Pareto front characterization for finite horizon optimal control problems with two different objectives

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Pareto Optimality

Pareto optimality

$$\begin{array}{ll} \text{minimize} & g(x) = (g_1(x), g_2(x)) \\ \text{suject to} & x \in X \end{array}$$

- $x^* \in X$ is a **Pareto solution** if there exists no $x \in X$ such that $g(x) \neq g(x^*)$ and

$$g_i(x) \leq g_i(x^*), \quad \text{for all } i = 1, 2.$$

- $x^* \in X$ is a **weak Pareto solution** if there exists no $x \in X$ such that

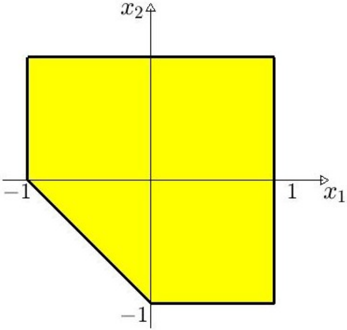
$$g_i(x) < g_i(x^*), \quad \text{for all } i = 1, 2.$$

- Set of Pareto and weak Pareto solutions: P and P_w
- Pareto and weak Pareto front:

$$\mathcal{F} = \{g(x) \mid x \in P\}, \qquad \mathcal{F}_w = \{g(x) \mid x \in P_w\}$$

Example: Pareto optimality

$$\text{Minimize} \quad g(x) = (x_1, x_2)$$



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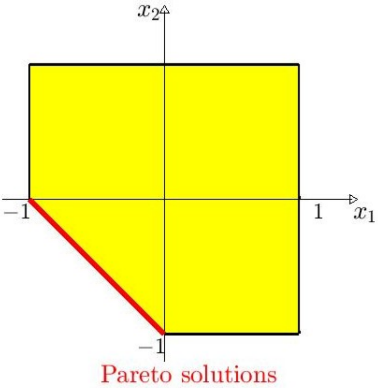

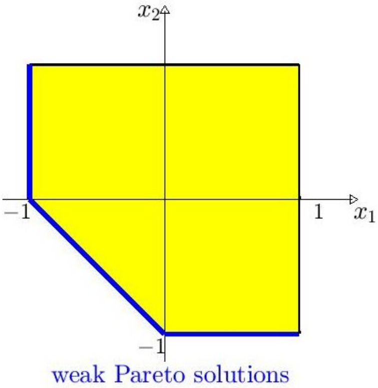

ε-Pareto optimality

Let $\varepsilon \geq 0$. We define the following sets of ε-Pareto solutions:

- $x \in \mathcal{P}^{1,\varepsilon}$ if there is no $y \in X$ such that

$$g(y) \neq g(x) - \varepsilon 1 \text{ and } g_i(y) \leq g_i(x) - \varepsilon, \quad \text{for all } i = 1, 2.$$

- $x \in \mathcal{P}^{2,\varepsilon}$ if there is $x^* \in \mathcal{P}$ such that $|g(x) - g(x^*)| \leq \varepsilon$.
- $x \in \mathcal{P}^{3,\varepsilon}$ if for some $y \in X$, $g(y) \neq g(x)$ and $g(y) \leq g(x)$, then

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Proposition (D.J. White, 1986)

If the feasible set X is a compact set and g is a continuous function, then $\mathcal{P} \subset \mathcal{P}^{3,\varepsilon} \subseteq \mathcal{P}^{2,\varepsilon} \subseteq \mathcal{P}^{1,\varepsilon}$.

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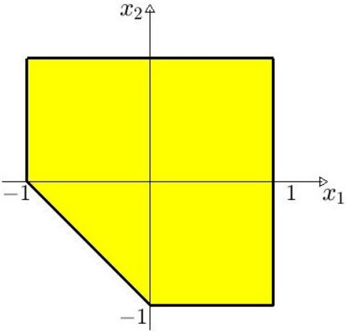

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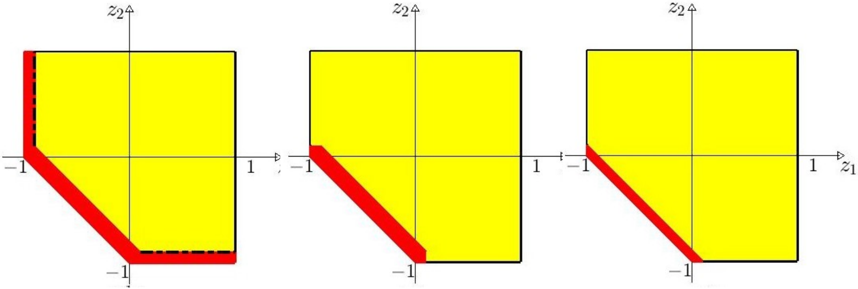

