A semi-Lagrangian scheme for second order degenerate Hamilton-Jacobi-Bellman equations in bounded domains

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We consider second order HJB equations in bounded domains:

$$\begin{cases} -\partial_t v + \sup_{a\in A} \mathcal{L}^a(t,x,D_x v, D_x^2 v) = 0 & \text{in } Q_T := [0,T)\times\Omega \\ v(t,x) = \Psi(t,x) & \text{in } \partial^* Q_T, \end{cases} \qquad (HJB)$$

where $\Omega \subset \mathbb{R}^d$ nonempty bounded domain, and

$$\partial^* Q_T := (\{T\}\times\overline{\Omega}) \cup ([0,T)\times\partial\Omega)$$

The operator $\mathcal{L}^a : \overline{Q}_T \times \mathbb{R}^d \times \mathbb{S}^d \to \mathbb{R}$ is given by

$$\mathcal{L}^a(t,x,p,Q) = \Big\{ -b(t,x,a)\cdot p - \frac{1}{2}\mathrm{Tr}[\sigma\sigma^T(t,x,a)Q] + f(t,x,a)\Big\}.$$

- $A \subset \mathbb{R}^m$: compact set;
- $T > 0$: terminal time;
- b, σ: continuous and Lipschitz in space;
- f, Ψ: continuous.

The HJB equation is related the value function of the following stochastic optimal control problem:

$$V(t,x) := \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_t^{T \wedge \tau_{t,x}^{\alpha}} f\left(s, X_{t,x}^{\alpha}(s), \alpha(s)\right) \mathrm{d}s + \Psi\left(T \wedge \tau_{t,x}^{\alpha}, X_{t,x}^{\alpha}(T \wedge \tau_{t,x}^{\alpha})\right)\right],$$

where $\mathcal{A} = \{A-$valued progressively measurable processes$\}$, $X_{t,x}^{\alpha}(\cdot)$ solves

$$\begin{cases} dX(s) = b(s, X(s), \alpha(s))\, ds + \sigma(s, X(s), \alpha(s))\, dW(s) \\ X(t) = x. \end{cases}$$

and $\tau_{t,x}^{\alpha}$ is the first exit time of $X_{t,x}^{\alpha}(\cdot)$ from Ω, i.e.

$$\tau_{t,x}^{\alpha} := \inf\{s \in [t, T] \mid X_{t,x}^{\alpha}(s) \notin \Omega\}.$$

Viscosity solutions for boundary problems: continuous value function

Definition: A locally bounded upper (resp. lower) semicontinuous function $u : \overline{Q}_T \to \mathbb{R}$ is a *viscosity subsolution* (resp. *supersolution*), to (HJB), in the strong sense, if

- u is a viscosity subsolution (resp. supersolution) to

$$-\partial_t v + \sup_{a \in A} \mathcal{L}^a(t, x, D_x v, D_x^2 v) = 0 \quad \text{in } Q_T.$$

- $u \leq \Psi$ (resp. $u \geq \Psi$) in $\partial^* Q_T$.

We say that u is a viscosity solution to (HJB), in the strong sense, if

$$u^*(t,x) := \limsup_{(s,y) \to (t,x)} u(s,y) \quad \text{and} \quad u_*(t,x) := \liminf_{(s,y) \to (t,x)} u(s,y)$$

are sub- and supersolutions to (HJB) in the strong sense.

◇ Note that, by definition, $V(t,x) = \Psi(t,x)$ for all $(t,x) \in \partial^* Q_T$.

◇ If the value function is continuous, i.e. $V \in C(\bar{Q}_T)$, then $V = V_* = V^*$ and, as a consequence of the dynamic programming principle

$$V(t,x) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_t^{\theta \wedge \tau^{t,x,\alpha}} f\left(s, Y^{x,t,\alpha}(s), \alpha(s)\right) ds + V\left(\theta \wedge \tau^{t,x,\alpha}, Y^{x,t,\alpha}(\theta \wedge \tau^{t,x,\alpha})\right) \right].$$

the value function is shown to be a viscosity solution to (HJB) in the strong sense.

◇ Moreover, V is the unique viscosity solution by a classical comparison principle between sub- and supersolutions to (HJB) (Jensen '86, Ishii '89, Crandall-Ishii '90).

◇ This implies that V is characterized by (HJB).

- In the current framework, the continuity of V does not hold in general.
- A typical assumption to ensure the continuity of V is the following (see e.g. Fleming-Soner '06)

(C)(i) $\partial\Omega$ is C^2.

