Trajectory Optimization of a Rolling Ball Robot Actuated by Internal Point Masses

ICIAM 2019 Valencia

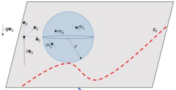
Stuart Rogers

Institute for Mathematics and its Applications
University of Minnesota, Twin Cities

July 17, 2019

Outline

Method



Derivation of Dynamics

- Newton's Laws
- Hamilton's Principle
- Lagrange-d'Alembert's Principle

ODE

$$\dot{\boldsymbol{x}} = \mathbf{f}\left(t, \boldsymbol{x}, \boldsymbol{u}\right)$$

$$\boldsymbol{\sigma} = \mathbf{0},\ \boldsymbol{\psi} = \mathbf{0},\ \mathbf{D} \leq \mathbf{0},\ J \equiv p + \int_a^b L \,\mathrm{d}t$$

Derivation of Controlled Equations of Motion

- Pontryagin's Minimum Principle

$$H \equiv L + \boldsymbol{\lambda}^{\mathsf{T}}\mathbf{f}$$

$$G \equiv p + \boldsymbol{\xi}^{\mathsf{T}}\boldsymbol{\sigma} + \boldsymbol{\nu}^{\mathsf{T}}\boldsymbol{\psi}$$

$$H_{\boldsymbol{uu}} > 0 \qquad \boldsymbol{u}_0$$

DAE TPBVP

$$\begin{aligned} \mathbf{0} &= H_{\boldsymbol{u}}^{\mathsf{T}}\left(t, \boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{u}\right) \\ \dot{\boldsymbol{x}} &= H_{\boldsymbol{\lambda}}^{\mathsf{T}}\left(t, \boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{u}\right) \\ \dot{\boldsymbol{\lambda}} &= -H_{\boldsymbol{x}}^{\mathsf{T}}\left(t, \boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{u}\right) \\ + &\qquad \text{B.C.s} \end{aligned}$$

ODE TPBVP

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{\pi}\left(t, \boldsymbol{x}, \boldsymbol{\lambda}\right) \\ \dot{\boldsymbol{x}} &= \hat{H}_{\boldsymbol{\lambda}}^{\mathsf{T}}\left(t, \boldsymbol{x}, \boldsymbol{\lambda}\right) \\ \dot{\boldsymbol{\lambda}} &= -\hat{H}_{\boldsymbol{x}}^{\mathsf{T}}\left(t, \boldsymbol{x}, \boldsymbol{\lambda}\right) \\ + &\qquad \text{B.C.s} \end{aligned}$$

Numerical Solution of Controlled Equations of Motion

- Automatic Differentiation
- Predictor-Corrector Path-Following
- Newton's Method
- MIRK & Collocation
- Deferred Corrections

Initial Solution

- Gradient Method
- Direct Method

GPOPS-II

State $\boldsymbol{x}$

Costate $\boldsymbol{\lambda}$

Control $\boldsymbol{u} = \boldsymbol{\pi}\left(t, \boldsymbol{x}, \boldsymbol{\lambda}\right)$

Indirect Method: Solvers

1. Use global instead of initial value method DAE/ODE TPBVP solvers.
   - Global methods (i.e. MIRK & collocation) converge more robustly than initial value methods (i.e. shooting).
2. Use ODE instead of DAE TPBVP solvers.
   - For path inequality constraints $\mathbf{D}(t, \boldsymbol{x}, \boldsymbol{u}) \leq 0$, the indirect method yields a DAE TPBVP, whose solution requires knowledge of the switching structure, which is very hard to guess.
   - Even if the switching structure is known, global method DAE TPBVP solvers are not readily available, especially in `MATLAB`. However, global method ODE TPBVP solvers are readily available, e.g. `bvptwp` and `sbvp`.
   - Enforce the path inequality constraints through penalty functions in the integrand cost function L and assume regularity $H_{\boldsymbol{uu}} > 0$, so that the indirect method yields an ODE TPBVP.

Direct Method

1. Discretization:

$$\begin{aligned} \boldsymbol{x}_m(t) &\approx \textstyle\sum_{r=1}^{R} \boldsymbol{\alpha}_{mr}\rho_r(t) \\ \boldsymbol{u}_m(t) &\approx \textstyle\sum_{r=1}^{R} \boldsymbol{\beta}_{mr}\rho_r(t) \end{aligned} \qquad t_{m-1} \leq t \leq t_m, \qquad 1 \leq m \leq M$$

2. Transcription:

$$\min_{a,\boldsymbol{x}(a),b,\boldsymbol{u}} J \;\text{ s.t. } \left\{ \begin{aligned} \dot{\boldsymbol{x}} &= \mathbf{f}\left(t, \boldsymbol{x}, \boldsymbol{u}\right), \\ \boldsymbol{\sigma}\left(a, \boldsymbol{x}(a)\right) &= \mathbf{0}, \\ \boldsymbol{\psi}\left(b, \boldsymbol{x}(b)\right) &= \mathbf{0}. \end{aligned} \right. \approx \min_{\boldsymbol{z}\in\mathbb{R}^Q} F(\boldsymbol{z}) \;\text{ s.t. } \left\{ \begin{aligned} &\mathbf{g}_{\mathrm{L}} \leq \mathbf{g}(\boldsymbol{z}) \leq \mathbf{g}_{\mathrm{U}}, \\ &\boldsymbol{z}_{\mathrm{L}} \leq \boldsymbol{z} \leq \boldsymbol{z}_{\mathrm{U}}. \end{aligned} \right.$$

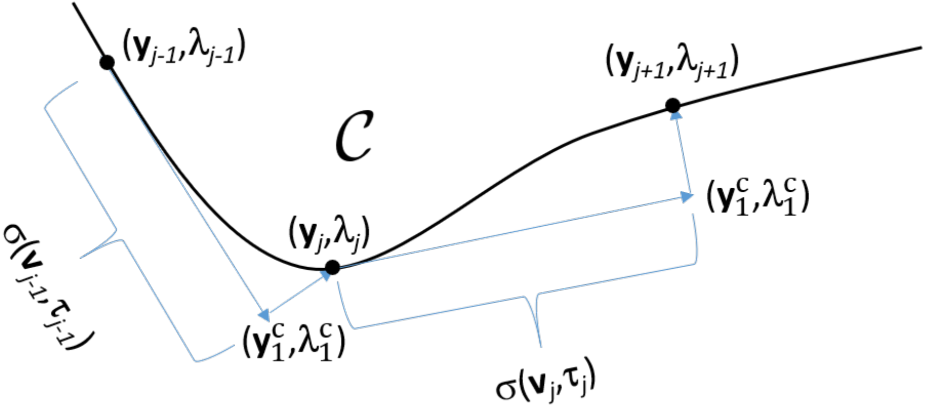
3. NLP solvers: Knitro, IPOPT, and SNOPT.

