Shortest Dubins Paths through Three Points

Zheng Chen and **Tal Shima**

DEPARTMENT OF AEROSPACE ENGINEERING
TECHNION
Israel Institute of Technology

Faculty of Aerospace Engineering

Technion - Israel Institute of Technology

Dubins vehicle and its kinematics

The Dubins vehicle moves only forward at a constant speed with a minimum turning radius.

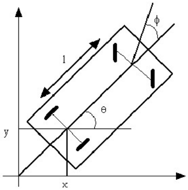
1. normalize the speed to one;
2. consider the minimum turning radius to be $r \in \mathbb{R}_+$.

The state $\boldsymbol{x} := (x, y, \theta) \in \mathbb{R}^2 \times \mathbb{S} =: \mathscr{X}$, also called configuration, consists of a position vector $(x, y) \in \mathbb{R}^2$ and a heading orientation angle $\theta \in \mathbb{S}$.

The kinematics is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x(t) \\ y(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} \cos\theta(t) \\ \sin\theta(t) \\ u(t)/r \end{pmatrix}$$

where $u \in [-1, 1]$ denotes the control.



Dubins car

Applications

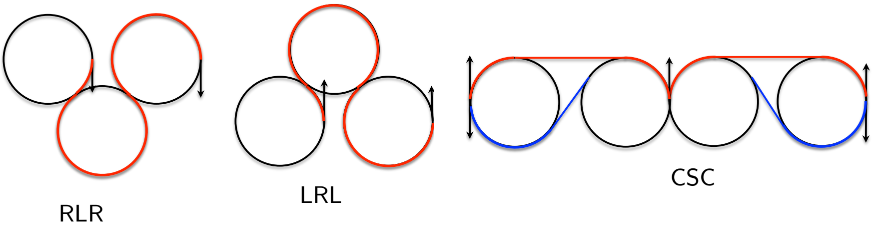
Many nonholonomic vehicles can be modelled by the Dubins vehicle, such as fixed-wing unmanned aerial vehicles, unmanned ground vehicles, ships, etc.

Shortest Dubins path between two configurations

The shortest Dubins path between two configurations belongs to six types in two families [*]:

- CCC = {RLR, LRL}, and
- CSC = {RSR, RSL, LSL, LSR}

where R (resp. L) denotes the corresponding circular arc with a right (resp. L) turning direction.



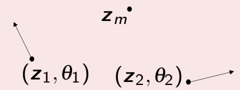
[*] Dubins, L.E. (July 1957). "On Curves of Minimal Length with a Constraint on Average Curvature, and with Prescribed Initial and Terminal Positions and Tangents". American Journal of Mathematics. 79 (3): 497–516.

3-Point Dubins Problem (3PDP)

3PDP

Given three different points $\boldsymbol{z}_1$, $\boldsymbol{z}_m$, and $\boldsymbol{z}_2$ in $\mathbb{R}^2$, let θ_1 and θ_2 in $[0,2\pi)$ be the fixed heading orientation angles at $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$, respectively. Then, the 3PDP consists of steering (Σ) by $u(\cdot) \in [-1,1]$ on $[0,t_f]$ from $(\boldsymbol{z}_1,\theta_1)$, pathing through $\boldsymbol{z}_m$ at $t_m \in (0,t_f)$, to $(\boldsymbol{z}_2,\theta_2)$ such that the final time $t_f > 0$ is minimized.

As the speed of the Dubins vehicle is a constant, solving the 3PDP is equivalent to finding the shortest path.



