Robust Bang-Bang Control through redundancy

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Dynamical Context

- ▷ Nonlinear control system in finite dimension

$$\dot{x}(t) = f(x(t), u(t))$$

with state $x \in \mathbb{R}^n$, control $u = (u_1, \cdots, u_m) \in \{0,1\}^m \subset \mathbb{R}^m$ and dynamics $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$.

Attitude control constraint

The attitude control system can only apply on/off thrusts, *another way of saying the controls have to be bang-bang*.

Initial trajectory

Solving an optimal control problem (minimum time, L^1-norm of control...) can yield bang-bang trajectories (while getting rid of possible singular control).

▷ Such trajectories have a minimal number of switching times

[KS89] Krener, Schättler. 1989.

- u a solution of optimal control problem $\dot{x}(t) = f(x(t), u(t))$ with $x(0) = x_0$ and $x(t_f) = x_f$
- When applied to corresponding real world system, modeling errors, perturbations...

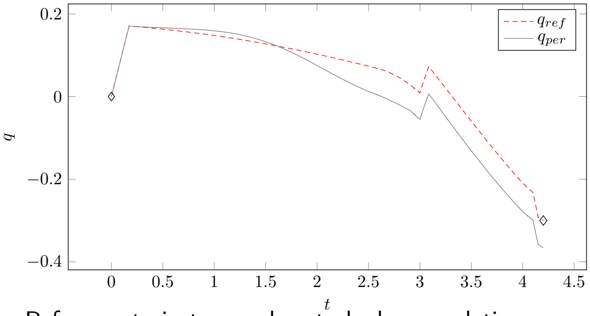

Figure – Reference trajectory and perturbed one ; relative error of 20 %

- Bang-bang control completely characterised by (initial value,) switching times and switching index : $((i_1, t_1), \ldots, (i_N, t_N))$.

Reduced End-Point Mapping

Let $u = (t_1, \ldots, t_N)$ be a bang-bang control. The reduced end-point mapping associated to the swicthing times is :

$$E(t_1, \ldots, t_N, t_f) = x_u(t_f)$$

where $x_u(\cdot)$ is solution to $\dot{x}_u(t) = f(x_u(t), u(t))$, with $x_u(0) = x_0$.

Aim at a perturbed target

Let $(\bar{t}_1, \ldots, \bar{t}_N)$ be a control s.t. $E(\bar{t}_1, \ldots, \bar{t}_N) = x_f$, and δx a perturbation. Find $(t_1, \cdots, t_N)$ s.t. $E(t_1, \ldots, t_N) = x_f + \delta x$.

- Let $\delta t = (t_1, \cdots, t_N) - (\bar{t}_1, \cdots, \bar{t}_N)$. Formally, we are looking for δt s.t. :

$$
\begin{aligned}
E((\bar{t}_1, \cdots, \bar{t}_N) + \delta t) &= x_f + \delta x \\
E(\bar{t}_1, \cdots, \bar{t}_N) + dE(\bar{t}_1, \cdots, \bar{t}_N) \cdot \delta t + o(\delta t) &= x_f + \delta x \\
dE(\bar{t}_1, \cdots, \bar{t}_N) \cdot \delta t &= \delta x \quad \text{(at order 1)}
\end{aligned}
$$

- Linear system with n equations and N unknowns.

Aim at a perturbed target

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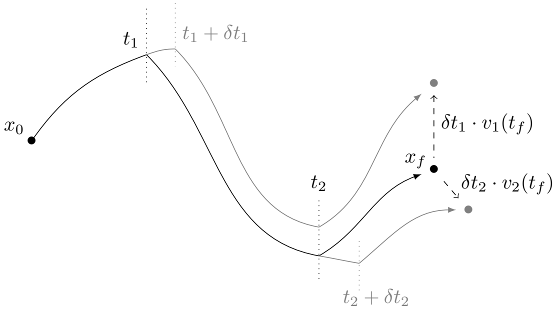
Proposition : End-point mapping differential

$E(\bar{t}_1+\delta t_1,\cdots,\bar{t}_N)=E(\bar{t}_1,\cdots,\bar{t}_N)+\delta t_1\cdot v_1(t_f)+o(\delta t_1)$, where v_1 is solution to the initial value problem

$$\dot{v}_1(t)=\frac{\partial f}{\partial x}(\bar{x}(t),\bar{u}(t))v_1(t)$$

+ a given initial condition at t_1.



