Optimal control of slow-fast mechanical systems

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Joint work with

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Slow-fast control system

Minimum time.

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \dot{I} = \varepsilon F_0(I,\varphi,\varepsilon) + \varepsilon \sum_{i=1}^{m} u_i F_i(I,\varphi,\varepsilon), \quad |u| = \sqrt{u_1^2 + \cdots + u_m^2} \leq 1,$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \dot{\varphi} = \omega(I) + \varepsilon G_0(I,\varphi,\varepsilon) + \varepsilon \sum_{i=1}^{m} u_i G_i(I,\varphi,\varepsilon),$$

with $I \in M$, $\varphi \in \mathbf{S}^1$, $u \in \mathbf{R}^m$, fixed extremities I_0, I_f, and free phases φ_0, φ_f.
Min. time Hamiltonian (Pontrjagin maximum principle):

$$H(I,\varphi,p_I,p_\varphi,\varepsilon) := p_\varphi \omega(I) + \varepsilon K(I,\varphi,p_I,p_\varphi,\varepsilon),$$

$$K := H_0 + \sqrt{\sum_{i=1}^{m} H_i^2},$$

$$H_i(I,\varphi,p_I,p_\varphi,\varepsilon) := p_I F_i(I,\varphi,\varepsilon) + p_\varphi G_i(I,\varphi,\varepsilon), \quad i = 0,\ldots,m.$$

Slow-fast control system

Averaged Hamiltonian. One defines

$$\overline{K} = \overline{H}_0 + \overline{K}_0, \quad \overline{H}_0 = \langle p_I, \overline{F}_0 \rangle,$$

$$\begin{aligned} \overline{K}_0(I, p_I) &:= \frac{1}{2\pi} \int_0^{2\pi} \sqrt{\sum_{i=1}^m H_i^2(I, \varphi, p_I, p_\varphi = 0, \varepsilon = 0)}\, \mathrm{d}\varphi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \sqrt{\sum_{i=1}^m \langle p_I, F_i(I, \varphi, \varepsilon = 0) \rangle^2}\, \mathrm{d}\varphi. \end{aligned}$$

Smooth on $\Omega := \complement \overline{\Sigma}$ where

$$\Sigma := \{(I, p_I, \varphi) \in T^*M \times \mathbf{S}^1 \,|\, (\forall i = 1, m) : \langle p_I, F_i(I, \varphi, \varepsilon = 0) \rangle = 0\},$$

$$\overline{\Sigma} := \varpi(\Sigma) \text{ (closed)}, \quad \varpi : T^*M \times \mathbf{S}^1 \to T^*M.$$

One also defines the submanifold of M

$$M_0 := \Pi(\Omega) \text{ (open and assumed to be connex).}$$

Slow-fast control system

Assuming

(A1) $\operatorname{rank}\{\partial^j F_i(I,\varphi,\varepsilon=0)/\partial\varphi^j,\ i=1,\dots,m,\ j\geq 0\}=n,\ (I,\varphi)\in M\times\mathbf{S}^1,$

one has $\overline{K}_0 : \Omega \subset T^*M \to \mathbf{R}$

$$\overline{K}_0(I,p_I) := \frac{1}{2\pi}\int_0^{2\pi}\sqrt{\sum_{i=1}^{m}\langle p_I, F_i(I,\varphi,\varepsilon=0)\rangle^2}\,\mathrm{d}\varphi,$$

positive definite (and 1-homogeneous): $\overline{K}_0$ defines a (symmetric) co-Finsler norm.

Remark. Condition related to the controllability of the original system without F_0. (Lie brackets of $F_1,\dots,F_m$ with $\partial/\partial\varphi$.)

Finsler geometry

Finsler norm. $F : TM \to \mathbf{R}$ smooth on $TM \backslash 0$ s.t.

