MASS TRANSPORTATION ON SUB-RIEMANNIAN STRUCTURES ADMITTING SINGULAR MINIMIZING GEODESICS

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Monge Transport Problem

Let M be a manifold of dimension $n \geq 2$. Let μ, ν be two probability measures compactly supported on M.

Definition 1 (Transport map)

A transport map between μ and ν is a measurable application $T : M \to M$ such that $T_\sharp \mu = \nu$,

i.e. $\nu(B) = \mu(T^{-1}(B)), \forall B \subset M$ *measurable*.

We consider the cost function given by $c(x, y), \forall x, y \in M$.

The Monge transport problem: Finding a transport map $T : M \to M$ minimizing the following

$$\int_M c(x, T(x)) \mathrm{d}\mu(x).$$

Kantorovich transport problem

Definition 2 (Transport plan)

A transport plan between μ and ν is a probability measure α on the product $M \times M$ such that

$$\alpha(A \times M) = \mu(A), \forall A \subset M \quad \alpha(M \times B) = \nu(B), \ \forall B \subset M.$$

$\Pi(\mu, \nu)$ is the set of transport plans between μ and ν.

The Kantorovich transport problem: Finding a transport plan $\alpha \in \Pi(\mu, \nu)$ minimizing the following

$$\int_{M \times M} c(x, y) \mathrm{d}\alpha(x, y).$$

Existence of optimal transport plans

Relaxation form

Let $T : M \to M$ be a transport map between μ and ν. The measure α defined as

$$\alpha := (id \times T)_\sharp \mu$$

is a transport plan between μ et ν.

Theorem 3

Let μ,ν be two probability measures compactly supported on M. Assume that c is continuous on $M \times M$. Then, there is at least one optimal transport plan, solution of the Kantorovich problem.

Dual problem

Dual problem : Finding $(\varphi, \varphi^c) \in L^1(\mu) \times L^1(\nu)$ such that

$$\begin{aligned} \varphi(x) &= \sup_{y \in M} \{\varphi^c(y) - c(x,y)\}, \quad \forall x \in M \\ \varphi^c(y) &= \inf_{x \in M} \{\varphi(x) + c(x,y)\}, \quad \forall y \in M \end{aligned}$$

and $$\sup_{\varphi^c(y) - \varphi(x) \leq c(x,y)} \left\{ \int_M \varphi^c(y) \mathrm{d}\nu(y) - \int_M \varphi(x) \mathrm{d}\mu(x) \right\}$$

- (φ, φ^c) is called Kantorovich potentials
- $\Gamma_\varphi := \left\{ (x,y) \in M \times M |\ \varphi^c(y) - \varphi(x) = c(x,y) \right\}$

Proposition

A transport plan $\alpha \in \Pi(\mu, \nu)$ is optimal $\Leftrightarrow$ supp $\alpha \subset \Gamma_\varphi$.

Sub-Riemannian structures

Let M be a smooth connected manifold of dimension n.

A **sub-Riemannian structure** on M of rank m is a pair (Δ, g) where

- Δ is a totally nonholonomic distribution of rank m $(m < n)$ on M, i.e. $\forall x \in M$, $\exists\ \mathcal{V}_x$ an open neighborhood of x and $\{X_x^1, \dots, X_x^m\}$ a local frame of Δ on $\mathcal{V}_x$ s.t.

$$Lie\{X_x^1, \dots, X_x^m\}(y) = T_y M, \forall y \in \mathcal{V}_x.$$

- g is a smooth Riemannian metric on Δ.

Minimizing geodesics

- An absolutely continuous path $\gamma : [0,1] \to M$ is said to be **horizontal** if
$$\dot{\gamma}(t) \in \Delta_{\gamma(t)}, a.e.\ t \in [0,1].$$
- **Sub-Riemannian distance:** $\forall x, y \in M$,
$d_{SR}(x,y) = \inf \left\{ length(\gamma) |\ \gamma \text{ horizontal path joining } x, y \right\}$
- A **minimizing geodesic** is a minimizing horizontal path with constant speed.
- **Hopf-Rinow Theorem:** Assume that (M, d_{SR}) is complete. Then, there is at least a minimizing geodesic joining two points in M.