Intermediary least-square problem

Let $(\bar{t}_1, \cdots, \bar{t}_N)$ be a control s.t. $E(\bar{t}_1, \cdots, \bar{t}_N) = x_f$, and δx a perturbation. Find $(t_1, \cdots, t_N)$ s.t. at the 1^{st} order $E(t_1, \cdots, t_N) = x_f + \delta x$

$$dE(\bar{t}_1, \cdots, \bar{t}_N) \cdot \delta t = \delta x,$$

or

$$\min_{\delta t \in \mathbb{R}^N} \|dE(\bar{t}_1, \cdots, \bar{t}_N) \cdot \delta t - \delta x\|^2.$$

Least-square solution - pseudo-inverse

The smallest norm solution is $\delta t = dE(\bar{t}_1, \cdots, \bar{t}_N)^\dagger \cdot \delta x$. With the estimate

$$\|\delta t\| \leq \|\delta x\| / \sigma_{min}$$

where σ_{min} is the smallest singular value of $dE(\bar{t}_1, \cdots, \bar{t}_N)$.

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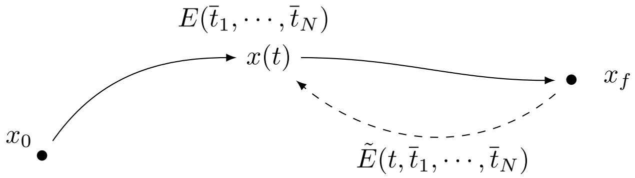
Retrograde end-point mapping

Let $u = (t_1, \cdots, t_N)$ be a bang-bang control and $t \in [0, t_f]$. The retrograde end-point mapping is defined as

$$\tilde{E}(t, t_1, \cdots, t_N) = \tilde{x}(t_f - t)$$

where $\tilde{x}(\cdot)$ solves the initial value problem

$$\dot{\tilde{x}}(t) = -f(\tilde{x}(t), u(t_f - t))$$
$$\tilde{x}(0) = x_f$$



- Let $t \in [0, t_f]$,and $\overline{x}(\cdot), (\overline{t}_1, \cdots, \overline{t}_N)$ be a reference trajectory. A perturbation $\delta x(t)$ is observed at time t.

Problem

How to change the switching times $(\overline{t}_1, \cdots, \overline{t}_N)$ so as to correct the trajectory and aim at the target x_f ?

With the previous notations, it writes as

$$\tilde{E}(t, (t_1, \cdots, t_N)) = \overline{x}(t) + \delta x(t)$$

That is $\quad \tilde{E}(t, (\overline{t}_1, \cdots, \overline{t}_N) + \delta t) = \overline{x}(t) + \delta x(t)$

So $\quad \delta t = d\tilde{E}(t, \overline{t}_1, \cdots, \overline{t}_N)^{\dagger} \cdot \delta x(t)$

With the norm estimate $\quad \|\delta t\| \leq \|\delta x(t)\| / \sigma_{min}(t)$

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Algorithm

Choose a subdivision $(\tau_1, \cdots, \tau_x)$ of $[0, t_f]$.

- ▷ Mesure the perturbation $\delta x(\tau_i)$.
- ▷ Apply the correction $\delta t = d\tilde{E}(\tau_i, \bar{t}_1, \cdots, \bar{t}_N)^\dagger \cdot \delta x(\tau_i)$, for which we have :

$$\|\delta t_i\| \leq \|\delta x(\tau_i)\|/\sigma_{min}(\tau_i)$$

- ▷ The singular values of $d\tilde{E}(\tau, \bar{t}_1, \cdots, \bar{t}_N)$ at time τ directly depend on the number of t_i s.t. $t_i > \tau$.
- ▷ Idea 1 : add switching times in order to make the control more robust...
- ▷ Idea 2 : so that the correction size δt is *controlled* (in particular does not change the switching times order)

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Robustness criterion

We define the cost to mesure the bang-bang trajectory as

$$C_r(u) = \int_0^{t_N} \frac{1}{\sigma_{min}(t)^2}\, dt$$

Robustification problem

Starting from a control $u = (t_1, \ldots, t_N)$, for instance solution of a minimum L^1-norm, add switching times $(s_1, \ldots, s_\ell)$ solution to the optimization problem :

$$\min_{s.t.\ E(t_1,\cdots,t_N,s_1,\cdots,s_\ell)=x_f} C_r(t_1,\cdots,t_N,s_1,\cdots,s_\ell) + \int_0^{t_f} \|u\|_1\, dt$$

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▷ Application to the equations of the angular velocity $\vec{\omega} = (p, q, r)$.

Simplified equations for the angular velocities

$$\dot{p}(t) = a_1 q(t) r(t) + \sum_{k=1}^{m} b_1^k u_k(t)$$

$$\dot{q}(t) = a_2 p(t) r(t) + \sum_{k=1}^{m} b_2^k u_k(t)$$

$$\dot{r}(t) = a_3 p(t) q(t) + \sum_{k=1}^{m} b_3^k u_k(t)$$

▷ Perturbations introduced with $a^{\varepsilon}_{1,2,3}(t) = a_{1,2,3} + \varepsilon h_i(t)$ with periodic $\|h_i\|_\infty = 1$.

