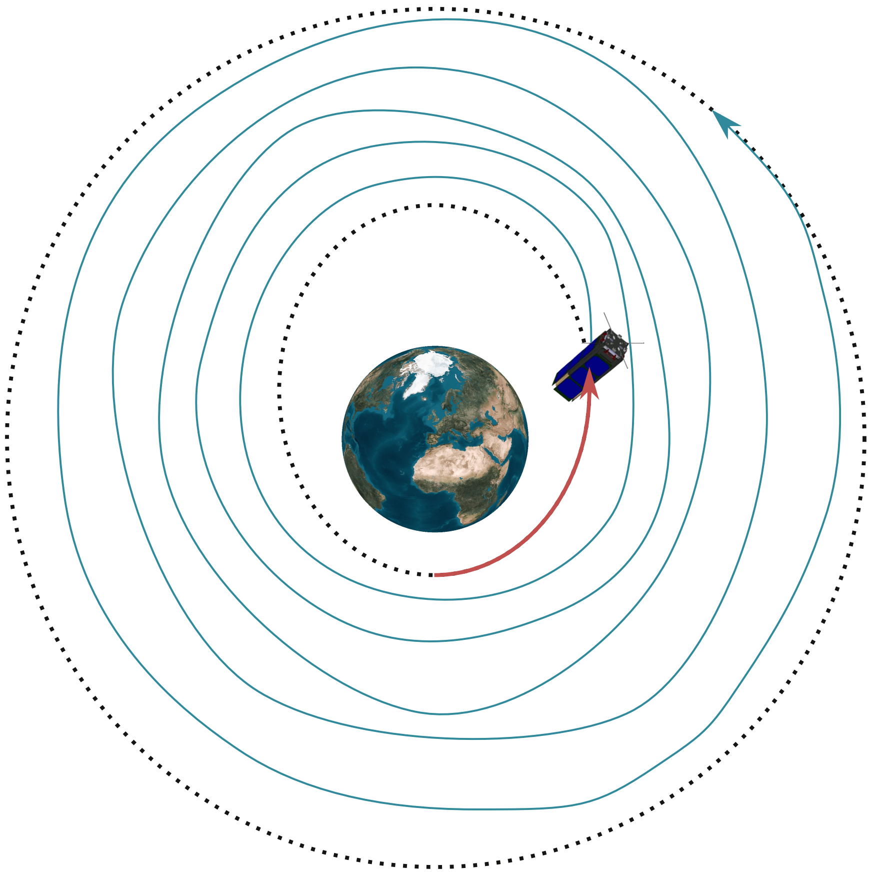
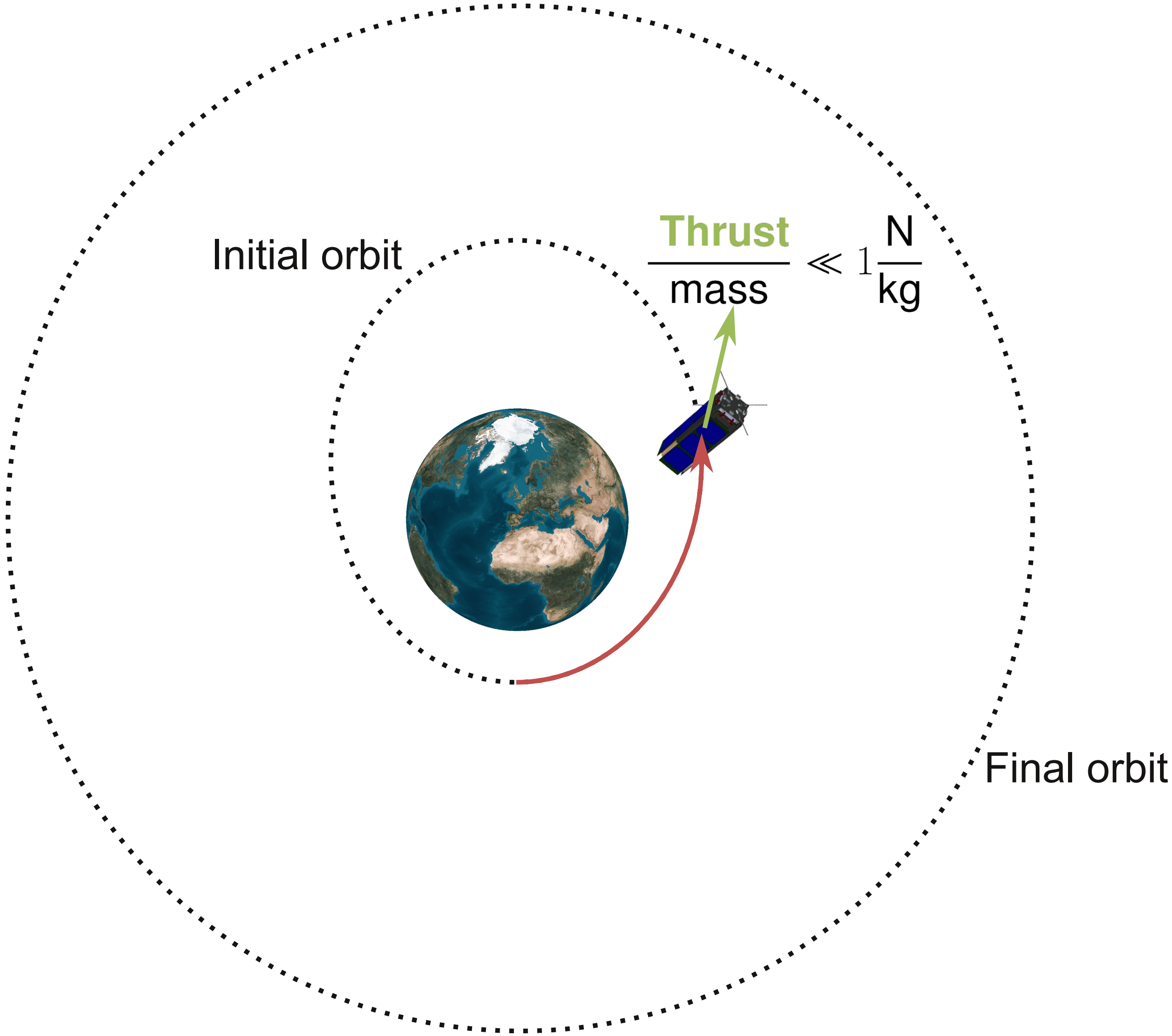
Two-Frequency Averaging of Optimal Control Problems with Application to Time-Optimal Orbital Transfer



