Shortest Dubins Paths through Three Points

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The Dubins vehicle moves only forward at a constant speed with a minimum turning radius.

1. normalize the speed to one;
2. consider the minimum turning radius to be $r \in \mathbb{R}_+$.

The state $\mathbf{x} := (x, y, \theta) \in \mathbb{R}^2 \times \mathbb{S} =: \mathcal{X}$, also called configuration, consists of a position vector $(x, y) \in \mathbb{R}^2$ and a heading orientation angle $\theta \in \mathbb{S}$.

The kinematics is

$$\frac{d}{dt} \begin{pmatrix} x(t) \\ y(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} \cos \theta(t) \\ \sin \theta(t) \\ u(t)/r \end{pmatrix}$$

where $u \in [-1, 1]$ denotes the control.
Many nonholonomic vehicles can be modelled by the Dubins vehicle, such as fixed-wing unmanned aerial vehicles, unmanned ground vehicles, ships, etc.
The shortest Dubins path between two configurations belongs to six types in two families [*]:

- **CCC** = \{RLR, LRL\}, and
- **CSC** = \{RSR, RSL, LSL, LSR\}

where \( R \) (resp. \( L \)) denotes the corresponding circular arc with a right (resp. left) turning direction.

3-Point Dubins Problem (3PDP)

Given three different points $z_1$, $z_m$, and $z_2$ in $\mathbb{R}^2$, let $\theta_1$ and $\theta_2$ in $[0,2\pi)$ be the fixed heading orientation angles at $z_1$ and $z_2$, respectively. Then, the 3PDP consists of steering ($\Sigma$) by $u(\cdot) \in [-1,1]$ on $[0,t_f]$ from $(z_1, \theta_1)$, pathing through $z_m$ at $t_m \in (0,t_f)$, to $(z_2, \theta_2)$ such that the final time $t_f > 0$ is minimized.

As the speed of the Dubins vehicle is a constant, solving the 3PDP is equivalent to finding the shortest path.

Given any two configurations $(y_1, \eta_1)$ and $(y_2, \eta_2)$ in $X$, denote by

$$F : X^2 \to \mathbb{R}, \quad [(y_1, \eta_1), (y_2, \eta_2)] \mapsto F[(y_1, \eta_1), (y_2, \eta_2)]$$

the length of the shortest Dubins path between them. As the heading orientation angles before and after $z_m$ are the same along the shortest path, we denote such a heading orientation angle by $\theta_m$, i.e.,

$$\theta_m := \operatorname{argmin}_{\theta \in [0,2\pi)} F[(z_1, \theta_1), (z_m, \theta)] + F[(z_m, \theta), (z_2, \theta_2)].$$
According to Bellman’s principle for optimality, the solution of 3PDP is the concatenation of the shortest Dubins paths between \((z_1, \theta_1)\) and \((z_m, \theta_m)\) and between \((z_m, \theta_m)\) and \((z_2, \theta_2)\). Hence, the solution path of 3PDP belongs to four families:

\[
CCC|CCC, \quad CCC|CSC, \quad CSC|CCC, \quad CSC|CSC,
\]

where the notation “|” denotes \(z_m\).

Up to \(6 \times 6 = 36\) possibilities.

- Once \(\theta_m\) is known, one needs to check 36 possibilities in order to solve the 3PDP.
- How to reduce the number of possibilities?
- How to compute \(\theta_m\)?
Characterization of the solution for 3PDP

1. Necessary conditions
2. Geometric properties
3. Common formula
Pontryagin Maximum Principle (PMP)

Denote by \( p = [p_x, p_y, p_\theta] \in T^*_x \mathcal{X} \) the costate of \( x = [x, y, \theta] \in \mathcal{X} \). The Hamiltonian is \( H(x, p, u, p^0) = p_x \cos(\theta) + p_y \sin(\theta) + p_\theta u/r + p^0 \).

Pontryagin maximum principle

Every minimizing trajectory \( x(\cdot) \) is the projection of an extremal \((x(\cdot), p(\cdot), p^0, u(\cdot))\) solution of

Canonical equation: \( \dot{x}(t) = \frac{\partial H}{\partial p^T}, \quad \dot{p}(t) = -\frac{\partial H}{\partial x^T} \),

Maximum principle: \( H(x, p, p^0, u) = \max_{\eta \in [-1,1]} H(x, p, p^0, \eta) \)

Transversality: \( 0 \equiv H(x(t), p(t), u(t), p^0) \),

\[
p_x(t_m^+) = p_x(t_m^-) + \lambda_x, \quad p_y(t_m^+) = p_y(t_m^-) + \lambda_y, \quad p_\theta(t_m^+) = p_\theta(t_m^-)
\]

An extremal is said normal if \( p^0 \neq 0 \), and abnormal if \( p^0 = 0 \) (abnormal extremals have been ruled out by Sussmann and Tang (1994)).