Direct vs Indirect Methods

	Direct	**Indirect**
Poor Initial State & Control Guess	Convergence	No Convergence
Initial Costate Guess	Not Required	Good Guess Required
Path Inequality Constraints	Handles	Cannot Handle
Predictor-Corrector Path-Following	Research	Available

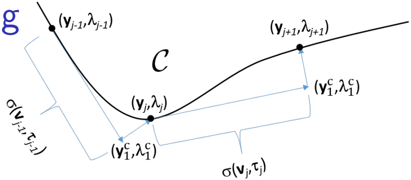
- A good initial guess must be supplied to the indirect method via gradient descent or the direct method.
- Path inequality constraints can be handled by the indirect method as soft constraints through penalty functions in the performance index.
- The direct and indirect methods only converge to a local minimum solution near the initial solution guess.
- Predictor-corrector path-following is a technique to obtain a solution to a difficult, nonconvex optimal control problem, starting from the solution to an easy optimal control problem.

Predictor-Corrector Path-Following



$$s \overset{\frac{t-a}{b-a}}{\longleftarrow} t \qquad \mathbf{y} \leftarrow \begin{bmatrix} \boldsymbol{x}^\top & \boldsymbol{\lambda}^\top & \boldsymbol{\xi}^\top & \boldsymbol{\nu}^\top & a & b \end{bmatrix}^\top \qquad \lambda \leftarrow \mu$$

Predictor-Corrector Path-Following



Seek to construct solutions of $\mathcal{N}(\mathbf{y},\lambda)=\mathbf{0}$, where $\mathcal{N}$ is an operator parameterized by λ. Let $\mathcal{C}\equiv\{(\mathbf{y},\lambda):\mathcal{N}(\mathbf{y},\lambda)=\mathbf{0}\}$.

1. $(\mathbf{y}_j,\lambda_j)$ is a solution on $\mathcal{C}$.
2. Find a unit tangent $(\mathbf{v}_j,\tau_j)$ to $\mathcal{C}$ at $(\mathbf{y}_j,\lambda_j)$ in order to construct a predictor $(\mathbf{y}_1^c,\lambda_1^c)\leftarrow(\mathbf{y}_j,\lambda_j)+\sigma(\mathbf{v}_j,\tau_j)$.

$$\begin{cases}\mathcal{N}_{\mathbf{y}}(\mathbf{y}_j,\lambda_j)\,\mathbf{v}_j+\mathcal{N}_\lambda(\mathbf{y}_j,\lambda_j)\,\tau_j=\mathbf{0}\\ \langle\mathbf{v}_j,\mathbf{v}_j\rangle+\tau_j^2=1\end{cases}$$

3. Push the predictor $(\mathbf{y}_1^c,\lambda_1^c)$ onto the next solution $(\mathbf{y}_{j+1},\lambda_{j+1})$ on $\mathcal{C}$ by Newton iteration.

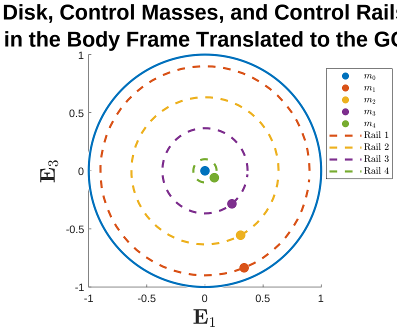
$$\begin{cases}\mathcal{N}_{\mathbf{y}}(\mathbf{y}_k^c,\lambda_k^c)\,\delta\mathbf{y}+\mathcal{N}_\lambda(\mathbf{y}_k^c,\lambda_k^c)\,\delta\lambda=-\mathcal{N}(\mathbf{y}_k^c,\lambda_k^c)\\ \langle\mathbf{v}_j,\delta\mathbf{y}\rangle+\tau_j\delta\lambda=0\end{cases}$$

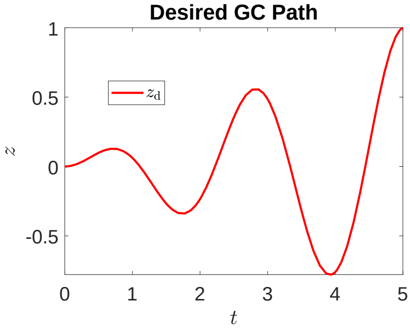
$$(\mathbf{y}_{k+1}^c,\lambda_{k+1}^c)\leftarrow(\mathbf{y}_k^c,\lambda_k^c)+(\delta\mathbf{y},\delta\lambda)$$

Trajectory Optimization

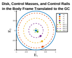
1. **Initialize** Construct an initial solution (e.g. via the gradient or direct method) for which all path inequality constraints but one are inactively satisfied, e.g. by using hard path inequality constraints $\mathbf{D}(t, \boldsymbol{x}, \boldsymbol{u}) \leq 0$ or by using softly-weighted penalty functions in L.
2. **Clamp** Enforce all the inactively satisfied path inequality constraints via heavily-weighted penalty functions in L.
3. **Wriggle** Starting from the initial solution, use predictor-corrector path-following to gradually enforce the lone violated path inequality constraint, without violating the inactively satisfied path inequality constraints.

Rolling Disk: Tracking a Sinusoidally-Modulated Linear Path





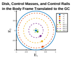
Rolling Disk: States, Controls, and Dynamics



$$\boldsymbol{x} \equiv \begin{bmatrix} \boldsymbol{\theta} \\ \dot{\boldsymbol{\theta}} \\ \phi \\ \dot{\phi} \end{bmatrix} \qquad\qquad \boldsymbol{u} \equiv \ddot{\boldsymbol{\theta}}$$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{\boldsymbol{\theta}} \\ \ddot{\boldsymbol{\theta}} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \mathbf{f}\left(t, \boldsymbol{x}, \boldsymbol{u}\right) \equiv \begin{bmatrix} \dot{\boldsymbol{\theta}} \\ \boldsymbol{u} \\ \dot{\phi} \\ \kappa\left(t, \boldsymbol{x}, \boldsymbol{u}\right) \end{bmatrix}$$