L. Dell'Elce, J.B. Caillau, and J.B. Pomet

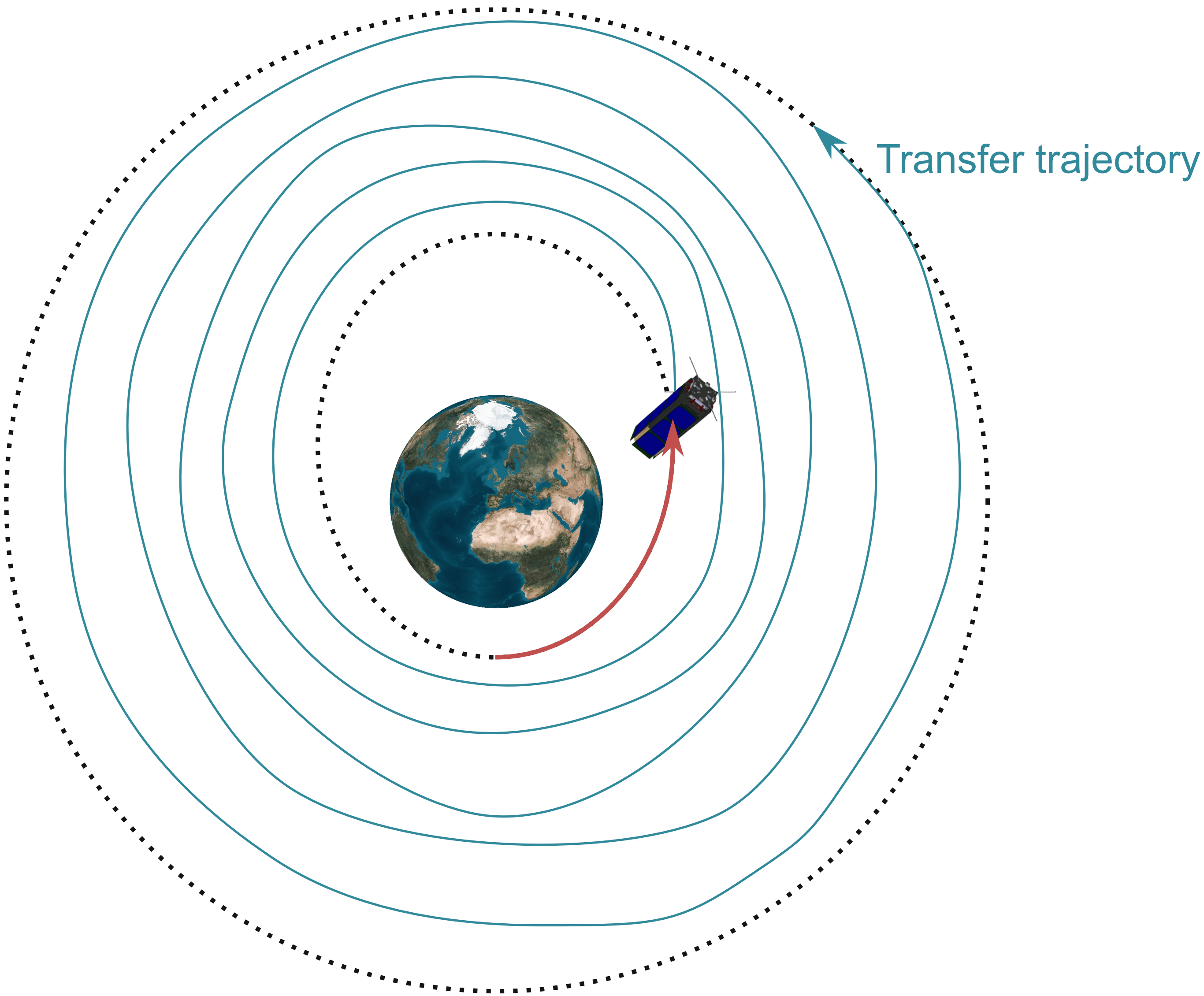
McTAO Days, Dijon, 05/12/2017