(a) $\mathcal{F}^{(1,\varepsilon)}$ (b) $\mathcal{F}^{(2,\varepsilon)}$ (c) $\mathcal{F}^{(3,\varepsilon)}$

Figure: ε-Pareto fronts for $\varepsilon = 0.1$

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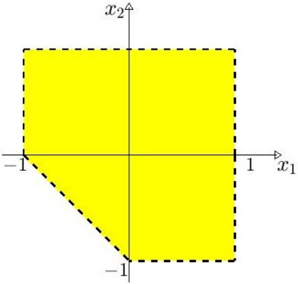

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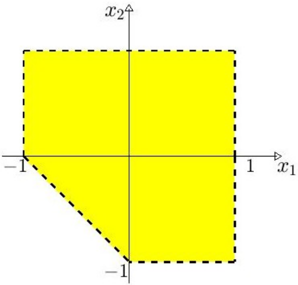

- For this example $\mathcal{P}$ and $\mathcal{P}_w$ are empty, so are $\mathcal{F}$ and $\mathcal{F}_w$ also empty.

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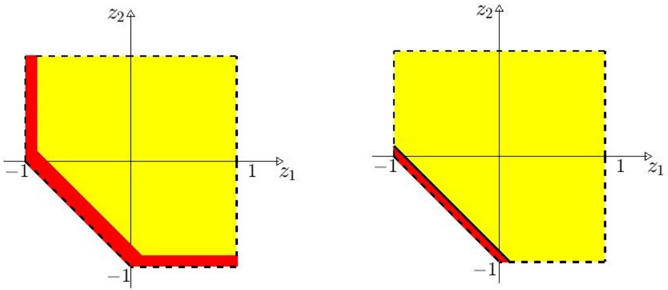

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Figure: ε-Pareto fronts for $\varepsilon = 0.1$

Consider the problem of minimize the objective functions over the closure of the feasible set X

$$\begin{cases} \text{Minimize} \;\; g(x) = (g_1(x), g_2(x)) \\ \text{subject to } x \in \overline{X} \end{cases}$$

- Denote the Pareto set and the weak Pareto set by $\mathcal{P}^{\#}$ and $\mathcal{P}_w^{\#}$, respectively.

Theorem [APC and Zidani, 2018]

Assume that f_i are Lipschitz continuous functions, with Lipschitz constant $L_i, \; i = 1, 2$.

(i) $\forall\, x^* \in \mathcal{P}^{\#}$, $\forall$ *varepsilon* > 0, $\exists\, x \in \mathcal{P}^{1,\varepsilon}$ such that

$$|x^* - x| \min_i(\varepsilon / L_i) \leq \quad \text{and} \quad |g(x^*) - g(x)| \leq \varepsilon.$$

(ii) $\forall\, x^* \in \mathcal{P}^{\#}_w$, $\forall$ *varepsilon* > 0, $\exists\, x \in \mathcal{P}^{1,\varepsilon}_w$ such that

$$|x^* - x| \leq \min_i(\varepsilon / L_i) \quad \text{and} \quad |g(x^*) - g(x)| \leq \varepsilon.$$

(iii) $\forall\, \varepsilon > 0$, $\forall\, x \in \mathcal{P}^{3,\varepsilon}$ there exists $x^* \in \mathcal{P}^{\#}$ such that

$$|g(x) - g(x^*)| \leq 2\varepsilon.$$

Bi-objective Optimal Control Problem

 

Figure Source: *http://www.esalq.usp.br and http://www.acupunctureinwestchase.com*

- **State variables:** population of a nuisance for humans and its predator.
- **Control Variable:** spraying chemical to poison the pest or not.
- **Dynamics:** (intertwined) growth of both populations.
- **Goal:** reduce the cost of spraying the chemical. (integral cost)

An illustrative example

 

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- **State variables:** population of a nuisance for humans and its predator.
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- **Goal:** reduce the cost of spraying the chemical. (integral cost)
- **Goal 2:** keep nuisance expansion under control by minimizing the maximum difference of certain proportion of both species.
  Objectives of different nature