Given any two configurations $(\boldsymbol{y}_1,\eta_1)$ and $(\boldsymbol{y}_2,\eta_2)$ in $\mathscr{X}$, denote by

$$F : \mathscr{X}^2 \to \mathbb{R},\ [(\boldsymbol{y}_1,\eta_1),(\boldsymbol{y}_2,\eta_2)] \mapsto F[(\boldsymbol{y}_1,\eta_1),(\boldsymbol{y}_2,\eta_2)]$$

the length of the shortest Dubins path between them. As the heading orientation angles before and after $\boldsymbol{z}_m$ are the same along the shortest path, we denote such a heading orientation angle by θ_m, i.e.,

$$\theta_m := \underset{\theta \in [0,2\pi)}{\operatorname{argmin}}\ F[(\boldsymbol{z}_1,\theta_1),(\boldsymbol{z}_m,\theta)] + F[(\boldsymbol{z}_m,\theta),(\boldsymbol{z}_2,\theta_2)].$$

Shortest path for 3PDP

According to Bellman's principle for optimality, the solution of 3PDP is the concatenation of the shortest Dubins paths between $(\mathbf{z}_1, \theta_1)$ and $(\mathbf{z}_m, \theta_m)$ and between $(\mathbf{z}_m, \theta_m)$ and $(\mathbf{z}_2, \theta_2)$. Hence, the solution path of 3PDP belongs to four families:

$$CCC|CCC, \; CCC|CSC, \; CSC|CCC, \; CSC|CSC,$$

where the notation "|" denotes $\mathbf{z}_m$.

Up to $6 \times 6 = 36$ possibilities.

- Once θ_m is known, one needs to check 36 possibilities in order to solve the 3PDP.
- How to reduce the number of possibilities?
- How to compute θ_m?

Characterization of the solution for 3PDP

1. Necessary conditions
2. Geometric properties
3. Common formula

Pontryagin Maximum Principle (PMP)

Denote by $\boldsymbol{p} = [p_x, p_y, p_\theta] \in T^*_{\boldsymbol{x}}\mathscr{X}$ the costate of $\boldsymbol{x} = [x, y, \theta] \in \mathscr{X}$. The Hamiltonian is $H(\boldsymbol{x}, \boldsymbol{p}, u, p^0) = p_x \cos(\theta) + p_y \sin(\theta) + p_\theta u/r + p^0$.

Pontryagin maximum prinicple

Every minimizing trajectory $\boldsymbol{x}(\cdot)$ is the projection of an extremal $(\boldsymbol{x}(\cdot), \boldsymbol{p}(\cdot), p^0, u(\cdot))$ solution of

Canonical equation: $\dot{\boldsymbol{x}}(t) = \frac{\partial H}{\partial \boldsymbol{p}^T}, \quad \dot{\boldsymbol{p}}(t) = -\frac{\partial H}{\partial \boldsymbol{x}^T},$

Maximum principle: $H(\boldsymbol{x}, \boldsymbol{p}, p^0, u) = \max\limits_{\eta \in [-1,1]} H(\boldsymbol{x}, \boldsymbol{p}, p^0, \eta)$

Transversality: $0 \equiv H(\boldsymbol{x}(t), \boldsymbol{p}(t), u(t), p^0),$

$$p_x(t_m^+) = p_x(t_m^-) + \lambda_x, \quad p_y(t_m^+) = p_y(t_m^-) + \lambda_y, \quad p_\theta(t_m^+) = p_\theta(t_m^-)$$

An extremal is said normal if $p^0 \neq 0$, and abnormal if $p^0 = 0$ (abnormal extremals have been ruled out by Sussmann and Tang (1994)).

In the normal case ($p^0 = -1$), the maximum Hamiltonian can be written as

$$H(\boldsymbol{x}, \boldsymbol{p}) = p_x \cos\theta + p_y \sin\theta + p_\theta u/r - 1$$

Pontryagin Maximum Principle

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} p_x(t) \\ p_y(t) \\ p_\theta(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ p_x(t)\sin[\theta(t)] - p_y(t)\cos[\theta(t)] \end{pmatrix}, \quad t \in [0, t_f] \setminus \{t_m\}.$$

