Controllability of a Magneto-Elastic Micro-Swimmer

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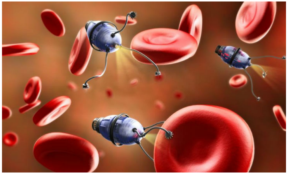
What is Micro-Swimming?

Concept
Micro-swimming is the study of propulsion in a fluid at microscopic scale.

Applications

- Biology: micro-organisms swimming behaviour (bacteria, sperm cells...)
- Micro-engineering
- Medicine: targeted drug delivery, micro-surgery



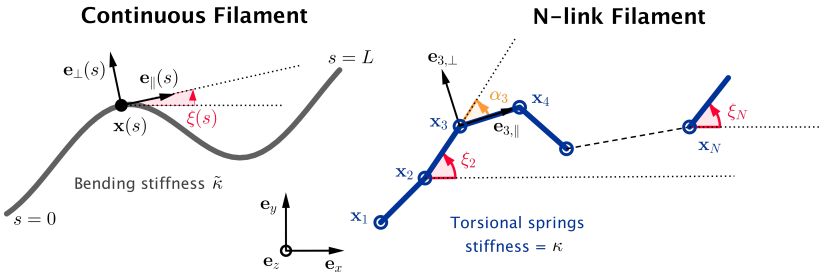
Our subject
Study a simple micro-swimmer model, describing a magnetised elastic filament immersed in a fluid, from the control theory point of view.

Outline

- Modeling an elastic filament in a fluid
- The two-link magnetic model
- Equations of the model
- Controllability results and sketch of the proof

Modelling an Elastic Filament

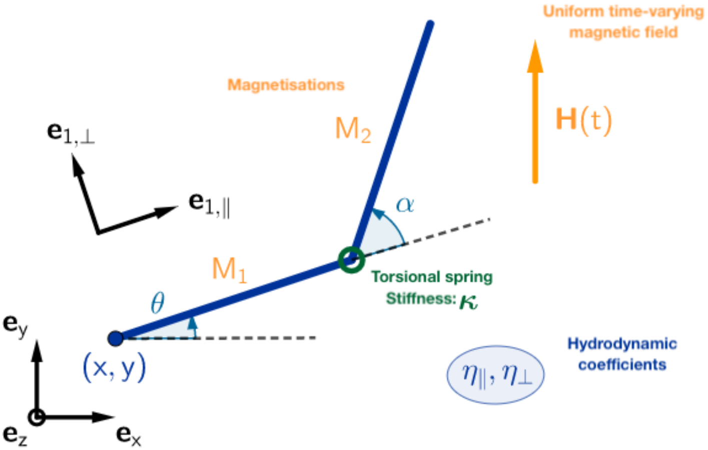
How to model an elastic filament immersed in a fluid at micro scale ?



- At this scale, the **inertial effects** are **negligible** compared to the **viscous effects** (low Reynolds number regime).
- Similar behaviour for the two models (Gadêlha, Moreau, An efficient model for studying the dynamics of an elastic microfilament, in preparation)

Model of a Two-Segment Magnetised Micro-Swimmer

Description of a 2-segment magnetized swimmer immersed in a fluid



Equations of the Model

Hydrodynamics effects: density of force at point $\mathbf{x}(s)$ given by the Resistive Force Theory [Gray, Hancock 1955]

$$\mathbf{f}_i(s) = \eta_{\|}(\dot{\mathbf{x}}(s) \cdot \mathbf{e}_{i,\|})\,\mathbf{e}_{i,\|} + \eta_{\perp}(\dot{\mathbf{x}}(s) \cdot \mathbf{e}_{i,\perp})\,\mathbf{e}_{i,\perp}, \quad i = 1, 2$$

- Hydrodynamic force

$$\mathbf{F}_i^h = \int_{S_i} \mathbf{f}_i(s) ds.$$

- Hence: hydrodynamic torque at point $\mathbf{x}$

$$\mathbf{T}_{i,\mathbf{x}}^h = \int_{S_i} (\mathbf{x}_i(s) - \mathbf{x}) \times \mathbf{f}_i(s) ds.$$

Magnetic effects: torque proportional to the magnetisations

$$\mathbf{T}_i^m = M_i \mathbf{e}_{i,\|} \times \mathbf{H}.$$

Internal elastic effect: torque proportional to the shape angle

$$\mathbf{T}^{el} = \kappa \alpha \mathbf{e}_z$$

Equations of the Model

Inertia negligible compared to the viscous effects (Low Reynolds number) $\rightarrow$ balance of forces and torques at all times

The system is described by the four state variables $\mathbf{z} = (x, y, \theta, \alpha)$.