Low-thrust transfer: a recent problem in astrodynamics



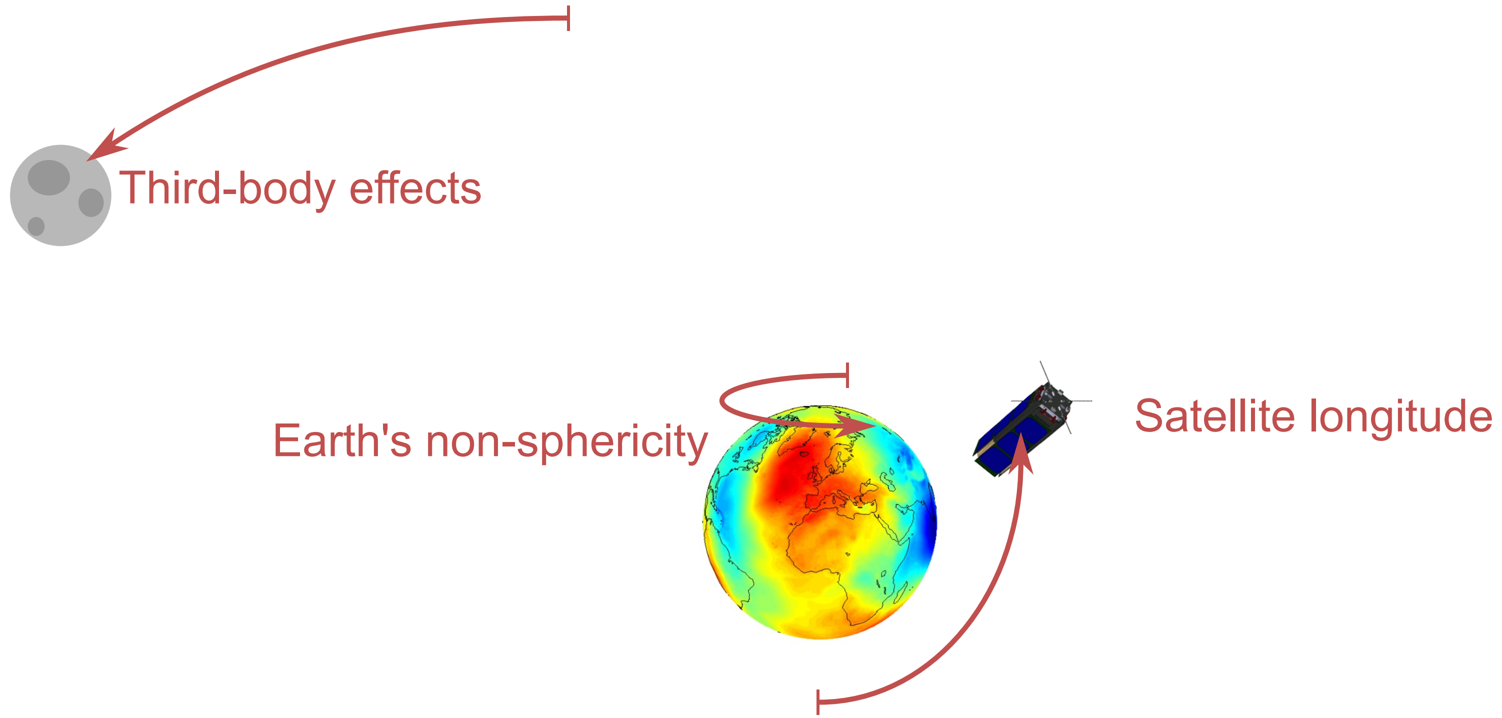
Fast motion of the satellite on a slowly-varying orbit