In the normal case \( (p^0 = -1) \), the maximum Hamiltonian can be written as

\[
H(x, p) = p_x \cos \theta + p_y \sin \theta + p_\theta u/r - 1
\]
Pontryagin Maximum Principle

\[
\frac{d}{dt} \begin{pmatrix} p_x(t) \\ p_y(t) \\ p_\theta(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ p_x(t) \sin[\theta(t)] - p_y(t) \cos[\theta(t)] \end{pmatrix}, \quad t \in [0, t_f] \setminus \{t_m\}.
\]

This set of equations indicates \(p_x\) and \(p_y\) are piecewise constant. Hence, we have

\[
p_\theta = \begin{cases} p_{x_0} y - p_{y_0} x + c_1, & t \in (0, t_m), \\ (p_{x_0} + \lambda_x) y - (p_{y_0} + \lambda_y) x + c_2, & t \in (t_m, t_f). \end{cases}
\]

If \(p_\theta \equiv 0\) on \([t_1, t_2]\), the graph of \((x(\cdot), y(\cdot))\) on \([t_1, t_2]\) forms a straight line segment, along which \(u \equiv 0\). Hence, we have

\[
u = \begin{cases} 1, & p_\theta > 0, \\ 0, & p_\theta \equiv 0, \\ -1, & p_\theta < 0. \end{cases}
\]
Geometric properties for the solution of 3PDP

**Theorem**

Let $C_1 T_2 C_3|C_4 T_5 C_6$ be the shortest path of 3PDP where $T \in \{S, C\}$. If none of its subarcs vanishes, then we have $C_3$ and $C_4$ have the same turning direction.

By contradiction, assume $C_3$ and $C_4$ have different turning directions, indicating $p_\theta(t_m) = 0$.

According to Lemma 3 in [*], we have $\beta = \gamma \in (\pi, 2\pi)$ and $\hat{\beta} = \hat{\gamma} \in (\pi, 2\pi)$.

Any three points lie on a straight line if $p_\theta = 0$ at the three points.

Geometric properties for the solution of 3PDP

According to Bellman’s principle for optimality, the solution path of 3PDP belongs to four families:

| CCC|CCC | CCC|CSC | CSC|CCC | CSC|CSC |
|-----|-----|-----|-----|-----|-----|-----|
| CCCCC | CCCSC | CSCCC | CSCSC |

Corollary

The shortest paths for 3PDP must be among 18 types in 4 families

\[ \mathcal{F} = \{ CCCCC, CSCCC, CCCSC, CSCSC \}, \]

where

- \( CCCCC = \{ RLRLR, LRLRL \} \),
- \( CCCSC = \{ RLRSR, RLRSL, LRLSL, LRLSR \} \),
- \( CSCCC = \{ RSRLR, LSRLR, RSLRL, LSLRL \} \),
- \( CSCSC = \{ RSRSR, LSRSR, RSRSL, LSRSL, LSLSL, RSLSL, LLSR, RLSLR \} \).
Geometric properties for the solution of 3PDP

36 possibilities reduce to 18.

- Once $\theta_m$ is known, one needs to check $\frac{18}{36}$ possibilities in order to solve the 3PDP.
- How to reduce the number of possibilities?
- How to compute $\theta_m$?
If the shortest path of 3PDP is of type $C_1 T_2 C_3 T_4 C_5$ ($T \in \{S, C\}$) such that none of its subarcs vanishes, then the angle $\theta_m \in [0, 2\pi)$ at $z_m$ takes such a value that

$$\frac{\cos(\theta_m - \phi_1)}{\cos(\alpha_1/2)} = \frac{\cos(\theta_m - \phi_2)}{\cos(\alpha_2/2)},$$

where

- if $T_2 = S$, then $\alpha_1 = 0$ and $\phi_1 \in [0, 2\pi)$ is the orientation angle of the line segment $T_2$ from its initial point to its final point;

- if $T_2 = C$, then $\alpha_1 \in (\pi, 2\pi)$ is the radian of $T_2$ such that $\cos^2(\alpha_1/2) = \frac{16r^2 - ||c^\mu_m - c^\mu_1||^2}{16r^2}$ and $\phi_1 \in [0, 2\pi)$ is the orientation angle of the vector $c^\mu_m - c^\mu_1$ where $\mu = r$ if $T_2 = R$ and $\mu = l$ otherwise;

- if $T_4 = S$, then $\alpha_2 = 0$ and $\phi_2 \in [0, 2\pi)$ is the orientation angle of the line segment $T_4$ from its initial point to its final point; and