One gets three differential equations (that give four scalar equations) by writing

- balance of forces (projected on x- and y-axis) over the swimmer
- balance of torques over the swimmer
- balance of torques on the **second** segment: here appears the internal elastic effect

$$
\left\{
\begin{array}{lllll}
\mathbf{F}_1^h + \mathbf{F}_2^h & & & & = 0 \\
\mathbf{T}_1^h + \mathbf{T}_2^h & + \mathbf{T}_1^m + \mathbf{T}_2^m & & & = 0 \\
\underbrace{\mathbf{T}_2^h}_{\text{hydrodynamics}} & \underbrace{+\mathbf{T}_2^m}_{\text{magnetism}} & + & \underbrace{\mathbf{T}^{el}}_{\text{elasticity}} & = 0
\end{array}
\right.
$$

Control System

This defines a control-affine system:

$$\dot{\mathbf{z}} = \mathbf{F}_0(\mathbf{z}) + H_{\parallel}\, \mathbf{F}_1(\mathbf{z}) + H_{\perp}\, \mathbf{F}_2(\mathbf{z})\,.$$

with $H_{\parallel}$ and $H_{\perp}$ seen as the controls. $\mathbf{F}_0$ is the **drift vector field**, due to elasticity, that tends to bring the swimmer back to **an aligned shape**.

Controllability goals:

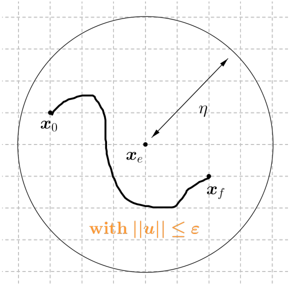
Is it possible to bring the swimmer from a given initial state to a given final state ?

- We look for **local controllability** around an equilibrium
- Equilibrium points in our system: $(x, y, \theta, 0)$

Small-Time Local Controllability

Stronger definition (Coron, *Control and Linearity*, 1992): A control system is **small-time locally controllable (STLC)** around an equilibrium $\mathbf{x}_e$ if for all $\epsilon > 0$, there exists $\eta > 0$ such that, for all initial and final states η-close to $\mathbf{x}_e$, there exists a trajectory that goes from one to the other state, using control functions bounded by ϵ, in time smaller than ϵ.

Weaker definition: A control system is K-**small-time locally controllable (K-STLC)** around an equilibrium $\mathbf{x}_e$ if there exists $K > 0$ such that for all $\epsilon > 0$, there exists $\eta > 0$ such that, for all initial and final states η-close to $\mathbf{x}_e$, there exists a trajectory that goes from one to the other state, using control functions bounded by $K + \varepsilon$, in time smaller than ϵ.



Is the 2-link swimmer STLC ?

Assume $M_1 \neq 0$, $M_2 \neq 0$, $M_1 + M_2 \neq 0$ and $\eta \neq \xi$. Then we have

Theorem 1 (Giraldi, Pomet, 2017)
The 2-link swimmer is K-STLC with $K = 2\,\kappa \left| \dfrac{M_2 + M_1}{M_2\, M_1} \right|$.

But...

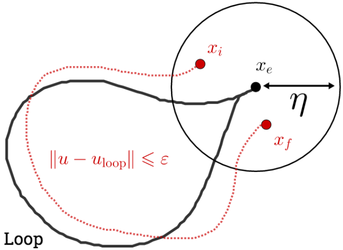
Theorem 2 (Giraldi, Lissy, Moreau, Pomet, 2017)
The 2-link swimmer is not STLC.

Theorem 1: Sketch of the proof

The classic methods (Kalman matrix on the linearised system and Sussmann conditions) do not work for this system.

Return method [Coron, 1992]:

1. Find a control u_{loop} that yields a loop trajectory:
   - the solution of the system with these controls begins and ends at the equilibrium point,
   - the controls are bounded by K.
2. Show that the linearised system is controllable along the loop trajectory.
3. Conclude that the system is K-STLC by small perturbation of the control u_{loop}.



It is possible to drive the system from and to states which are η-close to the equilibrium, by staying close to the loop.

