Input-Output system

$$\begin{cases} \frac{dx(t)}{dt} = A_1(t)x(t) + \sum_{i=0}^{N} B_i^1(t)y(t-\tau_i) + C_1(t)u(t) \\ y(t) = \sum_{i=1}^{N} B_i^2(t)y(t-\tau_i) + A_2(t)x(t) + C_2(t)u(t) \\ z(t) = \sum_{i=1}^{N} B_i^3(t)y(t-\tau_i) + A_3(t)x(t) + C_3(t)u(t), \ \ t \geq 0, \end{cases}$$

- $x(t), y(t), z(t) = 0$ for $t < 0$,
- Input $u \in L^2_{loc}([0, +\infty), \mathbb{R})$ current perturbation, output $z$ the voltage,
- All coefficients are $T - periodic$.

# Harmonic Transfer Function

- $X(t,\alpha)$ response at time $t$ to an impulse at time $\alpha$
  $z(t) = \int_0^t X(t,\alpha)u(\alpha)d\alpha$

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**Definition (Harmonic Transfer Function HTF)**

The infinite matrix $H(s)$ defined by $H_{m,n}(s) := G_{n-m}(s + \frac{2i\pi m}{T})$ for $s \in \mathbb{C}$ is called the harmonic transfer function.

# Harmonic Transfer Function

$$Z(s) := \int_0^{+\infty} z(t)e^{-st}dt \text{ and } U(s) := \int_0^{+\infty} u(t)e^{-st}dt.$$

$$\begin{pmatrix} \vdots \\ Z(s+i\omega_0) \\ Z(s) \\ Z(s-i\omega_0) \\ \vdots \end{pmatrix} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & G_2(s-i\omega_0) & G_1(s) & G_0(s+i\omega_0) & \cdots \\ \cdots & G_1(s-i\omega_0) & G_0(s) & G_{-1}(s+i\omega_0) & \cdots \\ \cdots & G_0(s-i\omega_0) & G_{-1}(s) & G_{-2}(s+i\omega_0) & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ U(s+i\omega_0) \\ U(s) \\ U(s-i\omega_0) \\ \vdots \end{pmatrix}$$

- HTF is an operator valued analytic map (values: continuous ops $l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$)
- its entries $\{G_n\}$ are complex valued analytic maps

# Structure of the Harmonic Transfer Function

Define $z_{j,k} = \frac{ln(\zeta_j)+2ik\pi}{T}$ for $j$ in $\{1...n\}$, $k$ in $\mathbb{Z}$.

**Theorem**

*In* $\{s \in \mathbb{C}, \Re(s) \geq \gamma\}$ *for some* $\gamma < 0$,

- *$H$ is a meromorphic operator $l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ wich possibly poles at $\{z_{j,k}, j \in \{1...n\}, k \in \mathbb{Z}\}$.*

*Under observability/controllability assumptions,*

- *for all $j$, there is at leat a $k$ such that $z_{j,k}$ is a pole of $H$, and also a pole of one $G_n$.*

*If no pole in right half plane, exponential $C^0$ stability.*

## Contribution

- Advances in stability of time-varying delay systems coming from a network of Telegrapher's equations
- Math. foundation of HF amplifiers stability decision in CAD
- Projection on the unstable part and rationnal approximation to find the poles

## Open questions

- Generalization of the Henry-Hale to the periodic difference delay system?
- For fixed $j$, which $z_{j,k}$ is a pole of which $G_n$?
(In practice, few $G_n$ are computed.)
- Bound on the number of unstable poles?
- May the (stable) singularities of the $G_n$'s be other than poles ?

Laurent Baratchart, Sébastien Fueyo, Gilles Lebeau, and Jean-Baptiste Pomet, *Sufficient Stability Conditions for Time-varying Networks of Telegrapher's Equations or Difference Delay Equations*, working paper or preprint, November 2019.

Yacine Chitour, Guilherme Mazanti, and Mario Sigalotti, *Stability of non-autonomous difference equations with applications to transport and wave propagation on networks*, Netw. Heterog. Media **11** (2016), no. 4, 563–601.

Daniel Henry, *Linear autonomous neutral functional differential equations*, J. Differential Equations **15** (1974), no. 1, 106–128.

Jack K. Hale and Sjoerd M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Applied Mathematical Sciences, vol. 99, Springer-Verlag, New York, 1993.

Pham Huu Anh Ngoc and Nguyen Dinh Huy, *Exponential stability of linear delay difference equations with continuous* *time*, Vietnam Journal of Mathematics **43** (2015), no. 2, 195–205.