The Monge quadratic sub-Riemannian problem

Let M be a manifold of dimension n equipped with a complete sub-Riemmanian structure (Δ, g) of rank m $(m < n)$.

Finding a transport map $T : M \to M$ minimizing the following

$$\int_M c(x, T(x)) \mathrm{d}\mu(x)$$

with

$$c(x, y) := d_{SR}^2(x, y), \forall (x, y) \in M \times M.$$

Previous results

Several results of [AR'04], [AL'09] et [FR'10] proved existence and uniqueness of solution for the Monge problem on sub-Riemannian structures **not admitting singular minimizing geodesics.**

A new technic [CH'14] solved the Monge problem on spaces satisfying **a measure contraction property**.

End-point map

Let $k = n(m+1)$ and $\{X^1, \ldots, X^k\}$ a global frame of Δ such that

$$\forall x \in M, \Delta(x) = Span\{X^1(x), \ldots, X^m(x)\}.$$

The End-point map End^x at $x \in M$ is defined by:

$$\begin{array}{rccl} End^x : & L^2([0,1], \mathbb{R}^k) & \to & M \\ & u & \mapsto & End^x(u) = \gamma_u(1) \end{array}$$

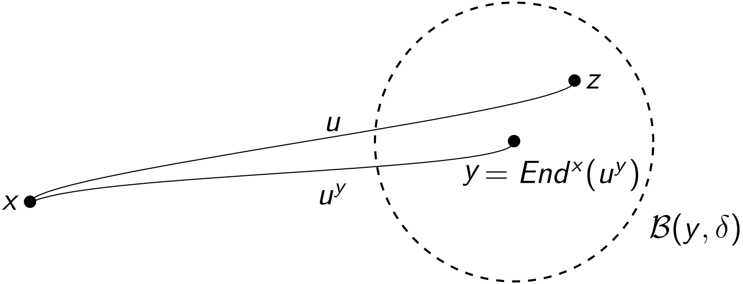
where $\gamma_u : [0,1] \to M$ is the unique solution of the following Cauchy problem

$$\left\{ \begin{array}{rcl} \dot{\gamma_u}(t) & = & \sum_{i=1}^{k} u_i(t) X^i(\gamma_u(t)); \\ \gamma_u(0) & = & x. \end{array} \right.$$

Singular horizontal curves

Definition 4

Given $x \in M$. An horizontal path γ_u starting at x is said to be singular if the control u associated to γ_u is a critical point of the End-point map End^x. Otherwise, we say that γ_u is regular.



$$\exists k > 0; \ ||u - u^y||_{L^2} \leq k \ |z - y|$$

PART I

Mass Transportation on sub-Riemannian structures of rank 2 in dimension 4

Existence and uniqueness of solutions

Theorem 5

Let M be an analytic manifold of dimension 4 and (Δ, g) be a complete analytic sub-Riemannian structure on M of rank 2 such that

$$\forall x \in M, \Delta(x) + \Big[\Delta, \Delta\Big](x) \text{ is of dimension 3.} \qquad (1)$$

Let μ,ν be two probability measures compactly supported on M such that μ is absolutely continuous with respect to the Lebesgue measure.

Then, there is a unique transport map from μ to ν for the sub-Riemannian quadratic cost d_{SR}^2.

Sketch of the proof

Let $c = d_{SR}^2$ be the cost function. Let μ,ν be two probability measures compactly supported on M.
We consider $\alpha \in \Pi(\mu, \nu)$ an optimal transport plan and (φ, φ^c) the Kantorovich potentials such that

$$supp(\alpha) \subset \Gamma_\varphi := \{(x, y)| \; \varphi^c(y) - \varphi(x) = c(x, y)\}.$$

$$\forall x \in M, \; \Gamma_\varphi(x) := \{y \in M \mid (x, y) \in \Gamma_\varphi\}.$$

Definition 6

The static set is $\mathcal{S} := \{x \in M \mid x \in \Gamma_\varphi(x)\}$

The mobile set is $\mathcal{M} := \{x \in M \mid x \notin \Gamma_\varphi(x)\}$

Static set

Proposition 7(Figalli-Rifford '10)

μ- a.e. $x \in \mathcal{S}$, we have $\Gamma_\varphi(x) = \{x\}$.