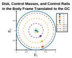
Rolling Disk: Initial and Final Boundary Conditions



$$\boldsymbol{\sigma}\left(a, \boldsymbol{x}(a)\right) \equiv \begin{bmatrix} \boldsymbol{\theta}(a) - \boldsymbol{\theta}_a \\ \dot{\boldsymbol{\theta}}(a) - \dot{\boldsymbol{\theta}}_a \\ \phi(a) - \phi_a \\ -r\dot{\phi}(a) - \dot{z}_a \end{bmatrix} = \mathbf{0}$$

$$\boldsymbol{\psi}\left(b, \boldsymbol{x}(b)\right) \equiv \begin{bmatrix} \left(\tilde{\Lambda}(\phi(b))\left[\frac{1}{M}\sum_{i=0}^{n} m_i \boldsymbol{\zeta}_i(\theta_i(b))\right]\right)_1 - \Delta_b \\ \dot{\boldsymbol{\theta}}(b) - \dot{\boldsymbol{\theta}}_b \\ z_a - r\left(\phi(b) - \phi_a\right) - z_b \\ -r\dot{\phi}(b) - \dot{z}_b \end{bmatrix} = \mathbf{0}$$

Rolling Disk: Path Inequality Constraints



Magnitude of Normal Force:

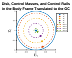
$$N = m_0 \left[g - \left(\ddot{\phi}\zeta_{0,3} + \dot{\phi}^2 \zeta_{0,1} \right) \sin\phi + \left(\ddot{\phi}\zeta_{0,1} - \dot{\phi}^2 \zeta_{0,3} \right) \cos\phi \right] - F_{\mathrm{e},3}$$

Static Friction:

$$-f_{\mathrm{s}} \boldsymbol{\sigma} = - \left\{ m_0 \left[r\ddot{\phi} + \left(\ddot{\phi}\zeta_{0,3} + \dot{\phi}^2 \zeta_{0,1} \right) \cos\phi + \left(\ddot{\phi}\zeta_{0,1} - \dot{\phi}^2 \zeta_{0,3} \right) \sin\phi \right] + F_{\mathrm{e},1} \right\} \mathbf{e}_1$$

1. GC Trajectory Tracking:
   $z_a - r(\phi - \phi_a) = z_{\mathrm{d}} \iff \left(z_a - r(\phi - \phi_a) - z_{\mathrm{d}} \right)^2 = 0$
2. No Detachment: $0 < N \Leftarrow \epsilon \le N$
3. No Slip: $f_{\mathrm{s}} \le \mu_{\mathrm{s}} N \iff f_{\mathrm{s}}^2 \le \mu_{\mathrm{s}}^2 N^2$
4. Bounded Acceleration Magnitudes:
   $\left| \dot{\theta}_i^2 \boldsymbol{\zeta}_i'' + \ddot{\theta}_i \boldsymbol{\zeta}_i' \right| \le M_i \iff \left| \dot{\theta}_i^2 \boldsymbol{\zeta}_i'' + \ddot{\theta}_i \boldsymbol{\zeta}_i' \right|^2 \le M_i^2 \quad 1 \le i \le n$

Rolling Disk: Endpoint Cost Function, Integrand Cost Function, and Performance Index

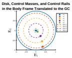


$$p\left(a, \boldsymbol{x}(a), b, \boldsymbol{x}(b)\right) \equiv 0$$

$$L\left(t, \boldsymbol{x}, \boldsymbol{u}\right) \equiv \overbrace{\sum_{i=1}^{n} \frac{\gamma_i}{2} \left|\dot{\theta}_i^2 \boldsymbol{\zeta}_i'' + \ddot{\theta}_i \boldsymbol{\zeta}_i'\right|^2}^{\text{Small Acceleration Magnitudes}} + \overbrace{\frac{\alpha}{2}\left(z_a - r(\phi - \phi_a) - z_{\mathrm{d}}\right)^2}^{\text{GC Trajectory Tracking}} + \overbrace{\kappa_1 \max{}^4 \left\{0, \epsilon - N\right\}}^{\text{No Detachment}}$$
$$+ \underbrace{\kappa_2 \max{}^4 \left\{0, f_{\mathrm{s}}^2 - \mu_{\mathrm{s}}^2 N^2\right\}}_{\text{No Slip}} + \underbrace{\sum_{i=1}^{n} \kappa_{i+2} \max{}^4 \left\{0, \left|\dot{\theta}_i^2 \boldsymbol{\zeta}_i'' + \ddot{\theta}_i \boldsymbol{\zeta}_i'\right|^2 - M_i^2\right\}}_{\text{Bounded Acceleration Magnitudes}}$$

$$J \equiv p\left(a, \boldsymbol{x}(a), b, \boldsymbol{x}(b)\right) + \int_a^b L\left(t, \boldsymbol{x}, \boldsymbol{u}\right)\, \mathrm{d}t$$
$$= \int_a^b \Bigg[\overbrace{\sum_{i=1}^{n} \frac{\gamma_i}{2} \left|\dot{\theta}_i^2 \boldsymbol{\zeta}_i'' + \ddot{\theta}_i \boldsymbol{\zeta}_i'\right|^2}^{\text{Small Acceleration Magnitudes}} + \overbrace{\frac{\alpha}{2}\left(z_a - r(\phi - \phi_a) - z_{\mathrm{d}}\right)^2}^{\text{GC Trajectory Tracking}} + \overbrace{\kappa_1 \max{}^4 \left\{0, \epsilon - N\right\}}^{\text{No Detachment}}$$
$$+ \underbrace{\kappa_2 \max{}^4 \left\{0, f_{\mathrm{s}}^2 - \mu_{\mathrm{s}}^2 N^2\right\}}_{\text{No Slip}} + \underbrace{\sum_{i=1}^{n} \kappa_{i+2} \max{}^4 \left\{0, \left|\dot{\theta}_i^2 \boldsymbol{\zeta}_i'' + \ddot{\theta}_i \boldsymbol{\zeta}_i'\right|^2 - M_i^2\right\}}_{\text{Bounded Acceleration Magnitudes}} \Bigg] \mathrm{d}t$$

Rolling Disk: Endpoint Function and Hamiltonian



$$G\left(a, \boldsymbol{x}(a), \boldsymbol{\xi}, b, \boldsymbol{x}(b), \boldsymbol{\nu}\right) \equiv p\left(a, \boldsymbol{x}(a), b, \boldsymbol{x}(b)\right) + \boldsymbol{\xi}^{\mathsf{T}} \boldsymbol{\sigma}\left(a, \boldsymbol{x}(a)\right) + \boldsymbol{\nu}^{\mathsf{T}} \boldsymbol{\psi}\left(b, \boldsymbol{x}(b)\right)$$

$$= \boldsymbol{\xi}^{\mathsf{T}} \begin{bmatrix} \boldsymbol{\theta}(a) - \boldsymbol{\theta}_a \\ \dot{\boldsymbol{\theta}}(a) - \dot{\boldsymbol{\theta}}_a \\ \phi(a) - \phi_a \\ -r\dot{\phi}(a) - \dot{z}_a \end{bmatrix} + \boldsymbol{\nu}^{\mathsf{T}} \begin{bmatrix} \left(\tilde{\Lambda}(\phi(b)) \left[\frac{1}{M} \sum_{i=0}^{n} m_i \boldsymbol{\zeta}_i(\theta_i(b))\right]\right)_1 - \Delta_b \\ \dot{\boldsymbol{\theta}}(b) - \dot{\boldsymbol{\theta}}_b \\ z_a - r\left(\phi(b) - \phi_a\right) - z_b \\ -r\dot{\phi}(b) - \dot{z}_b \end{bmatrix}$$