(ii) For every $(t,x) \in \partial^* Q_T$ there exists $a \in A$ such that the signed distance $\hat{\rho}$ to $\partial\Omega$ satisfies:

$$-\frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(t,x,a)D_x^2\hat{\rho}(x)] - b(t,x,a)\cdot D_x\hat{\rho}(x) > 0.$$

(iii) The fonction Ψ can be extended to a function $g \in C^{1,2}(\overline{Q}_T)$ satisfying

$$-\partial_t g(t,x) + \sup_{a\in A}\mathcal{L}^a(t,x,D_x g(t,x),D_x^2 g(t,x)) \leq 0 \quad \forall\,(t,x) \in \overline{Q}_T.$$

◇ The main idea behind assumption (C)(ii) is that, by Itô's formula,

$$\begin{aligned} d\hat{\rho}(X(s)) &= -\left(\tfrac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(s,X(s),a)D_x^2\hat{\rho}(X(s))] + b(t,X(s),a)\cdot D_x\hat{\rho}(X(s))\right)ds \\ &\quad +\sigma^\top(s,X(s),a)n(X(s))\cdot dW(s) \end{aligned}$$

and, hence, almost surely,

$$\inf\{s > t \mid X(s) \neq \overline{\Omega}\} = t.$$

◇ Assumption (C)(ii) is satisfied for instance in the uniformly elliptic case: there exists $c > 0$ such that

$$\sum_{i,j=1}^{d}(\sigma\sigma^\top)_{ij}(t,x,a)\xi_i\xi_j \geq c|\xi|^2 \;\; \forall\,(t,x) \in \overline{Q}_T,\; a \in A.$$

◇ In the deterministic case $\sigma = 0$, (C)(ii) is satisfied if there exists $c > 0$ such that for all $(t,x) \in [0,T] \times \partial\Omega$ we have

$$b(t,x,a)\cdot n(x) > c,$$

for some $a \in A$.

- In general, V is not continuous in $\overline{Q}_T$.
- One can prove that V is continuous on $\{T\} \times \overline{\Omega}$ but, in general, discontinuous in $[0, T) \times \partial\Omega$.
- However, it can be checked that V^* and V_* are sub- and supersolutions in Q_T, and, in $\partial^* Q_T$, we have

$$\min\Big(-\partial_t V^*(t,x) + \sup_{a\in A} \mathcal{L}^a(t,x,D_x V^*(t,x), D_x^2 V^*(t,x)), V^* - \Psi\Big) \leq 0,$$
$$\Big(\text{resp. } \max\Big(-\partial_t V_*(t,x) + \sup_{a\in A} \mathcal{L}^a(t,x,D_x V_*(t,x), D_x^2 V_*(t,x)), V_* - \Psi\Big) \geq 0\Big).$$

in the viscosity sense.

Definition: A locally bounded upper (resp. lower) semicontinuous function $u : \overline{Q}_T \to \mathbb{R}$ is a *viscosity subsolution* (resp. *supersolution*), to (HJB), in the weak sense, if

- u is a viscosity subsolution (resp. supersolution) to

$$-\partial_t v + \sup_{a \in A} \mathcal{L}^a(t, x, D_x v, D_x^2 v) = 0 \quad \text{in } Q_T.$$

- At every $(t, x) \in \partial^* Q_T$, the subsolution (resp. supersolution) u satisfies, in the viscosity sense,

$$\min\Big(-\partial_t u(t,x) + \sup_{a\in A} \mathcal{L}^a(t,x,D_x u(t,x), D_x^2 u(t,x)), u - \Psi \Big) \leq 0,$$
$$\Big(\text{resp. } \max\Big(-\partial_t u(t,x) + \sup_{a\in A} \mathcal{L}^a(t,x,D_x u(t,x), D_x^2 u(t,x)), u - \Psi \Big) \geq 0\Big).$$

Definition: We say that a strong comparison principle holds if for any sub- and supersolutions u_1 and u_2 to (HJB) in the weak sense, one has $u_1 \leq u_2 \in Q_T$.

- Strong comparison principles for (HJB) have been studied in Chaumont '04, following some techniques in Barles-Rouy '98 in the stationary framework and in Barles-Perthame '90 in the deterministic framework.

Semi-Lagrangian scheme in unbounded domains

Consider the HJB equation

$$
\begin{aligned}
-\partial_t v + \sup_{a\in A} \mathcal{L}^a(t,x,D_x v, D_x^2 v) = 0 \quad & \text{in } [0,T)\times\mathbb{R} \\
v(T,x) = \Psi(x) \quad & \text{in } \mathbb{R},
\end{aligned}
$$

Given a time step $h = T/N$, and a space step Δx, with associated space grid $\mathcal{G}_{\Delta x}$, the scheme reads as follows: for all $k = 0,\dots,N-1$

$$
\begin{aligned}
V_k(x_i) &= \inf_{a\in A}\left\{\theta f(t_k,x_i,a) + \gamma^+\mathcal{I}[V_{k+1}](y_h^+(t_k,x_i,a)) + \gamma^-\mathcal{I}[V_{k+1}](y_h^-(t_k,x_i,a))\right\} \\
V_N(x_i) &= \Psi(x_i)
\end{aligned}
$$

where

$$
y_h^{\pm}(t,x,a) := x + hb(t,x,a) \pm \sqrt{h}\sigma(t,x,a), \quad \gamma^{\pm} = \frac{1}{2}, \quad \theta = h.
$$

- The scheme can be seen as the DPP equation of a Markov decision problem that discretizes the optimal control problem.
- If $(\Delta x)^2/h \to 0$, then we get convergence to the value function (Menaldi '89, Camilli-Falcone '95).
- When $\sigma = 0$, an overview of this (an other methods) can be found in Falcone-Ferretti '14.