Sketch of the proof

- $\{X^1, X^2\}$ an orthonormal local frame of Δ.
- $\forall z \in M$, $\varphi(z) = \sup_{y \in M}\{\varphi^c(y) - d_{SR}^2(z,y)\}$ locally Lipschitz with respect to d_{SR}.
- **Pansu-Rademacher Theorem:** φ is differentiable a.e. with respect to X^1, X^2.
- $x \in \Gamma_\varphi(x) \Rightarrow \forall i = 1,2$, $X^i\varphi(x) = 0$, a.e. $x \in M$.

Mobile set

Let $x \in \mathcal{M}$ and $y \in \Gamma_\varphi(x)$, we can find singular or regular minimizing geodesics joining x and y.

Definition 8

$$\Gamma^S(x) := \left\{ y \in \Gamma_\varphi(x) | \exists \text{singular min. geo. joining } x, y \right\}$$

$$\Gamma^R(x) := \left\{ y \in \Gamma_\varphi(x) | \exists \text{regular min. geo. joining } x, y \right\}$$

$\mathcal{M}^S := \{x \in \mathcal{M} | \Gamma^S(x) \neq \emptyset\}$ and $\mathcal{M}^R := \{x \in \mathcal{M} | \Gamma^R(x) \neq \emptyset\}$

Regularity of d_{SR}

Let $x \in \mathcal{M}^R$ and $y \in \Gamma^R(x)$.

- $\exists$ open set $\mathcal{B}_x$ in $\mathbb{R}^4$ containing x and a function $\phi^{x,y} : B_x \to \mathbb{R}$ of class C^2 such that

$$\phi^{x,y}(z) \geq d_{SR}^2(z,y), \forall z \in \mathcal{B}_x \text{ and equality for } z = x.$$

- $\mathcal{M}^R \subset \bigcup_{k \in \mathbb{N}} W_k$

  où

$$W_k := \Big\{ x \in \mathcal{M} \,|\, \exists p_x \in \mathbb{R}^4; |p_x| \leq k \text{ and}$$
$$\varphi(x) \leq \varphi(z) - \langle p_x, x - z \rangle + k\,|x - z|^2,\ \forall z \in \bar{B}(x, 1/k) \Big\}$$

- We extend φ to a locally semiconcave function $\tilde{\varphi}$ on an open neighborhood $\mathcal{O}_x$ of x.

- We set

$$\rho^{x,y}(z) := \tilde{\varphi}(x) + d_{SR}^2(x,y) - \tilde{\varphi}(z), \forall z \in \mathcal{O}_x$$

such that $\rho^{x,y}$ is differentiable a.e. on $\mathcal{O}_x$, and

$$\rho^{x,y}(z) \leq d_{SR}^2(z,y), \forall z \in \mathcal{O}_x \text{ and equality } z = x.$$

- $\exists$ an open neighborhood $\mathcal{V}_x$ of x such that

$$\rho^{x,y}(z) \leq d_{SR}^2(z,y) \leq \phi^{x,y}(z), \forall z \in \mathcal{V}_x \text{ and equality } z = x.$$

Conclusion: there is a unique $y \in \Gamma^R(x)$ such that

$$y = exp_x(d_x\phi^{x,y}) = exp_x(d_x\rho^{x,y}) = exp_x(-d_x\tilde{\varphi}),$$

where exp_x is the exponential map at x.