Pest control example - mathematical formulation

$$
\begin{cases}
\inf & \left(\int_0^T 0.3\mathbf{u}(s)ds, \max\limits_{0 \le s \le T} 0.25(\mathbf{y}_1(s) - 0.7\mathbf{y}_2(s))^2 \right), \\
\\
\text{s.t.} & \dot{\mathbf{y}}_1(s) = \mathbf{y}_1(s) - \mathbf{y}_1(s)\mathbf{y}_2(s) - 0.4\mathbf{y}_1(s)\mathbf{u}(s), \\
& \dot{\mathbf{y}}_2(s) = -\mathbf{y}_2(s) + \mathbf{y}_1(s)\mathbf{y}_2(s) - 0.2\mathbf{y}_2(s)\mathbf{u}(s), \\
& \mathbf{y}(0) = x, \\
& \mathbf{u}(s) \in \{0, 1\}
\end{cases}
$$

Problem statement

$$\begin{cases} \inf & \left(\int_t^T \ell(\mathbf{y}(s), \mathbf{u}(s)) ds + \varphi(\mathbf{y}(T)), \max_{0 \le s \le T} 0.25(\mathbf{y}_1(s) - 0.7\mathbf{y}_2(s))^2 \right), \\ \text{s.t.} & \dot{\mathbf{y}}_1(s) = \mathbf{y}_1(s) - \mathbf{y}_1(s)\mathbf{y}_2(s) - 0.4\mathbf{y}_1(s)\mathbf{u}(s), \\ & \dot{\mathbf{y}}_2(s) = -\mathbf{y}_2(s) + \mathbf{y}_1(s)\mathbf{y}_2(s) - 0.2\mathbf{y}_2(s)\mathbf{u}(s), \\ & \mathbf{y}(0) = x, \\ & \mathbf{u}(s) \in \{0, 1\} \end{cases}$$

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$$\begin{cases} \inf & \left(\int_t^T \ell(\mathbf{y}(s), \mathbf{u}(s))ds + \varphi(\mathbf{y}(T)), \max\limits_{s\in[t,T]} \psi(\mathbf{y}(s)) \right), \\ \\ \text{s. t.} & \dot{\mathbf{y}}_1(s) = \mathbf{y}_1(s) - \mathbf{y}_1(s)\mathbf{y}_2(s) - 0.4\mathbf{y}_1(s)\mathbf{u}(s), \\ & \dot{\mathbf{y}}_2(s) = -\mathbf{y}_2(s) + \mathbf{y}_1(s)\mathbf{y}_2(s) - 0.2\mathbf{y}_2(s)\mathbf{u}(s), \\ & \mathbf{y}(0) = x, \\ & \mathbf{u}(s) \in \{0, 1\} \end{cases}$$

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\text{s. t.} & \dot{\mathbf{y}}(s) = f(\mathbf{y}(s), \mathbf{u}(s)), \qquad \text{a.e. } s \in [0, T], \\
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(MOP)\begin{cases}
\inf & \left(\displaystyle\int_t^T \ell(\mathbf{y}(s),\mathbf{u}(s))ds + \varphi(\mathbf{y}(T)),\ \max_{s\in[t,T]} \psi(\mathbf{y}(s))\right), \\
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 & \mathbf{y}(t) = x, \\
 & \mathbf{u} \in \mathcal{U}
\end{cases}
$$

where

- $\mathcal{U} = \left\{ \mathbf{u} : [0, +\infty) \to \mathbb{R}^M \text{ measurable, } \mathbf{u}(s) \in U \text{ a.e.} \right\}$

U is a compact subset of $\mathbb{R}^M$, $\ell : \mathbb{R}^N \times U \to \mathbb{R}$, $\varphi : \mathbb{R}^N \to \mathbb{R}$, $\psi : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R}^N \times U \mapsto \mathbb{R}^N$ are Lipschitz continuous and bounded functions.