- We consider now equation (HJB).
- Inspired by Milstein-Tretyakov '01, in the framework of semilinear parabolic equations, we consider the scheme: for all $k = 0, \dots, N-1$

$$V_k(x_i) = \inf_{a\in A} \Big\{ \theta^a f(t_k, x_i, a) + \gamma^{a,+}\Phi[V_{k+1}](y^+_{\lambda^{a,+}}(t_k, x_i, a)) + \gamma^{a,-}\Phi[V_{k+1}](y^-_{\lambda^{a,-}}(t_k, x_i, a)) \Big\},$$

where

- $\lambda^{a,\pm} := \max\{h, \inf\{\lambda \in [0,h] \mid y^\pm_\lambda(t_k, x_i, a) \notin \Omega\}\} \in [0,h]$.
- $\gamma^{a,\pm} := \frac{\sqrt{\lambda^{a,\mp}}}{\sqrt{\lambda^{a,-}}+\sqrt{\lambda^{a,+}}} \in (0,1)$
- $\theta^a := \gamma^{a,-}\lambda^{a,-} + \gamma^{a,+}\lambda^{a,+} \in (0,h]$.
- $$\Phi[V_{k+1}](y^\pm_{\lambda^{a,\pm}}(t_k, x_i, a)) := \begin{cases} \mathcal{I}[V_{k+1}](y^\pm_{\lambda^{a,\pm}}(t_k, x_i, a)) & \text{if } \lambda^{a,\pm} = h, \\ \Psi(t_k + \lambda^{a,\pm}, y^\pm_{\lambda^{a,\pm}}(t_k, x_i, a)) & \text{otherwise.} \end{cases}$$

- In the framework of a continuous value function we have the following result:

Theorem: Assume that (C) holds and that $(\Delta x)^2/h \to 0$. Then, the solution of the scheme converges uniformly in $\overline{Q}_T$ to the value function V.

- Related work: Reisinger-Rotaetxe '17; truncated SL for HJB equations: The scheme requires CFL conditions $\Rightarrow$ implicit schemes.

- In the framework of a discontinuous value function we have the following result:

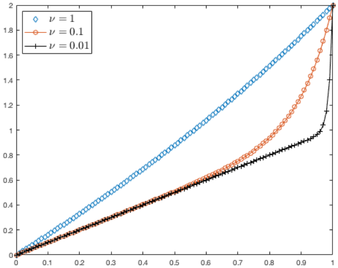
Theorem: Assume that a strong comparison principle holds, and that $(\Delta x)^2/h \to 0$. Then, the solution of the scheme converges uniformly on compact subsets of $[0, T) \times \Omega$ to the value function V.

- The proofs of both results rely on the techniques introduced in Barles-Souganidis '91.

- The same results hold for the stationary version of (HJB).

Let us consider the following stationary problem:

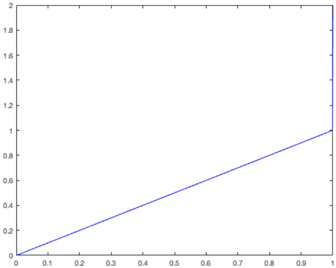
$$\begin{cases} |v_x| - \nu v_{xx} = 1 & x \in (0,1) \\ v(0) = 0, \quad v(1) = 2. \end{cases}$$



Non degenerate case ($\nu > 0$): the boundary condition Ψ is taken continuously.

About the boundary behavior: weak B.C.

In the degenerate case ($\nu = 0$):



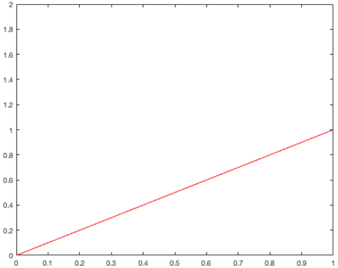
v is discontinuous at $\partial\Omega$,
$v_*(1) < v(1)$.

Suppose that instead of considering $\tau^{\alpha}_{t,x}$, we consider

$$\hat{\tau}^{\alpha}_{t,x} := \inf\{s \in [t, T] \mid X^{\alpha}_{t,x}(s) \notin \overline{\Omega}\}.$$

Then, the natural scheme is

$$V_k(x_i) = \inf_{a\in A}\left\{\theta^a f(t_k, x_i, a) + \gamma^{a,+}\mathcal{I}[V_{k+1}](y^{+}_{\lambda^{a,+}}(t_k, x_i, a)) \right.$$
$$\left. +\gamma^{a,-}\mathcal{I}[V_{k+1}](y^{-}_{\lambda^{a,-}}(t_k, x_i, a))\right\},$$



- Approximates the unique continuous solution in $\overline{\Omega}$:
  Numerically ✓
  Theoretically ??