Singular minimizing geodesics

We consider $M = \mathbb{R}^4$.

lemme 10

There is an open set $\mathcal{H}$ of full measure on M such that:
$\forall x \in \mathcal{H}, \;\; T_xM = \Delta(x) + [\Delta, \Delta](x) + [\Delta, [\Delta, \Delta]](x)$.

lemme 11

On $\mathcal{H}$, there is a horizontal vector field X such that singular horizontal paths are exactly the trajectories described by the flow of X.
Moreover, for any compact K of M, $\exists C > 0$;

$$|div_x X| \leq C|X(x)|, \;\forall x \in K.$$

Vector field X and its properties

$\forall z \in M, \Delta(z) = Span\{X^1(z), X^2(z)\}$ such that

$$\forall x \in M,\ \Delta(x) + [\Delta, \Delta](x) \text{ is of dimension } 3.$$

Characterization of singular curves: Let $\gamma_u : [0,1] \to M$ be a singular horizontal curve associated to $u \in L^2([0,1], \mathbb{R}^2)$. In local coordinates, $\exists\ p : [0,1] \to (\mathbb{R}^4)^* \backslash \{0\}$ satisfying $\forall i = 1, 2$:

$$\dot{p}(t) = -\sum_{i=1}^{m} u_i(t) p(t).D_{\gamma_u(t)} X^i,\ \ a.e.\ t \in [0,1]$$

$$p(t).X^1(\gamma_u(t)) = p(t).X^2(\gamma_u(t)) = 0,\ \ \forall t \in [0,1]$$

$$X = \alpha_1 X^1 + \alpha_2 X^2$$

$$\forall K \subset M \text{ compact}, \exists C > 0; |div_x X| \leq C|X(x)|,\ \ \forall x \in K$$

Contraction property

For $\varepsilon \in \{-1,+1\}$ fixed, let $(\varphi^X_{\varepsilon t})$ be the flow of X.
For any $K \subset M$ compact,

$$K_t = \varphi^X_{\varepsilon t}(K),\ \forall t \in [0,1] \text{ et } K_0 = K.$$

$$\begin{aligned}\mathcal{L}^4(K_t) &= \int_K exp\Big(\int_0^t div\ X(\varphi^X_s(x))\ \mathrm{d}s\Big)\mathrm{d}x \\ &\geq \int_K exp\Big(-C\int_0^t |X(\varphi^X_{\varepsilon s}(x))|\ \mathrm{d}s\Big)\mathrm{d}x\end{aligned}$$

$$I(K,K_t) := \sup_{x\in K}\int_0^t |X(\varphi^X_{\varepsilon s}(x))|\ \mathrm{d}s$$

For any $K \subset M$ compact,

$$\mathcal{L}^4(K_t) \geq \exp(-C\ I(K,K_t))\mathcal{L}^4(K), \forall t \in [0,1].$$

What happen on $\mathcal{H}^c$?

$$\mathcal{H}^c = \{x \in M | \Delta(x) + [\Delta, \Delta](x) + [\Delta, [\Delta, \Delta]](x) \neq T_x M\}.$$

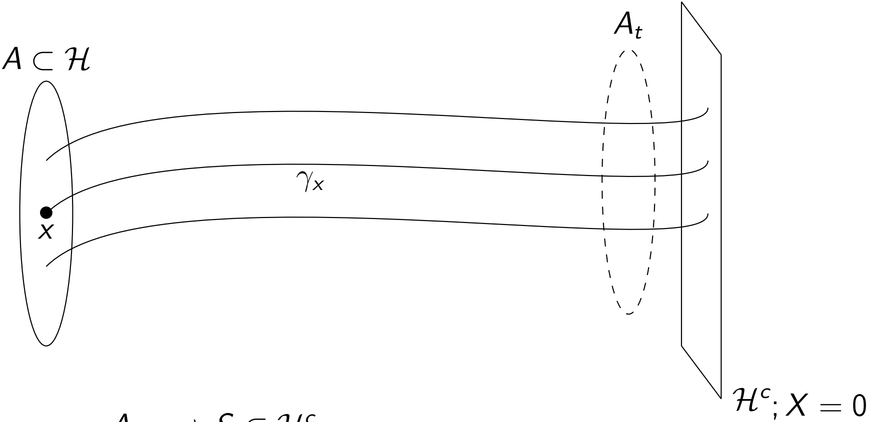
Lemma 12

Let A be a compact of $\mathcal{H}$ such that $\mathcal{L}^4(A) > 0$. The set

$$\{x \in A | \ \exists \gamma : [0, 1] \to M \text{ singular with } \gamma(1) \in \mathcal{H}^c\}$$

is of Lebesgue measure zero.