An equivalent bi-objective problem

Define:

- $\mathcal{A}(x,u) = M_\ell - \ell(x,u)$ and the set-valued function

$$G(x) = \left\{ \begin{pmatrix} f(x,u) \\ -\ell(x,u) - a \end{pmatrix},\ u \in U,\ 0 \leq a \leq \mathcal{A}(x,u) \right\}.$$

- set of trajectories:

$$\mathcal{S}_{[t,T]}(x,0) = \{(\mathbf{y},\mathbf{z}) :\ (\dot{\mathbf{y}}(s),\dot{\mathbf{z}}(s))^\intercal \in G(\mathbf{y}(s)),\ \text{for a.e. } s \in [t,T];$$
$$(\mathbf{y}(t),\mathbf{z}(t)) = (x,0)\}.$$

- bi-objective optimal control problem:

$$\begin{cases} \inf \left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s\in[t,T]} \psi(\mathbf{y}(s))\right) \\ \text{s.t } (\mathbf{y},\mathbf{z}) \in \mathcal{S}_{[t,T]}(x,0). \end{cases} \tag{1}$$

An equivalent bi-objective problem

- For any $(\mathbf{y}^*, \mathbf{z}^*)$ (weak) that is a Pareto optimal solution of (1) and $\mathbf{u}^*$, γ^* the respective controls. The pair $(\mathbf{y}^*, \mathbf{u}^*)$ is a (weak) Pareto optimal solution for problem (MOP).
- Moreover the (weak) Pareto front of two problems coincides.

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- Moreover the (weak) Pareto front of two problems coincides.

Without any additional assumption it is not possible to guarantee that the set of trajectories $\mathcal{S}_{[t,T]}(x, 0)$ is compact.

Relaxed problem

- Consider

$$\mathcal{S}^{\#}_{[t,T]}(x,0) = \{(\mathbf{y},\mathbf{z}) : \ (\dot{\mathbf{y}}(s),\dot{\mathbf{z}}(s))^{\intercal} \in \overline{co}\Big(G(\mathbf{y}(s))\Big), \text{ for a.e. } s \in [t,T];$$
$$(\mathbf{y}(t),\mathbf{z}(t)) = (x,z)\},$$

  where $\overline{co}(S)$ denotes the closed convex hull of the subset S, that is the minimal convex set that contains S.

- $\mathcal{S}^{\#}_{[t,T]}(x,0)$ is compact and equal the closure of $\mathcal{S}_{[t,T]}(x,z)$ in the space of continuous functions $C(t,T)$.
  [Frankowska and Rampazzo, 1999]

So we introduce the following convexified bi-objective optimal control problem

$$\begin{cases} \min \Big(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max\limits_{s\in[t,T]} \psi(\mathbf{y}(s))\Big) \\ \text{s.t } (\mathbf{y},\mathbf{z}) \in \mathcal{S}^{\#}_{[t,T]}(x,0). \end{cases} \tag{MORP}$$

Theorem [APC and Zidani, 2018]

Let $(t, x) \in [0, T] \times \mathbb{R}^N$.

(i) $\forall\ (\mathbf{y}^*, \mathbf{z}^*) \in \mathcal{P}^{\#}(t, x)$, $\forall\ \varepsilon > 0$, $\exists\ (\mathbf{y}, \mathbf{z}) \in \mathcal{P}^{1,\varepsilon}(t, x)$ such that

$$|(\mathbf{y}^*, \mathbf{z}^*) - (\mathbf{y}, \mathbf{z})| \leq \min\left(\frac{\varepsilon}{L_\varphi}, \frac{\varepsilon}{L_\psi}\right) \quad \text{and}$$

$$\left|\left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s\in[t,T]} \psi(\mathbf{y}^*(s))\right) - \left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s\in[t,T]} \psi(\mathbf{y}(s))\right)\right| \leq \varepsilon.$$

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(ii) $\forall\ (\mathbf{y}^*, \mathbf{z}^*) \in \mathcal{P}^{\#}_w(t,x)$, $\forall\ \varepsilon > 0$, $\exists\ (\mathbf{y}, \mathbf{z}) \in \mathcal{P}^{1,\varepsilon}_w(t,x)$ such that

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(iii) $\forall\ \varepsilon > 0$, $\forall\ (\mathbf{y}^*, \mathbf{z}^*) \in \mathcal{P}^{3,\varepsilon}(t, x)$, $\exists (\mathbf{y}, \mathbf{z}) \in \mathcal{P}^{\#}(t, x)$ such that

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Auxiliary control problem

$$w(t, x, z_1, z_2) = \min_{(\mathbf{y},\mathbf{z}) \in \mathcal{S}^{\#}_{[t,T]}(x,0)} \left[\Big(\varphi(\mathbf{y}(T)) - \mathbf{z}(T) \Big) \bigvee \max_{s \in [t,T]} (\psi(\mathbf{y}(s)) - z_2) \right]$$