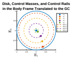
$$H\left(t, \boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{u}\right) \equiv L\left(t, \boldsymbol{x}, \boldsymbol{u}\right) + \boldsymbol{\lambda}^{\mathsf{T}} \mathbf{f}\left(t, \boldsymbol{x}, \boldsymbol{u}\right)$$

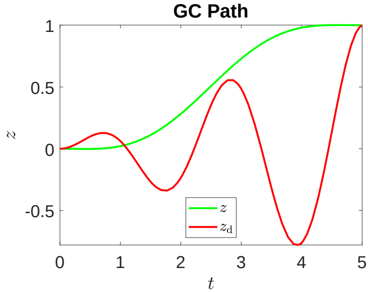
$$= \overbrace{\sum_{i=1}^{n} \frac{\gamma_i}{2} \left| \dot{\theta}_i^2 \boldsymbol{\zeta}_i'' + \ddot{\theta}_i \boldsymbol{\zeta}_i' \right|^2}^{\text{Small Acceleration Magnitudes}} + \overbrace{\frac{\alpha}{2} \left(z_a - r(\phi - \phi_a) - z_{\mathrm{d}}\right)^2}^{\text{GC Trajectory Tracking}} + \overbrace{\kappa_1 \max{}^4 \left\{0, \epsilon - N\right\}}^{\text{No Detachment}}$$

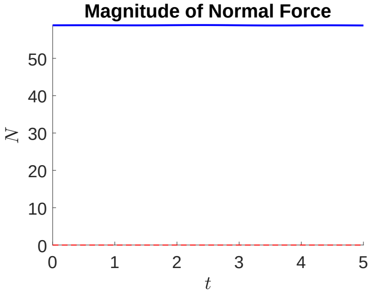
$$+ \underbrace{\kappa_2 \max{}^4 \left\{0, f_{\mathrm{s}}^2 - \mu_{\mathrm{s}}^2 N^2\right\}}_{\text{No Slip}} + \underbrace{\sum_{i=1}^{n} \kappa_{i+2} \max{}^4 \left\{0, \left| \dot{\theta}_i^2 \boldsymbol{\zeta}_i'' + \ddot{\theta}_i \boldsymbol{\zeta}_i' \right|^2 - M_i^2 \right\}}_{\text{Bounded Acceleration Magnitudes}}$$

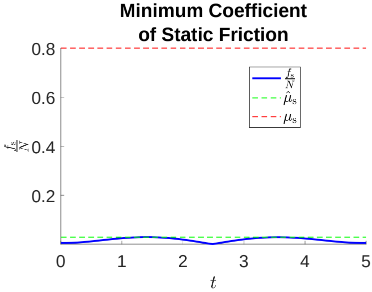
$$+ \boldsymbol{\lambda}^{\mathsf{T}} \begin{bmatrix} \dot{\boldsymbol{\theta}} \\ \boldsymbol{u} \\ \dot{\phi} \\ \kappa\left(t, \boldsymbol{x}, \boldsymbol{u}\right) \end{bmatrix}$$

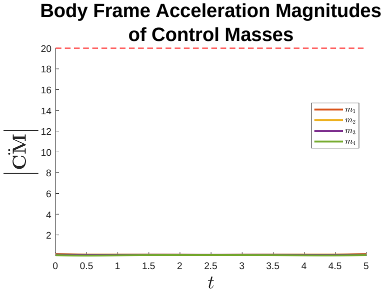
Direct Method



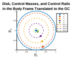


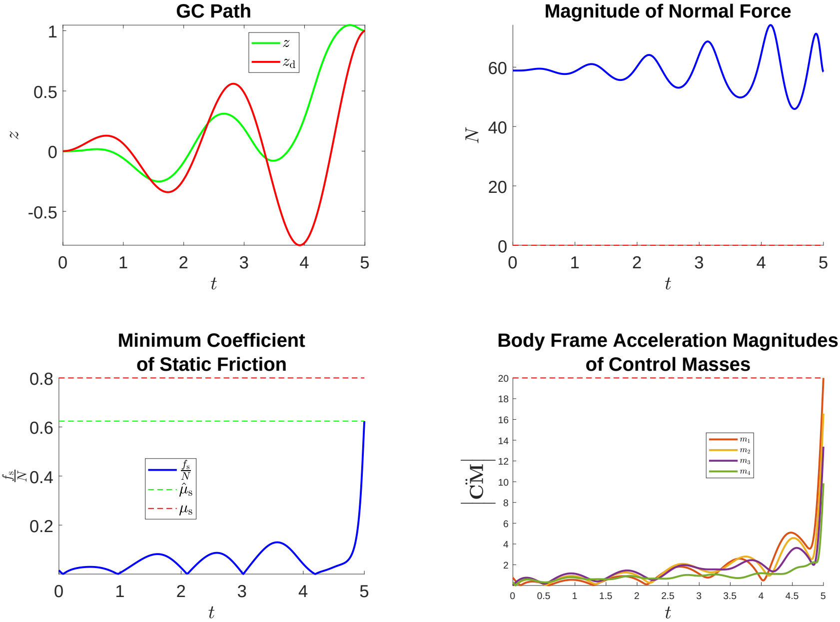




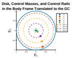


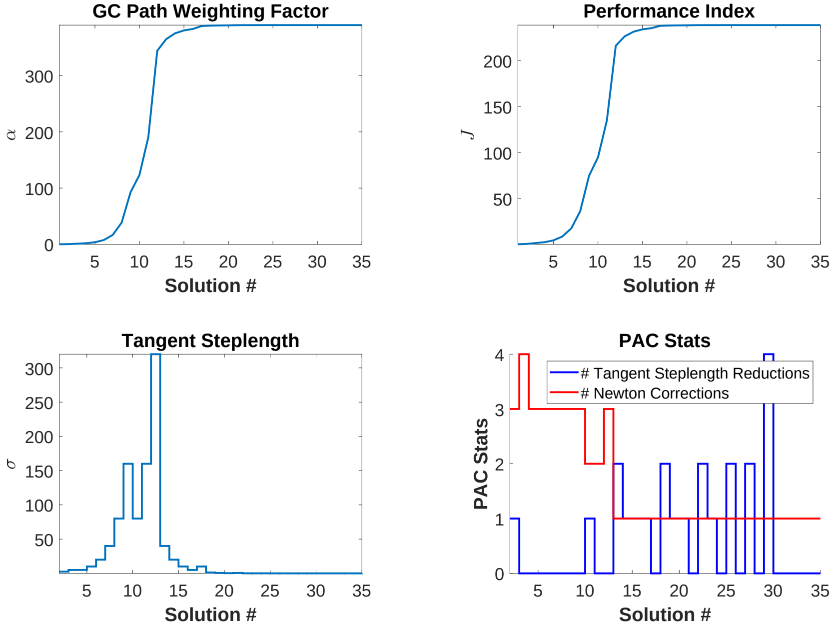
Predictor-Corrector Path-Following in $\alpha = 0 \nearrow 390$