Sketch of the proof



$$A_t \underset{t\to\infty}{\longrightarrow} S \subset \mathcal{H}^c$$

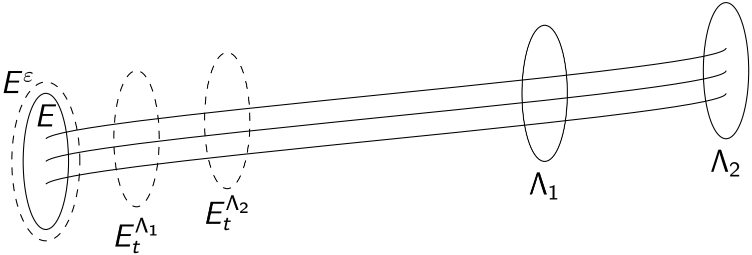
$$\mathcal{L}^4(A_t) \geq \; exp\Big(-CL\Big) \; \mathcal{L}^4(A), \;\; \forall t \in [0,1].$$

$$\Rightarrow \mathcal{L}^4(A) = 0.$$

Existence and uniqueness of solution on $\mathcal{M}^S$

We set

$$E := \{x \in \mathcal{M}^S | \Gamma^S(x) \text{ is not a singleton}\} \ .$$



$$\forall i = 1, 2, \ \forall t \in [0, 1],$$
$$E_t^{\Lambda_i} := \{\varphi_t^X(x) | \ \varphi_0^X(x) \in E, \ \varphi_\varepsilon^X(x) \in \Lambda_i\}.$$

$$
\begin{aligned}
\mathcal{L}^4(E) &= \limsup_{\varepsilon\to 0} \mathcal{L}^4(E^\varepsilon) \\
&\geq \lim_{t\to 0} \mathcal{L}^4(E_t^{\Lambda_1} \cup E_t^{\Lambda_2}) \\
&= \lim_{t\to 0} \mathcal{L}^4(E_t^{\Lambda_1}) + \mathcal{L}^4(E_t^{\Lambda_2}) \\
&\geq \lim_{t\to 0} \quad exp(-C\ l(E, E_t^{\Lambda_1}))\mathcal{L}^4(E) \\
&\qquad\qquad + exp(-C\ l(E, E_t^{\Lambda_2}))\mathcal{L}^4(E)
\end{aligned}
$$

We can choose $l(E, E_t^{\Lambda_i}) > 0$ sufficiently small,

$$exp(-C\ l(E, E_t^{\Lambda_i}) > \frac{1}{2}$$

$\Rightarrow \mathcal{L}^4(E) = 0.$

What happen on $\mathcal{M}^S \cap \mathcal{M}^R$?

Lemma 13

$\mathcal{M}^R \cap \mathcal{M}^S$ has Lebesgue measure zero.

Sketch of the proof

Assume that there is A of $\mathcal{M}^R \cap \mathcal{M}^S$ of positive measure.

- $A_t^R := \left\{\gamma_x(t) |\ \gamma_x \text{ regular with } x \in A \text{ and } \gamma_x^R(1) \in \Gamma^R(x)\right\}$,
  $\rightarrow$ measure contraction property
- $A_t^S := \varphi_{\varepsilon t}^X(A)$.
  $\rightarrow$ measure contraction property

As the geodesics are ANALYTICS, there is $\bar{t} \in]0,1[$ such that

$$A_s^R \cap A_s^S = \emptyset,\ \forall s \in]\bar{t}, 1[\text{ and } \bar{A} := A_{\bar{t}}^R \cap A_{\bar{t}}^S \neq \emptyset$$