Theorem 2: Sketch of the proof

Lemma: Loop trajectories
If a control system is STLC at (z_e, u_e), then, for any $\varepsilon > 0$, there exists a control $u^\varepsilon(\cdot)$ such that the solution $t \mapsto z^\varepsilon(t)$ of the system, defined for t in $[0, \varepsilon]$, satisfies

- $z^\varepsilon(0) = z^\varepsilon(\varepsilon) = z_e$,
- $z^\varepsilon(t) \neq z_e$ for at least one t in $[0, \varepsilon]$,

and, for almost all t in $[0, \varepsilon]$,

$$\|u^\varepsilon(t) - u_e\| \leqslant \varepsilon.$$

Theorem 2: Sketch of the proof

Illustration of how lemma 3 will be useful on a simple example:

$$\begin{cases} \dot{x}_1 & = x_2^2 \\ \dot{x}_2 & = u \end{cases}$$

Suppose there exists a loop defined on $[0, \varepsilon]$ as above. Then

$$\int_0^{\varepsilon} \dot{x}_1(t)dt = x_1(\varepsilon) - x_1(0) = 0.$$

But one also has

$$\int_0^{\varepsilon} \dot{x}_1(t)dt = \int_0^{\varepsilon} x_2^2.$$

Therefore $x_2(t) = 0$, and $x_1(t) = 0$ on all the interval. So the system is not controllable.

Idea for the swimmer system: find a quantity that forbids the existence of a small loop.

Theorem 2: Sketch of the proof

The control system reads

$$\dot{\mathbf{z}} = \mathbf{F}_0(\mathbf{z}) + H_{\|}\, \mathbf{F}_1(\mathbf{z}) + H_{\perp}\, \mathbf{F}_2(\mathbf{z})\,.$$

Asymptotic expansion: write the expansions of $\mathbf{F}_0$, $\mathbf{F}_1$ and $\mathbf{F}_2$ around the equilibrium.
Change of coordinates: $(x, y, \theta, \alpha) \to (x, y, z_3, z_4)$, with z_3 and z_4 linear combinations of θ, α such that

$$\dot{z}_3 = \underbrace{z_4}_{\text{dominant term}} + z_4 \left(z_4\, r_{0,3}(z_4) + H_{\perp}\, r_{1,3}(z_4) + H_{\|}\, r_{2,3}(z_4) \right),$$

$$\dot{z}_4 = \underbrace{H_{\perp}}_{\text{dominant term}} - z_4 \left(a_2 + z_4\, r_{0,4}(z_4) + H_{\perp}\, r_{1,4}(z_4) + H_{\|}\, r_{2,4}(z_4) \right).$$

where the $r_{i,j}$ are smooth functions. In these coordinates,

$$\dot{x} = \underbrace{c_3\, z_4^2}_{\substack{\text{to keep for}\\ \text{the bad loop}}} + \underbrace{(c_1 z_3 + c_2 z_4) H_{\perp}}_{\text{to make disappear}} + \underbrace{z_4^2 R_1(z_3, z_4) + z_3 z_4\, R_2(z_3, z_4) + z_3^2 R_3(z_3, z_4)}_{\text{higher order, will not be a problem (see later)}}.$$

Theorem 2: Sketch of the proof

Then, defining

$$\zeta = x - c_1 z_3 z_4 - \frac{1}{2} c_2 {z_4}^2,$$

one has

$$\dot{\zeta} = {z_4}^2 \left(c_0 + \widetilde{R}_1 \right) + z_3 z_4 \, \widetilde{R}_2 + {z_3}^2 \, \widetilde{R}_3,$$

with $c_0 = \frac{108\kappa}{\ell^8 \eta^3 \xi}(M_2^2 - M_1^2)(\eta - \xi)$, nonzero due to the assumptions made. From there, we suppose there exists a loop, defined on $[0, \varepsilon]$, such that the controls stay smaller than ε. Then, integrating on the time interval, we get $\int_0^\varepsilon \dot{\zeta}(t) \mathrm{d}t = 0$, i.e.

$$|c_0| \int_0^\varepsilon z_4^2 \leqslant \int_0^\varepsilon z_4^2 \underbrace{|\widetilde{R}_1|}_{\leqslant K_1 \varepsilon} + \underbrace{\int_0^\varepsilon |z_3 z_4 \widetilde{R}_2|}_{\leqslant K_2 \varepsilon \int_0^\varepsilon z_4^2} + \underbrace{\int_0^\varepsilon z_3^2 |\widetilde{R}_3|}_{\leqslant K_3 \varepsilon^2 \int_0^\varepsilon z_4^2} \leqslant K\varepsilon \int_0^\varepsilon z_4^2.$$

Therefore, for ε small enough, z_4 is identically zero along $[0, \varepsilon]$, and so are the other coordinates. Q.E.D.

Conclusion – Perspectives

Why is it hard to control the system around the equilibrium?

This lack of controllability can be intuitively explained by the fact that when the swimmer is aligned, the parallel component of the magnetic field has no effect on it.

Perspectives on control theory

Necessary condition for STLC: could it be generalised for other control systems? (Sussmann 1983, Kawski 1987, Krastanov 1998, Beauchard and Marbach 2017)

Perspectives on micro-swimming

- What about swimmers with three or more segments?
- How to avoid the problem at the equilibrium when trying to steer a filament with a magnetic field?