Inverse Linear-Quadratic problem

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Outline

1. Inverse optimal control
   - Motivation
   - Definition
   - Injectivity
2. Linear-Quadratic case
   - Direct LQ problem
   - Inverse LQ problem
   - Injectivity
   - Reconstruction

Outline

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   - Inverse LQ problem
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   - Reconstruction

Modeling of human movements

- goal-oriented walking
- arm reaching movements

Assumption in physiology: humans movements are optimal

What do we minimize?

Inverse optimal control problem = to recover the cost which we minimize

Outline

Inverse optimal control

(Direct) Optimal control problem

Given a dynamics $\dot{x} = f(x, u)$ and a cost $C(x_u)$:
for any time $T > 0$ and pair of points x_0, x_T, find a trajectory x_{u^*} solution of

$$\inf\{C(x_u) \; : \; x_u \text{ sol. of } \dot{x} = f(x, u) \text{ s.t. } x_u(0) = x_0, x_u(T) = x_T\}.$$

Optimal synthesis = the set of all optimal trajectories for all T, x_0, x_T.

Inverse optimal control problem

Given $\dot{x} = f(x, u)$ and a set Γ of trajectories:
find a cost $C(x_u)$ such that every $\gamma \in \Gamma$ is solution of

$$\inf\{C(x_u) \; : \; x_u \text{ sol. of } \dot{x} = f(x, u) \text{ s.t. } x_u(0) = \gamma(0), x_u(T) = \gamma(T)\}.$$

Is the cost C corresponding to Γ unique?

Outline

Injectivity

Fix $\mathcal{C}$ family of cost functions

Injectivity problem: is there a **unique** cost $C \in \mathcal{C}$ corresponding to a given optimal synthesis?

Definition

C_1, C_2 are **equivalent** if their optimal synthesis coincide

two costs C and $c \times C$ with constant $c > 0$ have always the same minimizers
$\Rightarrow$ C and $c \times C$ are **trivially equivalent**

Injectivity of the inverse problem $\iff$ no costs in $\mathcal{C}$ non-trivially equivalent

Outline

Linear-quadratic (LQ) case

- Dynamics: controllable linear system on $\mathbb{R}^n$

$$\dot{x} = Ax + Bu, \qquad u \in \mathbb{R}^m$$

- Class $\mathcal{L}$: class of costs defined by quadratic Lagrangians $L(x,u)$

$$\int_0^T L(x,u) = \int_0^T x^\top Q x + 2x^\top S u + u^\top R u$$

  - $Q = Q^\top \geq 0, \quad R = R^\top > 0, \quad Q - SR^{-1}S^\top \geq 0$
  - the Hamiltonian H has no pure imaginary eigenvalues

$$H = \begin{pmatrix} A - BR^{-1}S^\top & BR^{-1}B^\top \\ Q - SR^{-1}S^\top & -A^\top + SR^{-1}B^\top \end{pmatrix}$$

Inverse LQ: to determine (Q, R, S) from the given optimal synthesis

[Kalman 1964, Jameson-Kreindler 1973, Nori-Frezza 2004]

Outline

Solution of the direct problem

For any triple $T, x(0), x(T)$ there exists a unique optimal solution

Solution of direct LQ pb with fixed $T, x(0), x(T)$ [Ferrante 2005]

$$x(t) = e^{t(A-BK_+)}y_+ + e^{t(A-BK_-)}y_-, \quad t \in [0,T],$$

- K_+, K_- det. uniquely by (Q,R,S) and (A,B) via **algebraic Riccati eq.**
- y_+, y_- linear functions of $x(0), x(T)$

Parameterization of the optimal synthesis

denote $A_+ = A - BK_+$ and $A_- = A - BK_-$

$\Rightarrow$ each optimal solution writes $x(t) = e^{tA_+}y_+ + e^{tA_-}y_-$

Theorem

Matrices (A_+, A_-) are uniquely determined by the optimal synthesis

Inverse problem: to determine (Q, R, S) from the given (A_+, A_-)

Outline

Reduction to the canonical class of costs

Fix dynamics A, B

Definition (Canonical class of costs)

$$L_c(x,u) = (u + Kx)^\top \tilde{R}\,(u + Kx)$$

- K is s.t. $A - BK$ is stable
- $\tilde{R} = \tilde{R}^\top > 0$ is s.t. $\det(\tilde{R}) = 1$ (normalization)

$$\text{Each } x^\top Q x + 2x^\top S u + u^\top R u \text{ is equivalent to } (u + Kx)^\top \tilde{R}\,(u + Kx)$$

$\rightarrow$ take $K = K_+$ and $\tilde{R} = \frac{1}{\det(R)} R$

Canonical LQ problem

$$\text{Cost}: \quad L_c(x,u) = (u + Kx)^\top R\,(u + Kx)$$
$$\text{Dynamics}: \quad \dot{x} = Ax + Bu$$

Inverse problem: determine R, K from given A_+, A_-

$$A_+ \mapsto \text{unique } K = K_+$$

K_+ unique solution of $A_+ = A - BK_+$

Injectivity: is the matrix R corresponding to A_+, A_- unique?