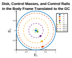


Predictor-Corrector Path-Following in $\alpha = 0 \nearrow 390$





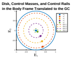
Rolling Disk Trajectory Optimization



1. **Initialize** Set all the constraint weights to zero (i.e. $\alpha = \kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = \kappa_5 = \kappa_6 = 0$) and use the direct method (i.e. `GPOPS-II` with IPOPT) to construct an initial solution which violates the GC trajectory tracking constraint, but which inactively satisfies the no detachment, no slip, and bounded acceleration magnitude constraints.

$$L(t, \boldsymbol{x}, \boldsymbol{u}) \equiv \overbrace{\sum_{i=1}^{4} \frac{1}{2} \left| \dot{\theta}_i^2 \boldsymbol{\zeta}_i'' + \ddot{\theta}_i \boldsymbol{\zeta}_i' \right|^2}^{\text{Small Acceleration Magnitudes}}$$

Rolling Disk Trajectory Optimization



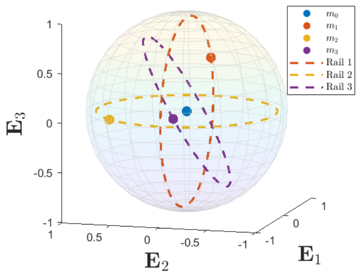
2. **Clamp** Enforce the inactively satisfied no detachment, no slip, and bounded acceleration magnitude constraints by setting $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = \kappa_5 = \kappa_6 = 1\mathrm{e}10$.

$$
\begin{aligned}
L\left(t, \boldsymbol{x}, \boldsymbol{u}\right) \equiv & \overbrace{\sum_{i=1}^{4} \frac{1}{2} \left| \dot{\theta}_i^2 \boldsymbol{\zeta}_i'' + \ddot{\theta}_i \boldsymbol{\zeta}_i' \right|^2}^{\text{Small Acceleration Magnitudes}} + \overbrace{\frac{\alpha}{2} \left(z_a - r(\phi - \phi_a) - z_\mathrm{d} \right)^2}^{\text{GC Trajectory Tracking}} + \overbrace{1\mathrm{e}10 \max{}^4 \left\{ 0, \epsilon - N \right\}}^{\text{No Detachment}} \\
& + \underbrace{1\mathrm{e}10 \max{}^4 \left\{ 0, f_\mathrm{s}^2 - \mu_\mathrm{s}^2 N^2 \right\}}_{\text{No Slip}} + \underbrace{\sum_{i=1}^{4} 1\mathrm{e}10 \max{}^4 \left\{ 0, \left| \dot{\theta}_i^2 \boldsymbol{\zeta}_i'' + \ddot{\theta}_i \boldsymbol{\zeta}_i' \right|^2 - M_i^2 \right\}}_{\text{Bounded Acceleration Magnitudes}}
\end{aligned}
$$

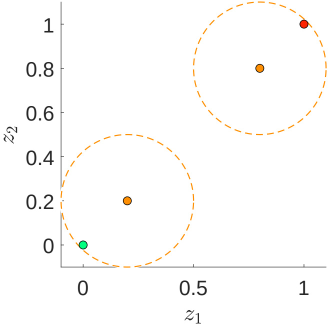
3. **Wriggle** Starting from the direct method solution, use the predictor-corrector path-following indirect method to gradually increase the GC path weighting factor α from 0 up to 390 in order to push the GC path towards the desired GC path.

Rolling Ball: Obstacle Avoidance

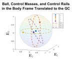
Ball, Control Masses, and Control Rails in the Body Frame Translated to the GC



Desired GC Path



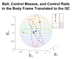
Rolling Ball: States, Controls, and Dynamics



$$\boldsymbol{x} \equiv \begin{bmatrix} \boldsymbol{\theta} \\ \dot{\boldsymbol{\theta}} \\ \mathfrak{q} \\ \boldsymbol{\Omega} \\ \boldsymbol{z} \end{bmatrix} \qquad\qquad \boldsymbol{u} \equiv \ddot{\boldsymbol{\theta}}$$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{\boldsymbol{\theta}} \\ \ddot{\boldsymbol{\theta}} \\ \dot{\mathfrak{q}} \\ \dot{\boldsymbol{\Omega}} \\ \dot{\boldsymbol{z}} \end{bmatrix} = \mathbf{f}\left(t, \boldsymbol{x}, \boldsymbol{u}\right) \equiv \begin{bmatrix} \dot{\boldsymbol{\theta}} \\ \boldsymbol{u} \\ \frac{1}{2}\mathfrak{q}\boldsymbol{\Omega}^{\sharp} \\ \boldsymbol{\kappa}\left(t, \boldsymbol{x}, \boldsymbol{u}\right) \\ \left(\left[\mathfrak{q}\boldsymbol{\Omega}^{\sharp}\mathfrak{q}^{-1}\right]^{\flat} \times r\mathbf{e}_3\right)_{12} \end{bmatrix}$$

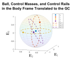
Rolling Ball: Initial and Final Boundary Conditions



$$\boldsymbol{\sigma}\left(a, \boldsymbol{x}(a)\right) \equiv \begin{bmatrix} \boldsymbol{\theta}(a) - \boldsymbol{\theta}_a \\ \dot{\boldsymbol{\theta}}(a) - \dot{\boldsymbol{\theta}}_a \\ \mathfrak{q}(a) - \mathfrak{q}_a \\ \boldsymbol{\Omega}(a) - \boldsymbol{\Omega}_a \\ \boldsymbol{z}(a) - \boldsymbol{z}_a \end{bmatrix} = \mathbf{0}$$

$$\boldsymbol{\psi}\left(b, \boldsymbol{x}(b)\right) \equiv \begin{bmatrix} \left(\left[\mathfrak{q}(b)\left[\frac{1}{M}\sum_{i=0}^{n} m_i \boldsymbol{\zeta}_i(\theta_i(b))\right]^\sharp \mathfrak{q}(b)^{-1}\right]^\flat\right)_{12} - \boldsymbol{\Delta}_b \\ \dot{\boldsymbol{\theta}}(b) - \dot{\boldsymbol{\theta}}_b \\ \boldsymbol{\Omega}(b) - \boldsymbol{\Omega}_b \\ \boldsymbol{z}(b) - \boldsymbol{z}_b \end{bmatrix} = \mathbf{0}$$