(i) $F(x, \lambda v) = \lambda F(x, v)$, $\lambda > 0$ (symmetric if $F(x, -v) = F(x, v)$)

(ii) $\partial^2 F^2(x, v)/\partial v^2 > 0$ (tensor depending on v)

Metric. For x and y in M, set $d(x, y) := \inf\ t_f$ on all $\mathscr{C}^1$ curves γ s.t.

$$\gamma(0) = x, \quad \gamma(t_f) = y,$$

$$F(\gamma(t), \dot{\gamma}(t)) \leq 1, \quad t \in [0, t_f].$$

Geodesics. Constant speed curves whose short segments are length minimizing.

Finsler geometry

Co-Finsler norm. $F^*: T^*M \to \mathbf{R}$ smooth on $T^*M\backslash 0$ s.t.

(i) $F^*(x, \lambda p) = \lambda F^*(x,p)$, $\lambda > 0$

(ii) $\partial^2 (F^*)^2(x,p)/\partial p^2 > 0$

Legendre transform. If F^* is a co-Finsler norm, then

$$F(x,v) := \max_{p \text{ s.t. } F^*(x,p) \leq 1} \langle p, v \rangle$$

defines a Finsler norm whose geodesics are integral curves of the Hamiltonian F^* restricted to the level $\{F^* = 1\}$.
One has $F^*(x,p) = F(x,v)$, $v := \ell_x^*(p)$, where $\ell_x^*: T_x^*M \to (T_x^*M)^* \simeq T_xM$ is the Legendre transform

$$\ell_x^* : p \mapsto \frac{1}{2} \frac{\partial^2 (F^*)^2}{\partial p^2}(x,p)(p,\cdot)$$

Slow-fast control system

$$\overline{K} = \overline{H}_0 + \overline{K}_0, \quad \overline{H}_0 = \langle p_I, \overline{F}_0 \rangle,$$

Assuming

(A2) $\overline{K}_0(I, \overline{F}_0^*(I)) < 1$, $I \in M_0$ (inverse Legendre transform of $\overline{F}_0$),

one has $\overline{K} = \overline{H}_0 + \overline{K}_0$ positive definite: $\overline{K}$ defines an asymmetric co-Finsler norm,

$$\overline{H}_0 := \langle p_I, \overline{F}_0 \rangle, \quad \overline{K}_0(I, p_I) := \frac{1}{2\pi} \int_0^{2\pi} \sqrt{\sum_{i=1}^{m} \langle p_I, F_i(I, \varphi, \varepsilon = 0) \rangle^2} \, \mathrm{d}\varphi.$$

Remark. For small $\varepsilon > 0$, this condition is related to the local controllability of the original system, F_0 included. (Ability of $F_1, \dots, F_m$ and their brackets with $\partial/\partial\varphi$ to compensate for the drift F_0.)

Slow-fast control system

Controllability, existence, convergence.

Proposition. For $\varepsilon > 0$ small enough, controllability holds on M_0.

(A3) The metric is geodesically convex on M_0.

(A4) Compactness assumptions (related to singularities).

Proposition. Let I_0 and I_f belong to M_0, $I_f \notin \mathrm{Cut}(I_0)$. For $\varepsilon > 0$ small enough, there exist min. time trajectories connecting I_0 to I_f.

Theorem. Let I_0 and I_f belong to M_0, $I_f \notin \mathrm{Cut}(I_0)$. Let $(I_\varepsilon, \varphi_\varepsilon, p_{I\varepsilon}, p_{\varphi_\varepsilon})_\varepsilon$ be a family of minimizing extremals, and let $(\bar{t}_f(\varepsilon))_\varepsilon$ be the associated min. times. Then $(z_\varepsilon := (I_\varepsilon, p_{I\varepsilon}))$,

$$\|z_\varepsilon - \bar{z}\|_\infty = O(\varepsilon) + O(k(\varepsilon) - 1), \quad \varepsilon \bar{t}_f(\varepsilon) \to d(I_0, I_f), \quad \varepsilon \to 0,$$

where $\bar{z}$ is the lift of the minimizing geodesic between I_0 and I_f.