PART II

Study of h-semiconcavity and MCP on Carnot groups

Carnot groups

- Let $\mathcal{G}$ be a Carnot group of dimension n of step r such that $\mathfrak{g} = V_1 + \cdots + V_r$.
- We consider V_1 of dimension m and $V_1 = Span\{X^1, \ldots, X^m\}$.
- We define the exponential map $\exp : \mathfrak{g} \to \mathcal{G}$ and its inverse $\begin{array}{cccc} \xi : & \mathcal{G} & \longrightarrow & \mathfrak{g} \\ & g & \longmapsto & \xi(g) = \xi_1(g) + \cdots + \xi_r(g) \end{array}$ such that $\xi_i : \mathcal{G} \to V_i$, for $i = 1, \ldots, r$.
- For $\lambda > 0$, family of dilations

$$\delta_\lambda(g) = \exp \circ \Delta_\lambda \circ \exp^{-1}(g), \ \forall g \in \mathcal{G}$$

where $\Delta_\lambda : \mathfrak{g} \to \mathfrak{g}$ is defined by $\Delta_\lambda(v_1 + \cdots + v_r) = \lambda v_1 + \cdots + \lambda^r v_r$.

h-semiconcavity on Carnot groups

- $\forall g \in \mathcal{G}$, $\mathcal{H}_g = \left\{ g' \in \mathcal{G} : \ g' = gh \text{ with } h \in \exp(V_1) \right\}$.
- $\forall g \in \mathcal{G}$, $\forall g' \in \mathcal{H}_g$, $\lambda \in [0,1] \mapsto g'\delta_\lambda(g^{-1}g') \in \mathcal{H}_g$ horizontal segment between g, g'.

Definition 14

We say that a function $f : \mathcal{G} \to \mathbb{R}$ is h-semiconcave on $\mathcal{G}$ if it is semiconcave on every horizontal segment, that is, there exists $C > 0$ such that

$$\lambda f(g') + (1-\lambda) f(g) \leq f\left(g\delta_\lambda(g^{-1}g')\right) + \lambda(1-\lambda) C |\xi_1(g') - \xi_1(g)|^2$$

$\forall g \in \mathcal{G}, \forall g' \ \in \mathcal{H}_g, \forall \lambda \in [0,1]$.

Properties of h-semiconcave functions

Let $f : \mathcal{G} \to \mathbb{R}$ be an h-semiconcave function with C as h-semiconcavity constant.

Theorem 15

- f is Lipschitz with respect to d_{SR}.
  Pansu-Rademacher Theorem: f is differentiable a.e. with respect to $X^i, i = 1, \ldots, m$.
- f is twice differentiable a.e. with respect to $X^i, i = 1, \ldots, m$.

Proposition 16

For any $g \in \mathcal{G}$, $(\nabla_h^2 f)^*(g) := \left[\frac{X^i X^j f + X^j X^i f}{2}(g)\right]_{i,j} \leq C I_m$ where I_m denotes the identity matrix $m \times m$

Let $\mathcal{G}$ be a Carnot group of dimension n whose first layer V_1 has dimension m such that $V_1 = Span\{X^1, \ldots, X^m\}$.

Definition 17

We say that **a sub-Riemannian structure is h-ideal** if it is complete and the sub-Riemannian distance d_{SR} is h-semiconcave on $\mathcal{G} \times \mathcal{G}$ outside the diagonal.

Up to a change of coordinates, we may assume that

$$X^1 = \partial_{x_1}, \text{ and } X^i = \partial_{x_i} + \sum_{j=m+1}^{n} \alpha_i^j \partial_{x_j}, \forall i = 2, \ldots, m$$

with $\alpha_i^j \in C^\infty(M)$.