▷ Exemple for 4 thrusters, i.e., 4 controls.

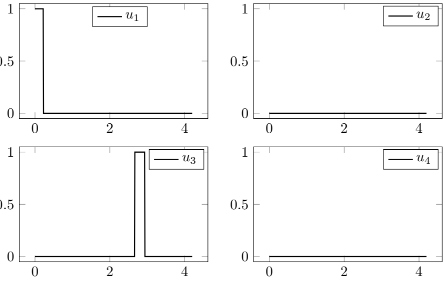

Figure – Initial Control

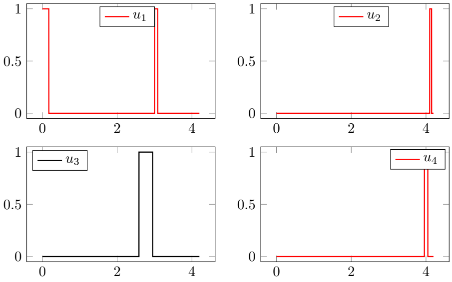

Figure – Robustified Control

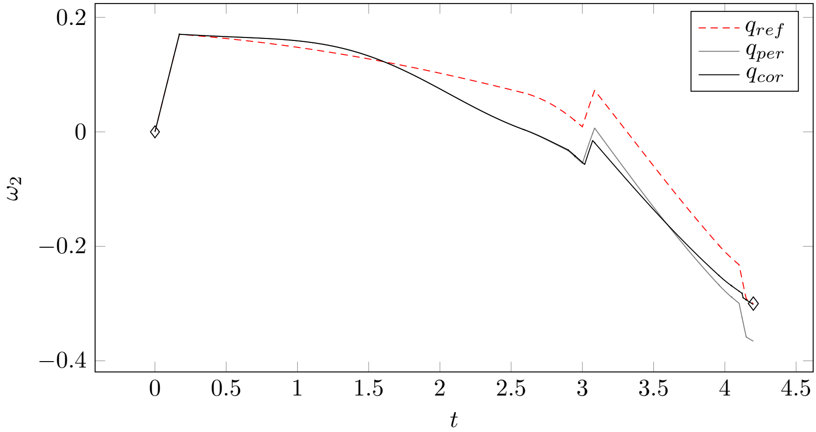

Figure – Corrected Trajectory. Relative error : $\|x_{per} - x_f\|/\|x_f\| = 1.3 \times 10^{-1}$, $\|x_{cor} - x_f\|/\|x_f\| = 5.5 \times 10^{-3}$.

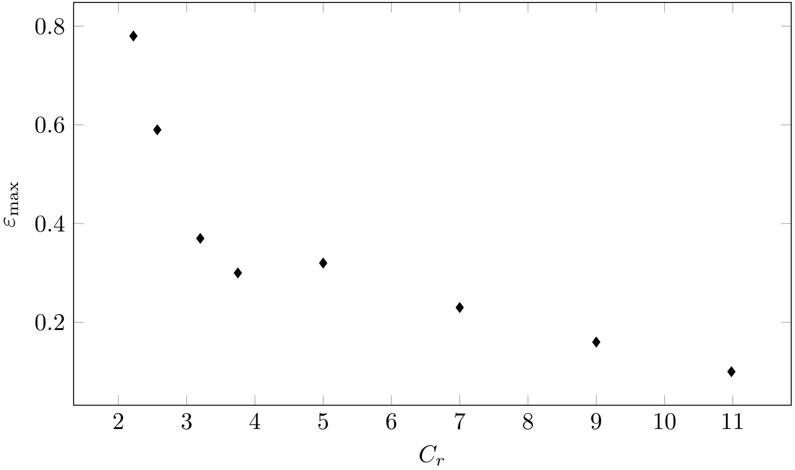

Figure – Maximum absorbed perturbation ε w.r.t. robustness criterion.

▷ Tracking fails = correction breaks original switching times order

Conclusion

Redundancy made off-line to maximize criterion for robustness.

Feedback control stays bang-bang.

Not yet adapted to large system.

Prospect

Need to *guess* the number of swicthing times to add.

Could be extended to multi (more than 2) valued controls.

[A018] Éric Bourgeois, Thomas Haberkorn, David-Alexis Handschuh, A. 0livier and Emmanuel Trélat *Redundancy implies robustness for bang-bang control strategy. Optimal Control Appl. Methods. Sept. 2018*

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Thank you for your attention.