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HJB equation

The function w is the unique viscosity solution of the following HJB equation:

$$\min\Big(\partial_t w(t,x,z) + \mathcal{H}^{\#}(x, D_x w, D_z w), w(t,x,z) - (g(x) - z_2)\Big) = 0,$$
$$\forall t \in [0,T),\ x \in \mathbb{R}^n,\ z \in \mathbb{R}^2$$
$$w(T,x,z) = \Big(\varphi(x) - z_1\Big) \bigvee \Big(g(x) - z_2\Big) \quad \forall\ x \in \mathbb{R}^n,\ z \in \mathbb{R}^2,$$

where the Hamiltonian $H^{\#}$ is given by

$$\mathcal{H}^{\#}(x,p,q) = \max_{(v_x,v_z)\in\overline{co}(G(x))} \Big(-v_x\cdot p - v_z\cdot q_1\Big).$$

Characterization of the Pareto front

Characterization of the weak Pareto front

Consider:

- $$z_1^*(t,x) = \inf\left\{\zeta \in \mathbb{R} \,\middle|\, \exists z \in \mathbb{R}^2 \; w(t,x,\zeta,z_2) \leq 0\right\}$$
$$= \min_{(\mathbf{y},\mathbf{z}) \in \mathcal{S}^{\#}_{[t,T]}(x,0)} \varphi(\mathbf{y}(T)) - \mathbf{z}(T) = \varphi(\mathbf{y}_1(T)) - \mathbf{z}_1(T);$$

- $$z_2^*(t,x) = \inf\left\{\zeta \in \mathbb{R} \,\middle|\, \exists z \in \mathbb{R}^2 \; w(t,x,z_1,\zeta) \leq 0\right\}$$
$$= \min_{(\mathbf{y},\mathbf{z}) \in \mathcal{S}^{\#}_{[t,T]}(x,0)} \max_{s \in [t,T]} \psi(\mathbf{y}(s)) = \max_{s \in [t,T]} \psi(\mathbf{y}_2(s));$$

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- $$\overline{z_1}(t,x) = \inf\left\{\zeta \in \mathbb{R} \,\middle|\, w(t,x,\zeta,z_2^*(t,x)) = 0\right\} = \varphi(\mathbf{y}_2(T)) - \mathbf{z}_2(T);$$

- $$\overline{z_2}(t,x) = \inf\left\{\zeta \in \mathbb{R} \,\middle|\, w(t,x,z_1^*(t,x),\zeta) = 0\right\} = \max_{s\in[t,T]} \psi(\mathbf{y}_1(s));$$

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- $$z_2^*(t,x) = \inf\left\{\zeta \in \mathbb{R} \;\middle|\; \exists z \in \mathbb{R}^2 \;\; w(t,x,z_1,\zeta) \leq 0\right\}$$
  $$= \min_{(\mathbf{y},\mathbf{z}) \in \mathcal{S}^{\#}_{[t,T]}(x,0)} \max_{s \in [t,T]} \psi(\mathbf{y}(s)) = \max_{s \in [t,T]} \psi(\mathbf{y}_2(s));$$

- $$\overline{z_1}(t,x) = \inf\left\{\zeta \in \mathbb{R} \;\middle|\; w(t,x,\zeta,z_2^*(t,x)) = 0\right\} = \varphi(\mathbf{y}_2(T)) - \mathbf{z}_2(T);$$

- $$\overline{z_2}(t,x) = \inf\left\{\zeta \in \mathbb{R} \;\middle|\; w(t,x,z_1^*(t,x),\zeta) = 0\right\} = \max_{s \in [t,T]} \psi(\mathbf{y}_1(s));$$

Assume that the ideal vector $\beta^*(t,x) = (z_1^*(t,x), z_2^*(t,x)) \in \mathbb{R}^2$ is not feasible, otherwise the Pareto front is reduced to this point.

Characterization of the weak Pareto front

Define:

- $\Omega = [z_1^*(t,x), \overline{z_1}(t,x)] \times [z_2^*(t,x), \overline{z_2}(t,x)]$.

Theorem (APC and Zidani,2018)

Let (t,x) be in $[0,T] \times \mathbb{R}^N$

(i) $\mathcal{F}^{\#}(t,x) \subset \mathcal{F}_w^{\#}(t,x) \cap \Omega \subset \left\{ z \in \Omega \mid w(t,x,z) = 0 \right\}$.