Application: J2 effect in orbit transfer

Controlled Kepler equation. Work with the French Space Agency (CNES) and Thales Alenia Space. Due to super-integrability of the $-1/|q|$ potential (Kepler), the min. time system

$$\ddot{q} = -\mu \frac{q}{|q|^3} + \frac{u}{M}, \quad |u| \leq T_{\max},$$

is a slow-fast control system with only one fast angle when restricted to transfer of a spacecraft between elliptical orbits.
Averaging $\Longrightarrow$ Analysis of a symmetric Finsler metric in dimension five (no drift)

J2 perturbation. Due to the Earth oblateness, one has to take into account a small drift F_0 on the slow variables (equinoctial elements $I = (a, e, \omega, \Omega, i)$), defined by the J_2 potential $(1/|q|^3)$

$$R_0 = \frac{\mu J_2 r_e^2 \left(1 - e^2\right)^{-3/2}}{|q|^3} \left(\frac{1}{2} - \frac{3}{4}\sin^2 i + \frac{3}{4}\sin^2 i \cos(2\omega + 2 - \varphi)\right)$$

Application: J2 effect in orbit transfer

Reduction to one small parameter. The system has two small parameters (depending on the initial conditions I_0), one for the J_2 effect, one for the control:

$$\varepsilon_0 = \frac{3}{2} J_2 \frac{r_e^2}{a_0^2}, \quad \varepsilon_1 = \frac{a_0^2 \, T_{\max}}{\mu M}.$$

r_e: Earth equatorial radius, a_0: Initial semi-major axis, $T_{\max}$: Maximum thrust level, M: Mass of the spacecraft

Write

$$\begin{aligned}
\frac{\mathrm{d}I}{\mathrm{d}t} &= \varepsilon_0 F_0(I,\varphi) + \varepsilon_1 \sum_{i=1}^{m} u_i F_i(I,\varphi), \\
&= \varepsilon \left(\lambda F_0(I,\varphi) + (1-\lambda) \sum_{i=1}^{m} u_i F_i(I,\varphi) \right),
\end{aligned}$$

with $\varepsilon := \varepsilon_0 + \varepsilon_1$ and $\lambda := \varepsilon_0 / (\varepsilon_0 + \varepsilon_1)$.

Application: J2 effect in orbit transfer

Critical ratio. Two regimes depending on whether

(i) the J_2 effect is small compared to the control ($\varepsilon_0 \ll \varepsilon_1$ and $\lambda \to 0$),

(ii) the J_2 effect dominates the control ($\varepsilon_0 \gg \varepsilon_1$ and $\lambda \to 1$).

In terms of the averaged system,

$$\overline{K} = \lambda \overline{H}_0 + (1-\lambda)\overline{K}_0$$

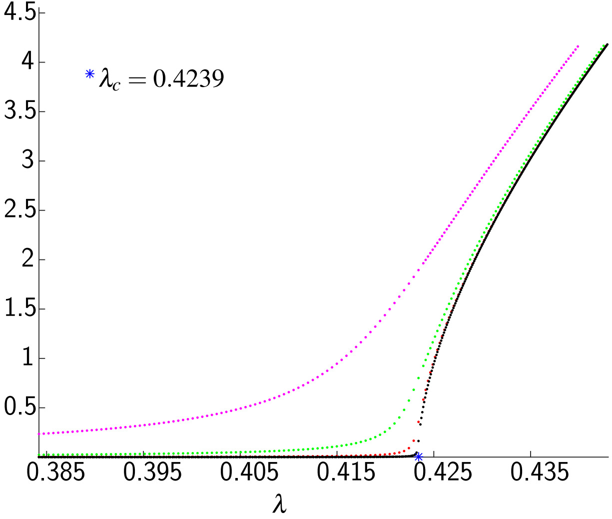
is a metric if and only if $\lambda < \lambda_c(I)$ with (*cf.* condition (A2))

$$\lambda_c(I) = \frac{1}{1+\overline{K}_0(I,\overline{F}_0^*(I))}.$$

$\Longrightarrow$ Relevance of this critical ratio for the qualitative analysis of the original system?