This set of equations indicates p_x and p_y are piecewise constant. Hence, we have

$$p_\theta = \begin{cases} p_{x_0} y - p_{y_0} x + c_1, \; t \in (0, t_m), \\ (p_{x_0} + \lambda_x) y - (p_{y_0} + \lambda_y) x + c_2, \; t \in (t_m, t_f). \end{cases}$$

If $p_\theta \equiv 0$ on $[t_1, t_2]$, the graph of $(x(\cdot), y(\cdot))$ on $[t_1, t_2]$ forms a straight line segment, along which $u \equiv 0$. Hence, we have

$$u = \begin{cases} 1, & p_\theta > 0, \\ 0, & p_\theta \equiv 0, \\ -1, & p_\theta < 0. \end{cases}$$

Geometric properties for the solution of 3PDP

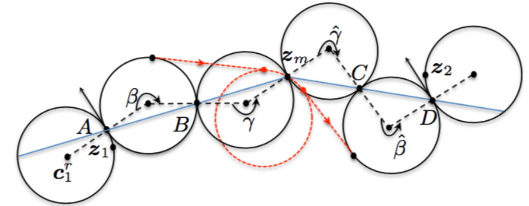
Theorem

Let $C_1T_2C_3|C_4T_5C_6$ be the shortest path of 3PDP where $T \in \{S, C\}$. If none of its subarcs vanishes, then we have C_3 and C_4 have the same turning direction.

By contradition, assume C_3 and C_4 have different turning directions, indicating $p_\theta(t_m) = 0$.

Any three points lie on a straight line if $p_\theta = 0$ at the three points.

According to Lemma 3 in [*], we have $\beta = \gamma \in (\pi, 2\pi)$ and $\hat{\beta} = \hat{\gamma} \in (\pi, 2\pi)$.



[*] X.-N. Bui, P. Souères, J.-D. Boissonnat, and J.-P. Laumond. Shortest path synthesis for Dubins non-holonomic robt. in 1994 IEEE international conference on Robotis and Automation, San Diego, CA, USA, May 1994.

Geometric properties for the solution of 3PDP

According to Bellman's principle for optimality, the solution path of 3PDP belongs to four families:

CCC|CCC	CCC|CSC	CSC|CCC	CSC|CSC
CCCCC	CCCSC	CSCCC	CSCSC

Corollary

The shortest paths for 3PDP must be among 18 types in 4 families

$$\mathscr{F} =: \{CCCCC, CSCCC, CCCSC, CSCSC\},$$

where

- $CCCCC$=$\{RLRLR, LRLRL\}$,
- $CCCSC$=$\{RLRSR, RLRSL, LRLSL, LRLSR\}$,
- $CSCCC$=$\{RSRLR, LSRLR, RSLRL, LSLRL\}$,
- $CSCSC$=$\{RSRSR, LSRSR, RSRSL, LSRSL, LSLSL, RSLSL, LSLSR, RSLSR\}$.

Geometric properties for the solution of 3PDP

36 possibilities reduce to 18.

- Once θ_m is known, one needs to check 18 $\cancel{36}$ possibilities in order to solve the 3PDP.
- ~~How to reduce the number of possibilities?~~
- How to compute θ_m?

Common formula

Theorem

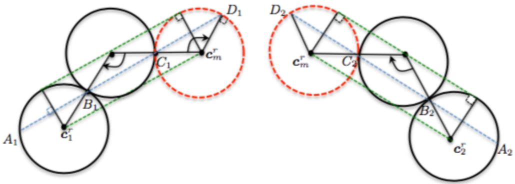
If the shortest path of 3PDP is of type $C_1 T_2 C_3 T_4 C_5$ ($T \in \{S, C\}$) such that none of its subarcs vanishes, then the angle $\theta_m \in [0, 2\pi)$ at $\boldsymbol{z}_m$ takes such a value that