- if $T_4 = C$, then $\alpha_2 \in (\pi, 2\pi)$ is the radian of $T_4$ such that $\cos^2(\alpha_2/2) = \frac{16r^2 - ||c^\mu_2 - c^\mu_m||^2}{16r^2}$ and $\phi_2 \in [0, 2\pi)$ is the orientation angle of the vector $c^\mu_2 - c^\mu_m$ where $\mu = r$ if $T_4 = R$ and $\mu = l$ otherwise.
Since \( p_\theta = 0 \) at \( B_1, C_1, B_2, \) and \( C_2 \), from \( H = p_x \cos \theta + p_y \sin \theta + p_\theta u/r - 1 = 0 \), we have

\[
\begin{align*}
    p_{x_0} \cos \theta + p_{y_0} \sin \theta - 1 &= 0, \\
    (p_{x_0} + \lambda_x) \cos \theta + (p_{y_0} + \lambda_y) \sin \theta - 1 &= 0,
\end{align*}
\]

\( \theta \) is the orientation at \( B_1 \) and \( C_1 \), and \( \theta \) is the orientation at \( B_2 \) and \( C_2 \).
Some results by common formula

The common formula reveals the relationship between $\theta_m$ and existing variables: $z_1, z_m, z_2, \theta_1, \theta_2,$ and $r$.

If the solution path is of type $C_1S_2C_3S_4C_5$, then we have

$$\cos(\theta_m - \phi_1) = \cos(\theta_m - \phi_2).$$

It means
- either the mid point $z_m$ bisects $C_3$,
- or the radian of $C_3$ is $2\pi$. 
Polynomial-based solution

The common formula

\[
\frac{\cos(\theta_m - \phi_1)}{\cos(\alpha_1/2)} = \frac{\cos(\theta_m - \phi_2)}{\cos(\alpha_2/2)}
\]

is a multivariable polynomial in terms of \(\cos \theta_m\) and \(\sin \theta_m\).

\[
\sin \theta_m = \frac{2\tan(\theta_m/2)}{1 + \tan^2(\theta_m/2)} \quad \text{and} \quad \cos \theta_m = \frac{1 - \tan^2(\theta_m/2)}{1 + \tan^2(\theta_m/2)}
\]

\(\tan(\theta_m/2)\) is a zero of some polynomials.

<table>
<thead>
<tr>
<th>Degree</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>LSLSL, RSRSR</td>
</tr>
<tr>
<td>6</td>
<td>RLRLR, LRLRL</td>
</tr>
<tr>
<td>8</td>
<td>{CSCSC}{RSRSR,LSLSL}, RLRSR, RSRLR, LSLRL, LRLSL</td>
</tr>
<tr>
<td>20</td>
<td>RLRSL, LSRLR, LRLSR, RSLRL</td>
</tr>
</tbody>
</table>
If the path is of type RSRLR, we have $\alpha_1 = 0$, $\cos(\theta_m - \phi_1) = \pm \frac{(\cos \theta_m, \sin \theta_m)(c'_m - c'_1)}{\|c'_m - c'_1\|}$, and $\cos(\theta_m - \phi_2) = \pm \frac{(\cos \theta_m, \sin \theta_m)(c'_2 - c'_m)}{\|c'_2 - c'_m\|}$. Substituting these equations into

$$\frac{\cos(\theta_m - \phi_1)}{\cos(\alpha_1/2)} = \frac{\cos(\theta_m - \phi_2)}{\cos(\alpha_2/2)}$$

and squaring the result yield

$$\frac{[(\cos \theta_m, \sin \theta_m)(c'_m - c'_1)]^2}{\|c'_m - c'_1\|^2} = \frac{[(\cos \theta_m, \sin \theta_m)(c'_2 - c'_m)]^2}{\|c'_2 - c'_m\|^2 \cos^2(\alpha_2/2)},$$

where $\cos^2(\alpha_2/2) = (\cos \alpha_2 + 1)/2 = (16r^2 - \|c'_2 - c'_m\|^2)/16r^2$. 
Polynomial for RSRLLR

\[ 0 = A_1 \cos^4 \theta_m + A_2 \cos^3 \theta_m \sin \theta_m + A_3 \cos^3 \theta_m + A_4 \cos^2 \theta_m \sin \theta_m 
+ A_5 \cos^2 \theta_m + A_6 \cos \theta_m \sin \theta_m + A_7 \cos \theta_m + A_8 \sin \theta_m + A_9, \]

where \( A_1 - A_9 \) are constant combinations of \( z_1, z_2, z_m, \theta_1, \theta_m, \) and \( r \).