(ii) *Let $z \in \Omega$ such that $w(t,x,z) = 0$. If there exists a admissible pair $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}^{\#}_{[t,T]}(x,0)$ such that $\varphi(\mathbf{y}(T)) - \mathbf{z}(T) = z_1$ and $\max\limits_{s \in [t,T]} \psi(\mathbf{y}(s)) = z_2$, then $z \in \mathcal{F}_w^{\#}(t,x)$.*

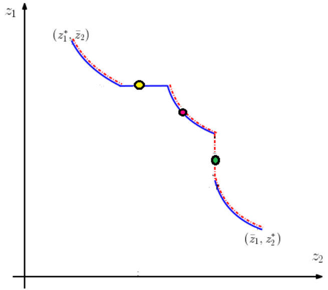
Characterization of the Pareto front

Introduce the projectors operators on $\mathbb{R}^2$:

$$\pi_1(z) = z_1,\ \pi_2(z) = z_2.$$

and the following functions

$$\eta_1 : [z_1^*, \overline{z_1}] \to [z_2^*, \overline{z_2}],\quad \eta_1(\gamma_1) := \inf\{\theta|\ w(t, x, \gamma_1, \theta) \leq 0\},$$
$$\eta_2 : [z_2^*, \overline{z_2}] \to [z_1^*, \overline{z_1}],\quad \eta_2(\gamma_2) := \inf\{\theta|\ w(t, x, \theta, \gamma_2) \leq 0\}.$$



Theorem [APC and Zidani,2018]

Let (t,x) (t,x) be in $[0, T] \times \mathbb{R}^N$.

(ii) $\mathcal{F}^{\#}(t,x) = \Big\{(\zeta, \eta_1(\zeta)),\ \zeta \in \mathrm{dom}(\eta_1)\Big\} \bigcap \Big\{(\eta_2(\zeta), \zeta),\ \zeta \in \mathrm{dom}(\eta_2)\Big\}.$

Theorem [APC and Zidani,2018]

Let (t, x) (t, x) be in $[0, T] \times \mathbb{R}^N$.

(ii) $\mathcal{F}^{\#}(t, x) = \Big\{(\zeta, \eta_1(\zeta)),\ \zeta \in \mathrm{dom}(\eta_1)\Big\} \bigcap \Big\{(\eta_2(\zeta), \zeta),\ \zeta \in \mathrm{dom}(\eta_2)\Big\}.$

(iii) For any $z \in \mathcal{F}^{\#}(t, x)$ let a trajectory $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}^{\#}_{[t,T]}(x, 0)$ that is optimal for the auxiliary problem. Then $(\mathbf{y}, \mathbf{z})$ is a Pareto optimal solution of (MORP).

Theorem [APC and Zidani,2018]

Let (t, x) be in $[0, T] \times \mathbb{R}^N$ and $\varepsilon > 0$.

(i) $\mathcal{F}^{1,\varepsilon}(t,x) \subset \mathcal{F}_w^{1,\varepsilon}(t,x) \subset \left\{ z \in \mathbb{R}^2 \;\middle|\; -\varepsilon \leq \; w(t,x,z) \leq 0 \right\}.$

Theorem [APC and Zidani,2018]

Let (t, x) be in $[0, T] \times \mathbb{R}^N$ and $\varepsilon > 0$.

(i) $\mathcal{F}^{1,\varepsilon}(t,x) \subset \mathcal{F}_w^{1,\varepsilon}(t,x) \subset \left\{ z \in \mathbb{R}^2 \;\middle|\; -\varepsilon \leq \; w(t,x,z) \leq 0 \right\}.$

(ii) Let $z_\varepsilon \in \left\{ z \in \mathbb{R}^2 \;\middle|\; -\varepsilon \leq \; w(t,x,z) \leq 0 \right\}$. If there exists $(\mathbf{y}_\varepsilon, \mathbf{z}_\varepsilon) \in \mathcal{S}_{[t,T]}(x, 0)$ that is optimal for the auxiliary control problem w. Then $(\mathbf{y}_\varepsilon, \mathbf{z}_\varepsilon) \in \mathcal{P}_w^{1,\varepsilon}(t,x)$ of problem (1).