ASSUMPTION 1

$$\forall i = 1, \ldots, m, X^i(\alpha_i^j) = 0, \forall j = m+1, \ldots, n.$$

MCP on Carnot groups

Proposition 18

Let $\mathcal{G}$ be a Carnot group whose first layer is h-ideal and satisfies ASSUMPTION 1. Then, there is $N > 0$ such that for every $k \in \mathbb{N}$ and for every measurable set

$$A \subset B_{SR}(0, \frac{1}{2^k}) \backslash B_{SR}(0, \frac{1}{2^{k+1}})$$

with $0 < \mathcal{L}^n(A) < +\infty$, we have

$$\mathcal{L}^n(A_s) \geq s^N \mathcal{L}^n(A), \ \forall s \in [0,1]$$

where

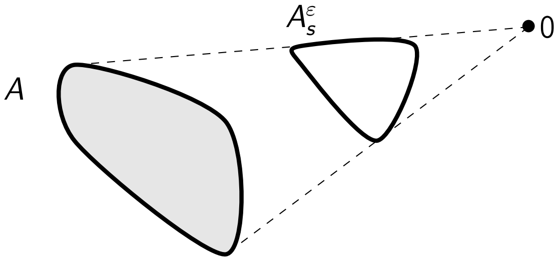
$$A_s := \Big\{ \gamma(s) | \ \gamma : [0,1] \to \mathcal{G} \text{ min. geo.}, \ \gamma(0) = 0, \gamma(1) \in A \Big\}.$$

Sketch of the proof

We may assume that $\mathcal{G} = \mathbb{R}^n$.

- $\forall g \in \mathcal{G}$, $f(g) := d_{SR}(0, g)$ and $Z(g) := -\nabla_h f(g)$.
- For $\varepsilon > 0$, $f_\varepsilon = \phi_\varepsilon * f$ smooth and h-semiconcave functions on $\mathcal{G} \setminus \{0\}$.
- For $g \in \mathcal{G}$, we define $\mathcal{Z}_\varepsilon(g) := -\nabla_h f_\varepsilon(g)$.
- For $A \subseteq B_{SR}(0,1) \setminus B_{SR}(0,1/2)$, we denote by

$$A_s^\varepsilon := \varphi_{1-s}^{\mathcal{Z}_\varepsilon}(A), \ \forall s \in [1/2, 1]$$



- $$div \mathcal{Z}_\varepsilon(g) =$$
  $$-E - \sum_{i=2}^{m} \sum_{l=m+1}^{n} X^i(\alpha_i^l)\partial_{x_l} f_\varepsilon - \sum_{i=1}^{m} X^i f_\varepsilon(g) \ div \ X^i(g)$$
  where E is given in terms of $(\nabla_h f_\epsilon)^*(g)$.
- $\exists \ C > 0,$
  $$div \mathcal{Z}_\varepsilon(g) \geq C, a.e. \ g \in A.$$
- $$\frac{d}{dt}\Big\{\mathcal{L}^n(A^\varepsilon_{1-t})\Big\} \geq -\int_{\varphi_t^{\mathcal{Z}_\varepsilon}(A)} C \mathrm{d}x = -C\mathcal{L}^n(A^\varepsilon_{1-t}).$$

- **Gronwall Theorem:** there is $N > 0$ such that

$$\mathcal{L}^n(A^{\varepsilon}_{1-t}) \geq t^N \mathcal{L}^n(A), \ \forall t \in [1/2, 1].$$

- $\varepsilon \to 0$,

$$\mathcal{L}^n(A_{1-t}) \geq t^N \mathcal{L}^n(A), \ \forall t \in [1/2, 1].$$

- By dilations properties, for every $k \in \mathbb{N}$, we have

$$\delta_{2^k}(A) \subset B_{SR}(0,1) \backslash B_{SR}(0,1/2)$$

$$\delta_{2^k}(A_s) = \big(\delta_{2^k}(A)\big)_s, \ \forall s \in [0,1].$$

- For any $A \subset B_{SR}(0, 1/2^k) \backslash B_{SR}(0, 1/2^{k+1})$, $\exists N > 0$ such that

$$\mathcal{L}^n(A_s) \geq s^N \mathcal{L}^n(A), \ \forall s \in [0,1]$$

Perspectives

- Examples of h-ideal sub-Riemannian structures on Carnot groups ??
- Proof of the MCP property on more general sub-Riemannian structures ??

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Thank you for your attention !!