Rolling Ball: Path Inequality Constraints



Magnitude of Normal Force:

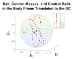
$$N = Mg + \left\langle \sum_{i=0}^{n} m_i \left[\dot{\boldsymbol{\Omega}} \times \mathbf{s}_i + \boldsymbol{\Omega} \times \left(\boldsymbol{\Omega} \times \boldsymbol{\zeta}_i + 2\dot{\theta}_i \boldsymbol{\zeta}_i' \right) + \dot{\theta}_i^2 \boldsymbol{\zeta}_i'' + \ddot{\theta}_i \boldsymbol{\zeta}_i' \right], \boldsymbol{\Gamma} \right\rangle - F_{\mathrm{e},3}$$

Static Friction:

$$-f_{\mathrm{s}} \boldsymbol{\sigma} = \begin{bmatrix} \left(\Lambda \sum_{i=0}^{n} m_i \left[\dot{\boldsymbol{\Omega}} \times \mathbf{s}_i + \boldsymbol{\Omega} \times \left(\boldsymbol{\Omega} \times \boldsymbol{\zeta}_i + 2\dot{\theta}_i \boldsymbol{\zeta}_i' \right) + \dot{\theta}_i^2 \boldsymbol{\zeta}_i'' + \ddot{\theta}_i \boldsymbol{\zeta}_i' \right] - \mathbf{F}_{\mathrm{e}} \right)_{12} \\ 0 \end{bmatrix}$$

1. Obstacle Avoidance:
   $\rho_j \le |\boldsymbol{z} - \boldsymbol{v}_j| \iff \rho_j^2 \le |\boldsymbol{z} - \boldsymbol{v}_j|^2 \quad 1 \le j \le K$
2. No Detachment: $0 < N \Leftarrow \epsilon \le N$
3. No Slip: $f_{\mathrm{s}} \le \mu_{\mathrm{s}} N \iff f_{\mathrm{s}}^2 \le \mu_{\mathrm{s}}^2 N^2$
4. Bounded Acceleration Magnitudes:
   $\left| \dot{\theta}_i^2 \boldsymbol{\zeta}_i'' + \ddot{\theta}_i \boldsymbol{\zeta}_i' \right| \le M_i \iff \left| \dot{\theta}_i^2 \boldsymbol{\zeta}_i'' + \ddot{\theta}_i \boldsymbol{\zeta}_i' \right|^2 \le M_i^2 \quad 1 \le i \le n$

Rolling Ball: Endpoint Cost Function, Integrand Cost Function, and Performance Index

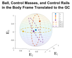


$$p\left(a, \boldsymbol{x}(a), b, \boldsymbol{x}(b)\right) \equiv 0$$

$$\begin{aligned} L\left(t, \boldsymbol{x}, \boldsymbol{u}\right) \equiv & \overbrace{\sum_{i=1}^{n} \frac{\gamma_i}{2} \left|\dot{\theta}_i^2 \boldsymbol{\zeta}_i'' + \ddot{\theta}_i \boldsymbol{\zeta}_i'\right|^2}^{\text{Small Acceleration Magnitudes}} + \overbrace{\sum_{j=1}^{K} h_j \max{}^4 \left\{0, \rho_j^2 - \left|\boldsymbol{z} - \boldsymbol{v}_j\right|^2\right\}}^{\text{Obstacle Avoidance}} + \overbrace{\kappa_1 \max{}^4 \left\{0, \epsilon - N\right\}}^{\text{No Detachment}} \\ & + \underbrace{\kappa_2 \max{}^4 \left\{0, f_{\mathrm{s}}^2 - \mu_{\mathrm{s}}^2 N^2\right\}}_{\text{No Slip}} + \underbrace{\sum_{i=1}^{n} \kappa_{i+2} \max{}^4 \left\{0, \left|\dot{\theta}_i^2 \boldsymbol{\zeta}_i'' + \ddot{\theta}_i \boldsymbol{\zeta}_i'\right|^2 - M_i^2\right\}}_{\text{Bounded Acceleration Magnitudes}} \end{aligned}$$

$$\begin{aligned} J \equiv & \; p\left(a, \boldsymbol{x}(a), b, \boldsymbol{x}(b)\right) + \int_a^b L\left(t, \boldsymbol{x}, \boldsymbol{u}\right) \, \mathrm{d}t \\ = & \int_a^b \Bigg[\overbrace{\sum_{i=1}^{n} \frac{\gamma_i}{2} \left|\dot{\theta}_i^2 \boldsymbol{\zeta}_i'' + \ddot{\theta}_i \boldsymbol{\zeta}_i'\right|^2}^{\text{Small Acceleration Magnitudes}} + \overbrace{\sum_{j=1}^{K} h_j \max{}^4 \left\{0, \rho_j^2 - \left|\boldsymbol{z} - \boldsymbol{v}_j\right|^2\right\}}^{\text{Obstacle Avoidance}} + \overbrace{\kappa_1 \max{}^4 \left\{0, \epsilon - N\right\}}^{\text{No Detachment}} \\ & + \underbrace{\kappa_2 \max{}^4 \left\{0, f_{\mathrm{s}}^2 - \mu_{\mathrm{s}}^2 N^2\right\}}_{\text{No Slip}} + \underbrace{\sum_{i=1}^{n} \kappa_{i+2} \max{}^4 \left\{0, \left|\dot{\theta}_i^2 \boldsymbol{\zeta}_i'' + \ddot{\theta}_i \boldsymbol{\zeta}_i'\right|^2 - M_i^2\right\}}_{\text{Bounded Acceleration Magnitudes}} \Bigg] \mathrm{d}t \end{aligned}$$