Pest control example

$$\begin{cases} \inf & \left(\int_0^T 0.3\mathbf{u}(s)ds, \max_{0 \le s \le T} 0.25(\mathbf{y}_1(s) - 0.7\mathbf{y}_2(s))^2 \right), \\ \\ \text{s.t.} & \dot{\mathbf{y}}_1(s) = \mathbf{y}_1(s) - \mathbf{y}_1(s)\mathbf{y}_2(s) - 0.4\mathbf{y}_1(s)\mathbf{u}(s), \\ & \dot{\mathbf{y}}_2(s) = -\mathbf{y}_2(s) + \mathbf{y}_1(s)\mathbf{y}_2(s) - 0.2\mathbf{y}_2(s)\mathbf{u}(s), \\ & \mathbf{y}(0) = (0.7, 0.2), \\ & \mathbf{u}(s) \in \{0, 1\} \end{cases}$$

In order to obtain a relaxed problem we are going to consider $\mathbf{u}(s) \in [0, 1]$. The corresponding value function w is solution of the following HJB equation

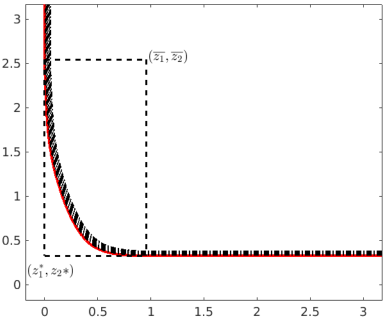
$$\min \Big(\partial_t w(t,x,z) + \mathcal{H}(x, D_x w, D_z w), w(t,x,z) - (0.25(x_1 - 0.7x_2)^2 - z_2) \Big) = 0,$$

$$\text{for } t \in [0, T), x, z \in \mathbb{R}^2$$

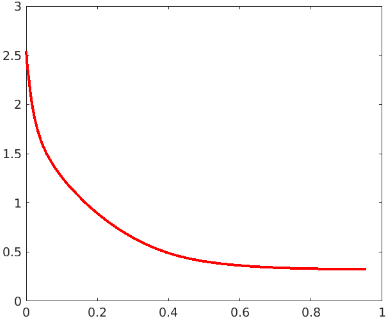
$$w(T,x,z) = -z_1 \bigvee (0.25(x_1 - 0.7x_2)^2 - z_2) \qquad \text{for } x \in \mathbb{R}^2, z \in \mathbb{R}^2.$$

- The HJB equation was solved by a finite difference method implement at C++ HJB-solver "ROC-HJ", available in *http://uma.ensta-paristech.fr/soft/ROC-HJ*
- Grid of 75^4 nodes on the domain $[0, 3.5] \times [0, 3.5] \times [0, 3] \times [0, 3]$.

Pest control example

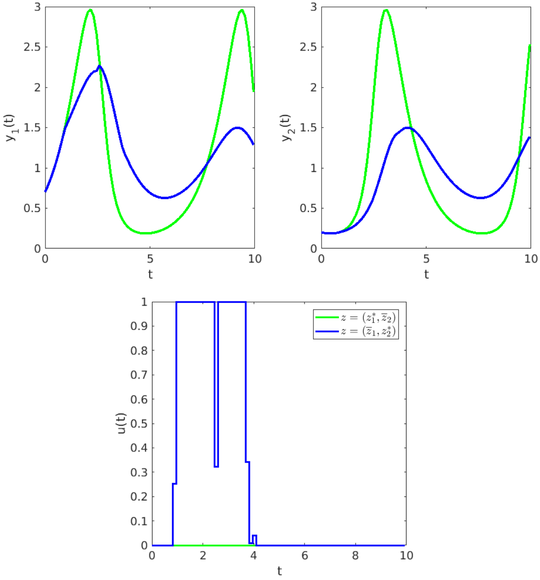


(a) Set that contains $\mathcal{F}_w^{1,\varepsilon}(t, x_0)$ and $\mathcal{F}_w^{\#}(t, x_0)$