$$\frac{\cos(\theta_m - \phi_1)}{\cos(\alpha_1/2)} = \frac{\cos(\theta_m - \phi_2)}{\cos(\alpha_2/2)},$$

where

- *if $T_2 = S$, then $\alpha_1 = 0$ and $\phi_1 \in [0, 2\pi)$ is the orientation angle of the line segment T_2 from its initial point to its final point;*
- *if $T_2 = C$, then $\alpha_1 \in (\pi, 2\pi)$ is the radian of T_2 such that $\cos^2(\alpha_1/2) = \frac{16r^2 - \|\mathbf{c}_m^\mu - \mathbf{c}_1^\mu\|^2}{16r^2}$ and $\phi_1 \in [0, 2\pi)$ is the orientation angle of the vector $\mathbf{c}_m^\mu - \mathbf{c}_1^\mu$ where $\mu = r$ if $T_2 = R$ and $\mu = l$ otherwise;*
- *if $T_4 = S$, then $\alpha_2 = 0$ and $\phi_2 \in [0, 2\pi)$ is the orientation angle of the line segment T_4 from its initial point to its final point; and*
- *if $T_4 = C$, then $\alpha_2 \in (\pi, 2\pi)$ is the radian of T_4 such that $\cos^2(\alpha_2/2) = \frac{16r^2 - \|\mathbf{c}_m^\mu - \mathbf{c}_2^\mu\|^2}{16r^2}$ and $\phi_2 \in [0, 2\pi)$ is the orientation angle of the vector $\mathbf{c}_2^\mu - \mathbf{c}_m^\mu$ where $\mu = r$ if $T_4 = R$ and $\mu = l$ otherwise.*

Common formula



LRLRL

Since $p_\theta = 0$ at B_1, C_1, B_2, and C_2, from $H = p_x \cos\theta + p_y \sin\theta + p_\theta u/r - 1 = 0$, we have

$$
\begin{aligned}
p_{x_0}\cos\theta + p_{y_0}\sin\theta - 1 = 0, &\qquad \theta \text{ is the orientation at } B_1 \text{ and } C_1,\\
(p_{x_0}+\lambda_x)\cos\theta + (p_{y_0}+\lambda_y)\sin\theta - 1 = 0, &\qquad \theta \text{ is the orientation at } B_2 \text{ and } C_2.
\end{aligned}
$$

Some results by common formula

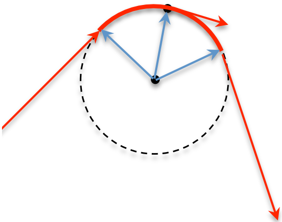
The common formula reveals the relationship between θ_m and existing variables: $\boldsymbol{z}_1$, $\boldsymbol{z}_m$, $\boldsymbol{z}_2$, θ_1, θ_2, and r.

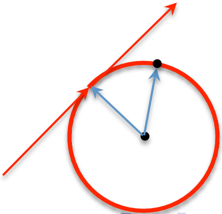
If the solution path is of type $C_1 S_2 C_3 S_4 C_5$, then we have

$$\cos(\theta_m - \phi_1) = \cos(\theta_m - \phi_2).$$

It means

- either the mid point $\boldsymbol{z}_m$ bisects C_3,
- or the radian of C_3 is 2π.





Polynomial-based solution

The common formula

$$\frac{\cos(\theta_m - \phi_1)}{\cos(\alpha_1/2)} = \frac{\cos(\theta_m - \phi_2)}{\cos(\alpha_2/2)}$$

is a multivariable polynomial in terms of $\cos\theta_m$ and $\sin\theta_m$.

$$\sin\theta_m = \frac{2\tan(\theta_m/2)}{1+\tan^2(\theta_m/2)} \text{ and } \cos\theta_m = \frac{1-\tan^2(\theta_m/2)}{1+\tan^2(\theta_m/2)}$$

$\tan(\theta_m/2)$ is a zero of some polynomials.