Rolling Ball: Endpoint Function and Hamiltonian



$$G\left(a, \boldsymbol{x}(a), \boldsymbol{\xi}, b, \boldsymbol{x}(b), \boldsymbol{\nu}\right) \equiv p\left(a, \boldsymbol{x}(a), b, \boldsymbol{x}(b)\right) + \boldsymbol{\xi}^{\mathsf{T}} \boldsymbol{\sigma}\left(a, \boldsymbol{x}(a)\right) + \boldsymbol{\nu}^{\mathsf{T}} \boldsymbol{\psi}\left(b, \boldsymbol{x}(b)\right)$$

$$= \boldsymbol{\xi}^{\mathsf{T}} \begin{bmatrix} \boldsymbol{\theta}(a) - \boldsymbol{\theta}_a \\ \dot{\boldsymbol{\theta}}(a) - \dot{\boldsymbol{\theta}}_a \\ \mathfrak{q}(a) - \mathfrak{q}_a \\ \boldsymbol{\Omega}(a) - \boldsymbol{\Omega}_a \\ \boldsymbol{z}(a) - \boldsymbol{z}_a \end{bmatrix} + \boldsymbol{\nu}^{\mathsf{T}} \begin{bmatrix} \left(\left[\mathfrak{q}(b) \left[\frac{1}{M} \sum_{i=0}^{n} m_i \boldsymbol{\zeta}_i(\theta_i(b)) \right]^{\sharp} \mathfrak{q}(b)^{-1} \right]^{\flat} \right)_{12} - \boldsymbol{\Delta}_b \\ \dot{\boldsymbol{\theta}}(b) - \dot{\boldsymbol{\theta}}_b \\ \boldsymbol{\Omega}(b) - \boldsymbol{\Omega}_b \\ \boldsymbol{z}(b) - \boldsymbol{z}_b \end{bmatrix}$$

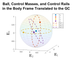
$$H\left(t, \boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{u}\right) \equiv L\left(t, \boldsymbol{x}, \boldsymbol{u}\right) + \boldsymbol{\lambda}^{\mathsf{T}} \mathbf{f}\left(t, \boldsymbol{x}, \boldsymbol{u}\right)$$

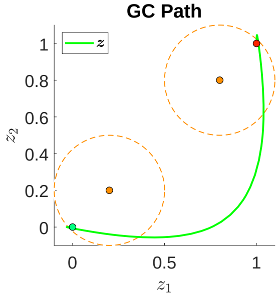
$$= \overbrace{\sum_{i=1}^{n} \frac{\gamma_i}{2} \left| \dot{\theta}_i^2 \boldsymbol{\zeta}_i'' + \ddot{\theta}_i \boldsymbol{\zeta}_i' \right|^2}^{\text{Small Acceleration Magnitudes}} + \overbrace{\sum_{j=1}^{K} h_j \max{}^4 \left\{ 0, \rho_j^2 - \left| \boldsymbol{z} - \boldsymbol{v}_j \right|^2 \right\}}^{\text{Obstacle Avoidance}} + \overbrace{\kappa_1 \max{}^4 \left\{ 0, \epsilon - N \right\}}^{\text{No Detachment}}$$

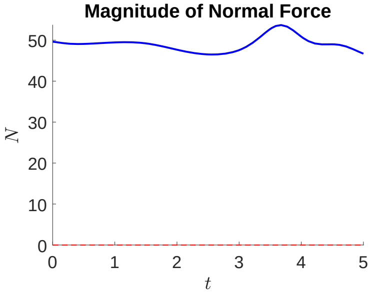
$$+ \underbrace{\kappa_2 \max{}^4 \left\{ 0, f_{\mathrm{s}}^2 - \mu_{\mathrm{s}}^2 N^2 \right\}}_{\text{No Slip}} + \underbrace{\sum_{i=1}^{n} \kappa_{i+2} \max{}^4 \left\{ 0, \left| \dot{\theta}_i^2 \boldsymbol{\zeta}_i'' + \ddot{\theta}_i \boldsymbol{\zeta}_i' \right|^2 - M_i^2 \right\}}_{\text{Bounded Acceleration Magnitudes}}$$

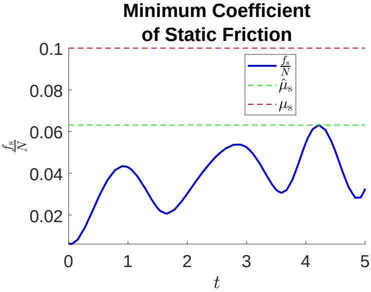
$$+ \boldsymbol{\lambda}^{\mathsf{T}} \begin{bmatrix} \dot{\boldsymbol{\theta}} \\ \boldsymbol{u} \\ \frac{1}{2} \mathfrak{q} \boldsymbol{\Omega}^{\sharp} \\ \boldsymbol{\kappa}\left(t, \boldsymbol{x}, \boldsymbol{u}\right) \\ \left(\left[\mathfrak{q} \boldsymbol{\Omega}^{\sharp} \mathfrak{q}^{-1} \right]^{\flat} \times r \mathbf{e}_3 \right)_{12} \end{bmatrix}$$