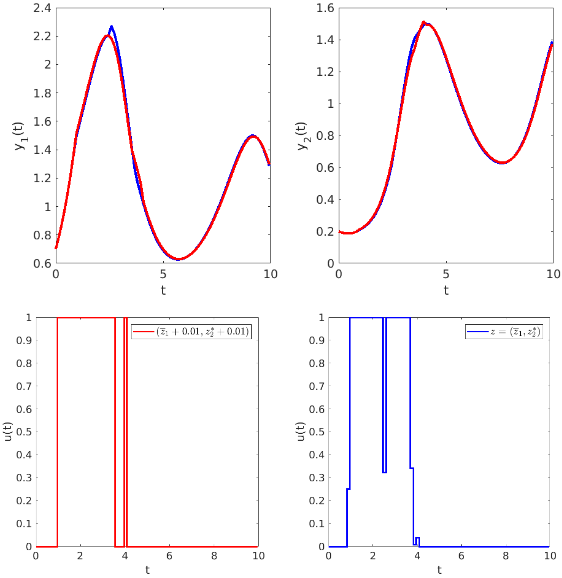


(b) zoom of $\mathcal{F}^{\#}(t, x_0)$

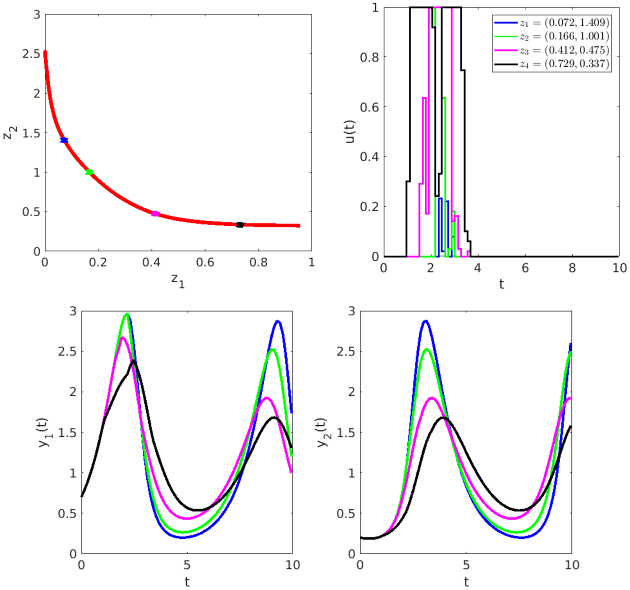
Optimal trajectories for single objective problems



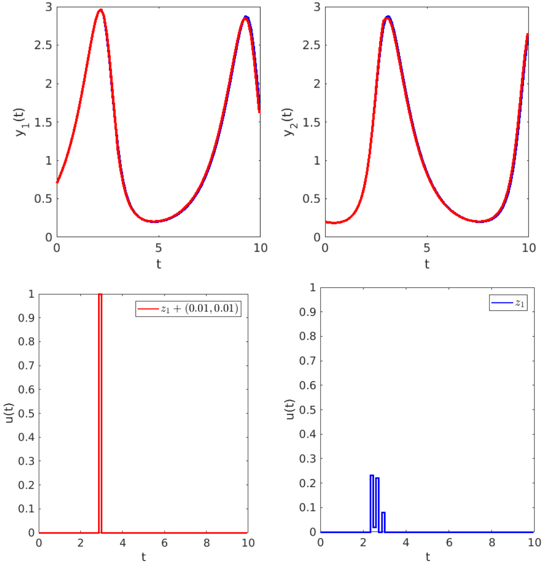
ε-Pareto Optimal trajectories for the original problem



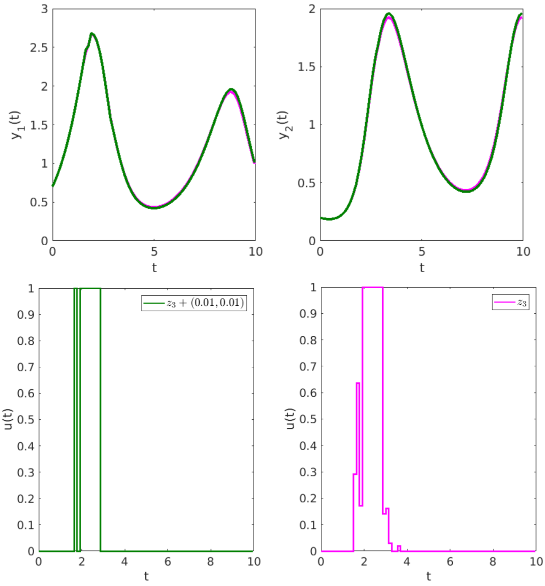
Pareto Optimal trajectories for the relaxed problem



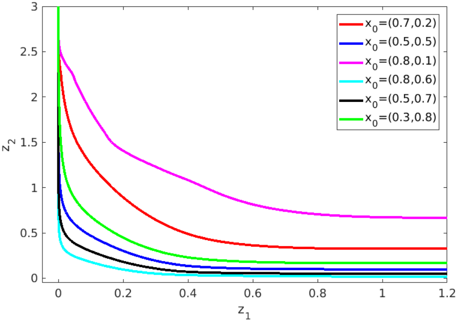
ε-Pareto Optimal trajectories for the original problem



ε-Pareto Optimal trajectories for the original problem



Different initial states



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Thank you for your attention