The degree of polynomial for each type in $\mathscr{F}$.

Degree	Type
4	LSLSL, RSRSR
6	RLRLR, LRLRL
8	{CSCSC}\{RSRSR,LSLSL}, RLRSR, RSRLR, LSLRL, LRLSL
20	RLRSL, LSRLR, LRLSR, RSLRL

Polynomial for RSRLR

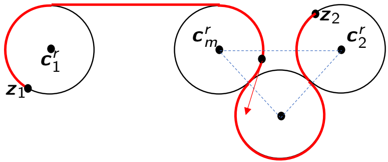
If the path is of type *RSRLR*, we have $\alpha_1 = 0$, $\cos(\theta_m - \phi_1) = \pm \frac{(\cos\theta_m, \sin\theta_m)(\mathbf{c}_m^r - \mathbf{c}_1^r)}{\|\mathbf{c}_m^r - \mathbf{c}_1^r\|}$, and $\cos(\theta_m - \phi_2) = \pm \frac{(\cos\theta_m, \sin\theta_m)(\mathbf{c}_2^r - \mathbf{c}_m^r)}{\|\mathbf{c}_2^r - \mathbf{c}_m^r\|}$. Substituting these equations into

$$\frac{\cos(\theta_m - \phi_1)}{\cos(\alpha_1/2)} = \frac{\cos(\theta_m - \phi_2)}{\cos(\alpha_2/2)}$$

and squaring the result yield

$$\frac{[(\cos\theta_m, \sin\theta_m)(\mathbf{c}_m^r - \mathbf{c}_1^r)]^2}{\|\mathbf{c}_m^r - \mathbf{c}_1^r\|^2} = \frac{[(\cos\theta_m, \sin\theta_m)(\mathbf{c}_2^r - \mathbf{c}_m^r)]^2}{\|\mathbf{c}_2^r - \mathbf{c}_m^r\|^2 \cos^2(\alpha_2/2)},$$

where $\cos^2(\alpha_2/2) = (\cos\alpha_2 + 1)/2 = (16r^2 - \|\mathbf{c}_2^r - \mathbf{c}_m^r\|^2)/16r^2$.



Polynomial for RSRLR

$$
\begin{aligned}
0 &= A_1 \cos^4\theta_m + A_2 \cos^3\theta_m \sin\theta_m + A_3 \cos^3\theta_m + A_4 \cos^2\theta_m \sin\theta_m \\
&+ A_5 \cos^2\theta_m + A_6 \cos\theta_m \sin\theta_m + A_7 \cos\theta_m + A_8 \sin\theta_m + A_9,
\end{aligned}
$$

where A_1–A_9 are constant combinations of $\boldsymbol{z}_1$, $\boldsymbol{z}_2$, $\boldsymbol{z}_m$, θ_1, θ_m, and r.

$$
\sin\theta = \frac{2\tan(\theta/2)}{1+\tan^2(\theta/2)} \quad \text{and} \quad \cos\theta = \frac{1-\tan^2(\theta/2)}{1+\tan^2(\theta/2)}
$$

$$
\begin{aligned}
0 &= B_1 \tan^8(\theta_m/2) + B_2 \tan^7(\theta_m/2) + B_3 \tan^6(\theta_m/2) + B_4 \tan^5(\theta_m/2) \\
&+ B_5 \tan^4(\theta_m/2) + B_6 \tan^3(\theta_m/2) + B_7 \tan^2(\theta_m/2) + B_8 \tan(\theta_m/2) + B_9,
\end{aligned}
$$

where B_1–B_9 are constant combinations of A_1–A_9.

$$
\theta_m = 2\arctan(\text{root})
$$

Numerical Simulations

——Test of Polynomial-Based Method

Table: Normalizing the time of solving the Dubins problem between two configurations to one, this table shows the normalized time to solve polynomials with degrees in $\{4, 6, 8, 20\}$.