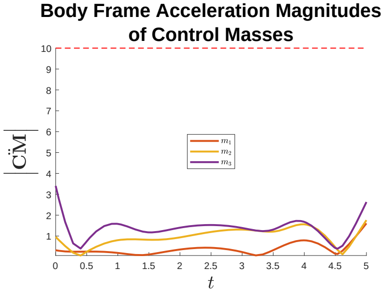
Direct Method



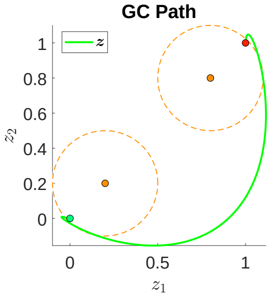


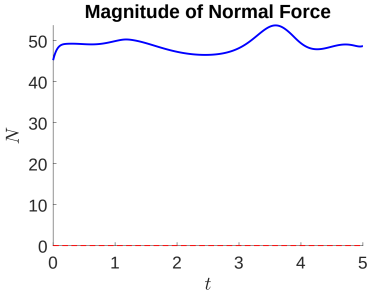


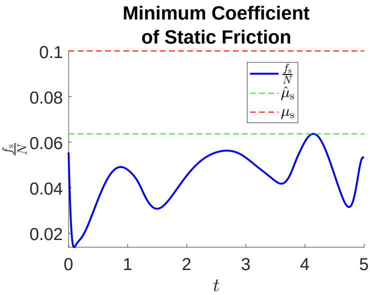


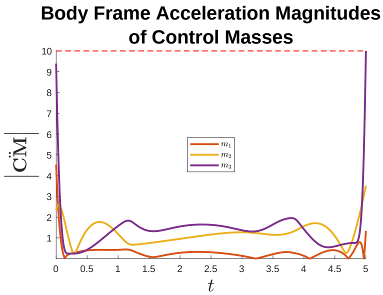


Predictor-Corrector Path-Following in $h_1 = h_2 = 0 \nearrow 5e9$

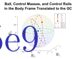


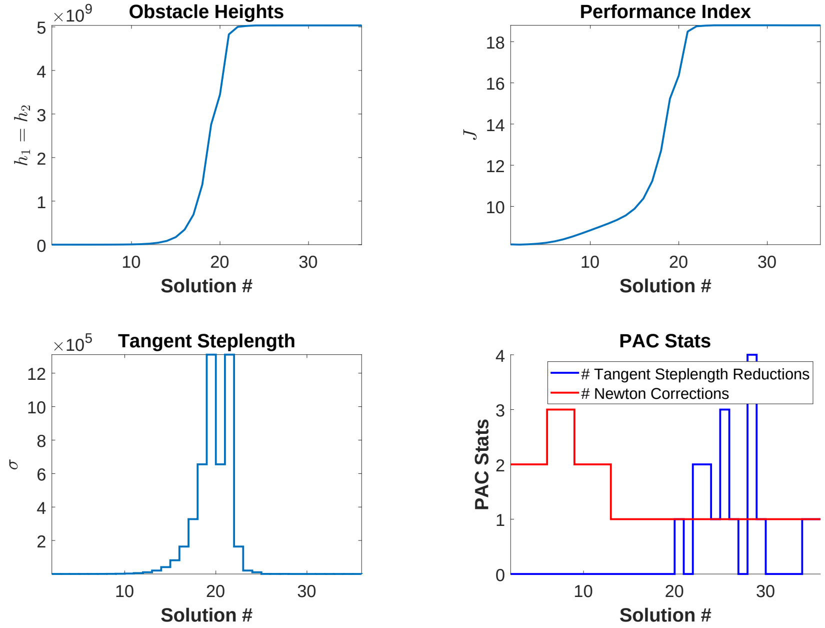




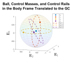


Predictor-Corrector Path-Following in $h_1 = h_2 = 0 \nearrow 5e9$





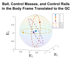
Rolling Ball Trajectory Optimization



1. **Initialize** Set all the constraint weights to zero (i.e. $h_1 = h_2 = \kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = \kappa_5 = 0$) and use the direct method (i.e. `GPOPS-II` with SNOPT) to construct an initial solution which violates the obstacle avoidance constraint, but which inactively satisfies the no detachment, no slip, and bounded acceleration magnitude constraints.

$$L\left(t, \boldsymbol{x}, \boldsymbol{u}\right) \equiv \overbrace{\sum_{i=1}^{3} \frac{1}{2} \left| \dot{\theta}_i^2 \boldsymbol{\zeta}_i'' + \ddot{\theta}_i \boldsymbol{\zeta}_i' \right|^2}^{\text{Small Acceleration Magnitudes}}$$

Rolling Ball Trajectory Optimization



2. **Clamp** Enforce the inactively satisfied no detachment, no slip, and bounded acceleration magnitude constraints by setting $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = \kappa_5 = 1\mathrm{e}10$.

$$
\begin{aligned}
L\left(t, \boldsymbol{x}, \boldsymbol{u}\right) \equiv\ & \overbrace{\sum_{i=1}^{3} \frac{1}{2} \left| \dot{\theta}_i^2 \boldsymbol{\zeta}_i'' + \ddot{\theta}_i \boldsymbol{\zeta}_i' \right|^2}^{\text{Small Acceleration Magnitudes}} + \overbrace{\sum_{j=1}^{2} h_j \max{}^4 \left\{0, \rho_j^2 - \left| \boldsymbol{z} - \boldsymbol{v}_j \right|^2 \right\}}^{\text{Obstacle Avoidance}} + \overbrace{1\mathrm{e}10 \max{}^4 \left\{0, \epsilon - N\right\}}^{\text{No Detachment}} \\
& + \underbrace{1\mathrm{e}10 \max{}^4 \left\{0, f_{\mathrm{s}}^2 - \mu_{\mathrm{s}}^2 N^2 \right\}}_{\text{No Slip}} + \underbrace{\sum_{i=1}^{3} 1\mathrm{e}10 \max{}^4 \left\{0, \left| \dot{\theta}_i^2 \boldsymbol{\zeta}_i'' + \ddot{\theta}_i \boldsymbol{\zeta}_i' \right|^2 - M_i^2 \right\}}_{\text{Bounded Acceleration Magnitudes}}
\end{aligned}
$$

3. **Wriggle** Starting from the direct method solution, use the predictor-corrector path-following indirect method to gradually increase the obstacle heights $h_1 = h_2$ from 0 up to $5\mathrm{e}9$ in order to push the GC path outside the obstacles.

Future Work

- In these numerical experiments, the predictor-corrector path-following method evolves the continuation parameter monotonically.
    - Are there scenarios which reveal turning points in the continuation parameter?
- The predictor-corrector path-following method stagnates once the boundary of a path inequality constraint, enforced via a heavily-weighted penalty function, is encountered.
    - Is it possible to avoid or proceed beyond such obstacles?

MATLAB Software Tools

- The ODE TPBVP solvers `bvptwp` and `sbvp` are used by the indirect method.
  - Both `bvptwp` and `sbvp` are 8^{th}-order accurate and can exploit vectorized ODE functions and ODE Jacobians.
  - `bvptwp` uses mono-implicit Runge-Kutta (MIRK) or collocation methods with deferred corrections and offers 4 non-continuation variants, `twpbvp_m`, `twpbvpc_m`, `twpbvp_l`, and `twpbvpc_l`, and 2 monotonic continuation variants, `acdc` and `acdcc`.
  - `sbvp` uses collocation methods.
- `GPOPS-II` is the direct method solver used.
  - `GPOPS-II` uses *hp*-adaptive mesh refinement and pseudospectral collocation methods to approximate the states and controls.
  - `GPOPS-II` provides estimates of the costates.
  - `GPOPS-II` can exploit vectorized Jacobians and Hessians.
  - `GPOPS-II` uses IPOPT or SNOPT to solve the NLP problems.
- `ADiGator` supplies 1^{st} and 2^{nd} vectorized forward mode automatic derivatives to `bvptwp`, `sbvp`, and `GPOPS-II`.

Relevant Publications

- Vakhtang Putkaradze and Stuart Rogers. "On the Dynamics of a Rolling Ball Actuated by Internal Point Masses." *Meccanica* 53.15 (2018): 3839–3868.
- Vakhtang Putkaradze and Stuart Rogers. "On the Normal Force and Static Friction Acting on a Rolling Ball Actuated by Internal Point Masses." *Regular and Chaotic Dynamics* 24.2 (2019): 145–170.
- Vakhtang Putkaradze and Stuart Rogers. "On the Optimal Control of a Rolling Ball Robot Actuated by Internal Point Masses." *arXiv preprint arXiv:1708.03829* (2017). Submitted to *Journal of Dynamic Systems, Measurement, and Control*.
- Vakhtang Putkaradze and Stuart Rogers. "Numerical Simulations of a Rolling Ball Robot Actuated by Internal Point Masses." *arXiv preprint arXiv:1904.13027* (2019). Submitted to *Numerical Algebra, Control and Optimization*.