Outline

Intermediate reduction

Canonical LQ problem

$$\begin{aligned} \text{Cost}: &\quad L_c(x,u) = (u + K_+x)^\top R\,(u + K_+x) \\ \text{Dynamics}: &\quad \dot{x} = Ax + Bu \end{aligned}$$

$v = u - K_+x \;\rightarrow$ new reduced LQ problem:

Reduced LQ problem

$$\begin{aligned} \text{Cost}: &\quad L_R(u) = u^\top Ru \\ \text{Dynamics}: &\quad \dot{x} = (A - BK_+)x + Bu = A_+x + Bu \end{aligned}$$

Is the reduced LQ problem injective?

Example of equivalent costs

Product structure of reduced LQ problem (with 2 elements)

(x_1, x_2) separation of variable $x \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and (u_1, u_2) of $u \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$

$$\text{Let the dynamics be:} \qquad \begin{cases} \dot{x}_1 = A_1 x_1 + B_1 u_1, \\ \dot{x}_2 = A_2 x_2 + B_2 u_2. \end{cases}$$

- Let R_1, R_2 be positive-definite, s.t. $\det(R_1) = \det(R_2) = 1$
- For an arbitrary $\alpha > 0$ consider R^α:

$$u^\top R^\alpha u = \alpha u_1^\top R_1 u_1 + \frac{1}{\alpha} u_2^\top R_2 u_2,$$

- $u = (u_1, u_2)$ minimizes $u^\top R^\alpha u \iff u_1$ min. $u_1^\top R_1 u_1$ and u_2 min. $u_2^\top R_2 u_2$

  minimizing traj. $\gamma = (\gamma_1, \gamma_2)$ does not depend on α

$\Rightarrow$ For any $\alpha \neq \beta$, R^α and R^β are equivalent

Structure of non-injective case

Theorem

R admits an equivalent cost $\iff$ the reduced LQ problem admits product str.

Ideas of the proof:

- Pontryagin extremals of equivalent costs project to the same trajectories
- there exists a diffeomorphism between extremals of equivalent costs
- the diffeomorphism is a constant linear map
- its characteristic spaces determine decomposition in the reduced LQ pb

Injectivity criterion

Theorem

Reduced LQ problem admits product structure with N components $\iff$ in some coordinates A_+, A_- have respective block-diagonal forms with N blocks

$\Rightarrow$ injectivity condition reposes on matrices A_+, A_-

Corollary

The LQ problem on the canonical class of costs is injective $\iff$ there is no coordinates in which A_+, A_- have respective block-diagonal forms

Outline

Reconstruction of R

Injective case: R is the unique $R = R^\top > 0$ and $\det(R) = 1$ satisfying

$$X = \int_0^\infty e^{tA_+} B R^{-1} B^\top e^{t(A_+)^\top} dt$$

$$(A_- - A_+)\, X = B R^{-1} B^\top \quad \text{and} \quad A_+ X = -X A_-^\top$$

Non-injective case?

Non-injective case: $A_+, A_- \ \mapsto\ N$ respective blocks $(A_+^1, A_-^1), \dots, (A_+^N, A_-^N)$

Product structure of non-injective case

$$u^\top R u = u_1^\top R_1 u_1 + \cdots + u_N^\top R_N u_N$$

$$\begin{cases} \dot{x}_1 = A_+^1 x_1 + B_1 u_1 \\ \vdots \\ \dot{x}_N = A_+^N x_N + B_N u_N \end{cases}$$

$\Rightarrow$ the inverse problem on R can be reduced to N injective problems

Recover unique R_i from A_+^i, A_-^i

$$\text{Cost}: \quad L_i = u_i^\top R_i u_i$$

$$\text{Dynamics}: \quad \dot{x}_i = A_+^i x_i + B_i u_i$$

Canonical cost reconstruction

- Given data: (A, B) and Γ (set of traj.)
- Output: canonical cost $L_c(x,u) = (u + Kx)^\top R\,(u + Kx)$

Steps of the reconstruction:

1. recover unique A_+, A_- from Γ;
2. recover unique $K = K_+$ from $A_+ = A - BK_+$;
3. check from A_+, A_- if the inverse problem is injective
    - in the injective case: find R as the unique solution of alg. equations
    - in the non-injective case: the pb is reduced to several injective problems

Thank you for your attention!