Degree	4	6	8	20
Normalized Time	1/11.16	1/9.19	1/7.99	1/2.94

Discretized method

$$\theta_m = \operatorname*{argmin}_{\theta \in \Phi} F[(z_1, \theta_1), (z_m, \theta)] + F[(z_m, \theta), (z_2, \theta_2)]$$

where $\Phi = \{2(i-1)\pi/l : i = 1, \ldots, l\}$ and $l \in \mathbb{N}$ is the discretization level.

Table: The improvement factors of time consumption of PBM compared to discretised method with $l = 360$.

d_m	$> 4r$	$= 3r$	$= 2r$	$= r$	$< r$
Factor	45.69	24.36	27.19	32.66	36.98

Numerical Example

——Solving Curvature-Constrained Shortest-Path Problem (CCSPP)

CCSPP

Given a sequence of waypoints $(\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_n)$ with the order fixed, let the heading orientation angles at $\boldsymbol{z}_1$ and $\boldsymbol{z}_n$ be fixed at θ_1 and θ_n, respectively. Then, the CCSPP consists of finding the shortest Dubins path starting from $(\boldsymbol{z}_1, \theta_1)$, passing through $\boldsymbol{z}_i$ in order, finally reaching $(\boldsymbol{z}_n, \theta_n)$.

$$\text{CCSPP}: \quad \min_{(\theta_2,\ldots,\theta_{n-1})\in[0,2\pi]^{n-2}} \sum_{i=1}^{n-1} F[(\boldsymbol{z}_i, \theta_i), (\boldsymbol{z}_{i+1}, \theta_{i+1})]$$

Coordinate Descent Algorithm (CDA)

What is the CDA?

Given an objective function $f : \mathbb{R}^n \to \mathbb{R}$, $\boldsymbol{x} \mapsto f(\boldsymbol{x})$, the CDA works as:

- starting with initial variable values $\boldsymbol{x}^0 = (x_1^0, \ldots, x_n^0)$,
- round $k+1$ defines $\boldsymbol{x}^{k+1}$ from $\boldsymbol{x}^k$ by iteratively solving the single variable optimization problems

$$x_i^{k+1} = \operatorname*{argmin}_{y \in \mathbb{R}} f(x_1^{k+1}, \ldots, x_{i-1}^{k+1}, y, x_{i+1}^k, \ldots, x_n^k)$$

for each variable x_i of $\boldsymbol{x}$, for i from 1 to n.

Numerical example

Given the 100 random targets, this table shows the lengths of the paths generated by AA, SVA, LAA, and CDA.

Algorithms	Radius, r						
	1	2	3	4	5	6	7
AA	963.58	1143.35	1489.33	1849.38	2344.17	2668.66	3186.98
SVA	952.04	1166.95	1483.71	1940.22	2431.62	3024.21	3617.90
LAA	874.23	957.96	1068.12	1305.00	1615.74	1953.80	2296.52
CDA	870.81	938.05	1048.42	1276.28	1544.23	1832.88	2123.03
	8	9	10	15	20	30	40
AA	3403.61	3978.35	4426.86	6539.55	8683.31	13369.95	16401.86
SVA	4196.64	4884.91	5319.43	8685.27	11672.90	18130.99	24426.06
LAA	2579.38	2954.04	3351.87	5074.41	6865.72	10388.09	13703.96
CDA	2451.60	2858.30	3190.46	4974.17	6685.20	10319.27	13625.59

Conclusions

The solution of 3PDP is synthesized:

1. Reduce 36 possibilities to 18;
2. A common formula is established for the 18 types;
3. A polynomial-based method is proposed to solve the 3PDP;
4. The result allows to use gradient-free CDA.

Future work includes:

- exploring the properties of the 18 types so that less possibilities are checked in order to solve the 3PDP, and
- application to motion planning.

Thanks! & Questions?

Email: zheng.chen@technion.ac.il