Application: J2 effect in orbit transfer

Qualitative analysis on the min. time. Curve $\lambda \mapsto \tau_f(\lambda)$, $\tau_d \to 0$



Example : $a = 30$ Mm, $e = 0.5$, $\omega = \Omega = 0$, $i = 51$ deg (inclined medium orbit)

Application: J2 effect in orbit transfer

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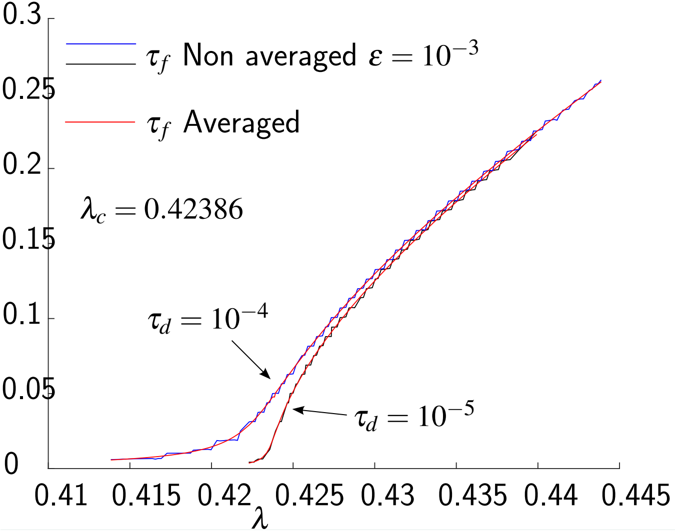

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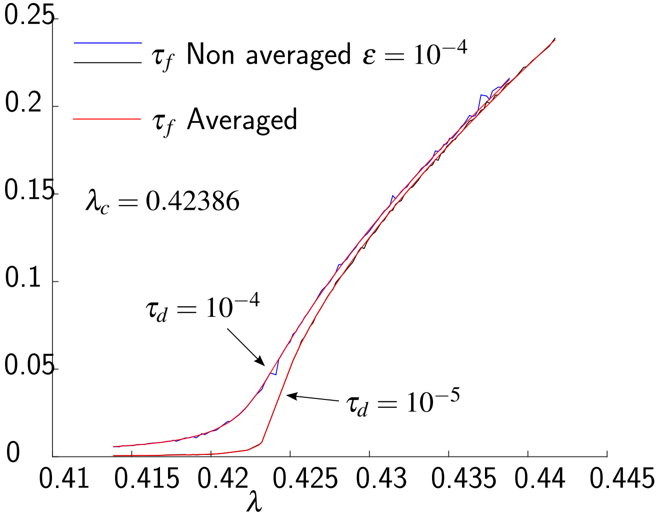

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Conclusion and ongoing work

- Well defined relevant limiting approximation: Asymmetric Finsler metric

  Caillau, J.-B.; Pomet, J.-B.; Rouot, J. Metric approximation of minimum time control systems. Preprint (2017).

- Complementary approaches (1/2): Filtering

  Caillau, J.-B.; Dargent, T.; Nicolau, F. Approximation by filtering in optimal control and applications. *IFAC PapersOnLine* **50** (2017), no. 1, 1649–1654. Proceedings of the 20th IFAC world congress, Toulouse, July 2017.

- Complementary approaches (2/2): Higher order averaging

  Dargent, T.; Pomet, J.-B.; Nicolau, F. Periodic averaging with a second order integral error. *IFAC PapersOnLine* **50** (2017), no. 1, 2892–2897. Proceedings of the 20th IFAC world congress, Toulouse, July 2017.

- Ongoing work: Multi-phase averaging for min. time

  Dell'Elce, L.; Caillau, J.-B.; Pomet, J.-B. Optimal low-thrust orbital transfer by averaging multiple frequencies. *7th International Meeting on Celestial Mechanics (CELMEC)*, San Martino, September 2017.

  PEPS MSI (UCA) with Thales Alenia Space and Inria