\[
\sin \theta = \frac{2 \tan(\theta / 2)}{1 + \tan^2(\theta / 2)} \quad \text{and} \quad \cos \theta = \frac{1 - \tan^2(\theta / 2)}{1 + \tan^2(\theta / 2)}
\]

\[ 0 = B_1 \tan^8(\theta_m / 2) + B_2 \tan^7(\theta_m / 2) + B_3 \tan^6(\theta_m / 2) + B_4 \tan^5(\theta_m / 2) 
+ B_5 \tan^4(\theta_m / 2) + B_6 \tan^3(\theta_m / 2) + B_7 \tan^2(\theta_m / 2) + B_8 \tan(\theta_m / 2) + B_9, \]

where \( B_1 - B_9 \) are constant combinations of \( A_1 - A_9 \).

\[ \theta_m = 2 \arctan(\text{root}) \]
Numerical Simulations

---Test of Polynomial-Based Method

**Table:** Normalizing the time of solving the Dubins problem between two configurations to one, this table shows the normalized time to solve polynomials with degrees in \( \{4, 6, 8, 20\} \).

<table>
<thead>
<tr>
<th>Degree</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normalized Time</td>
<td>1/11.16</td>
<td>1/9.19</td>
<td>1/7.99</td>
<td>1/2.94</td>
</tr>
</tbody>
</table>

Discretized method

\[
\theta_m = \arg\min_{\theta \in \Phi} F[(z_1, \theta_1), (z_m, \theta)] + F[(z_m, \theta), (z_2, \theta_2)]
\]

where \( \Phi = \{2(i - 1)\pi/l : i = 1, \ldots, l\} \) and \( l \in \mathbb{N} \) is the discretization level.

**Table:** The improvement factors of time consumption of PBM compared to discretised method with \( l = 360 \).

<table>
<thead>
<tr>
<th>( d_m )</th>
<th>( &gt; 4r )</th>
<th>( = 3r )</th>
<th>( = 2r )</th>
<th>( = r )</th>
<th>( &lt; r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factor</td>
<td>45.69</td>
<td>24.36</td>
<td>27.19</td>
<td>32.66</td>
<td>36.98</td>
</tr>
</tbody>
</table>
Problem statement

Solution characterization

Polynomial solution

Examples

Numerical Example

—–Solving Curvature-Constrained Shortest-Path Problem (CCSPP)

CCSPP

Given a sequence of waypoints \((z_1, z_2, \ldots, z_n)\) with the order fixed, let the heading orientation angles at \(z_1\) and \(z_n\) be fixed at \(\theta_1\) and \(\theta_n\), respectively. Then, the CCSPP consists of finding the shortest Dubins path starting from \((z_1, \theta_1)\), passing through \(z_i\) in order, finally reaching \((z_n, \theta_n)\).

\[
\text{CCSPP} : \min_{(\theta_2, \ldots, \theta_{n-1}) \in [0,2\pi]^{n-2}} \sum_{i=1}^{n-1} F[(z_i, \theta_i), (z_{i+1}, \theta_{i+1})]
\]
Coordinate Descent Algorithm (CDA)

What is the CDA?

Given an objective function \( f : \mathbb{R}^n \to \mathbb{R}, \ x \mapsto f(x) \), the CDA works as:

- starting with initial variable values \( x^0 = (x_1^0, \ldots, x_n^0) \),
- round \( k + 1 \) defines \( x^{k+1} \) from \( x^k \) by iteratively solving the single variable optimization problems

\[
x_i^{k+1} = \arg\min_{y \in \mathbb{R}} f(x_1^{k+1}, \ldots, x_{i-1}^{k+1}, y, x_i^k, \ldots, x_n^k)
\]

for each variable \( x_i \) of \( x \), for \( i \) from 1 to \( n \).
Given the 100 random targets, this table shows the lengths of the paths generated by AA, SVA, LAA, and CDA.

<table>
<thead>
<tr>
<th>Algorithms</th>
<th>Radius, ( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>AA</td>
<td>963.58</td>
</tr>
<tr>
<td>SVA</td>
<td>952.04</td>
</tr>
<tr>
<td>LAA</td>
<td>874.23</td>
</tr>
<tr>
<td>CDA</td>
<td>870.81</td>
</tr>
<tr>
<td></td>
<td>8</td>
</tr>
<tr>
<td>AA</td>
<td>3403.61</td>
</tr>
<tr>
<td>SVA</td>
<td>4196.64</td>
</tr>
<tr>
<td>LAA</td>
<td>2579.38</td>
</tr>
<tr>
<td>CDA</td>
<td>2451.60</td>
</tr>
</tbody>
</table>
The solution of 3PDP is synthesized:

1. Reduce 36 possibilities to 18;
2. A common formula is established for the 18 types;
3. A polynomial-based method is proposed to solve the 3PDP;
4. The result allows to use gradient-free CDA.

Future work includes:

- exploring the properties of the 18 types so that less possibilities are checked in order to solve the 3PDP, and
- application to motion planning.
Thanks! & Questions?

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