Orbital period $\ll$ Maneuvering time



What about the dynamics of **adjoint variables**?

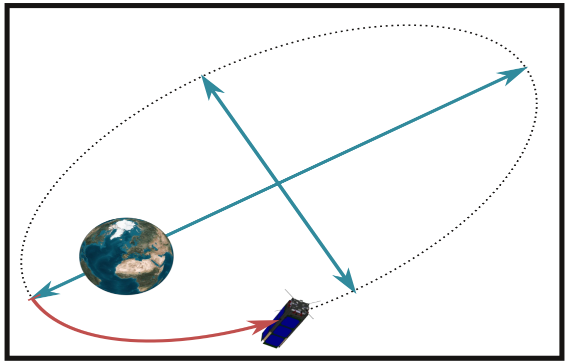
Orbital perturbations may introduce new frequencies



Challenges: Do adjoint variables introduce additional fast dynamics?

Is simple averaging enough when resonances are crossed?

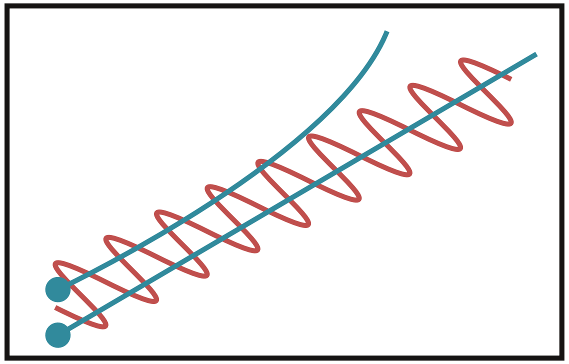
Outline



1. Dynamical systems with slow & fast dynamics

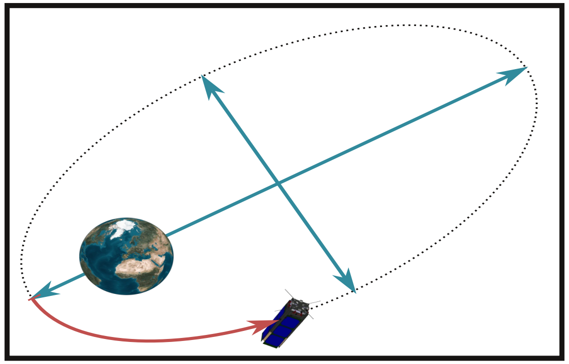
$$\min_{u \in \mathcal{U}} \mathcal{H}\left(p, q, u\right)$$

2. Averaging the two-phase optimal control problem



3. Near identity transformation of the initial state

Outline



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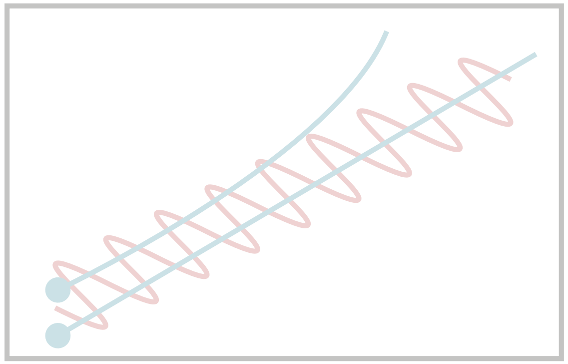

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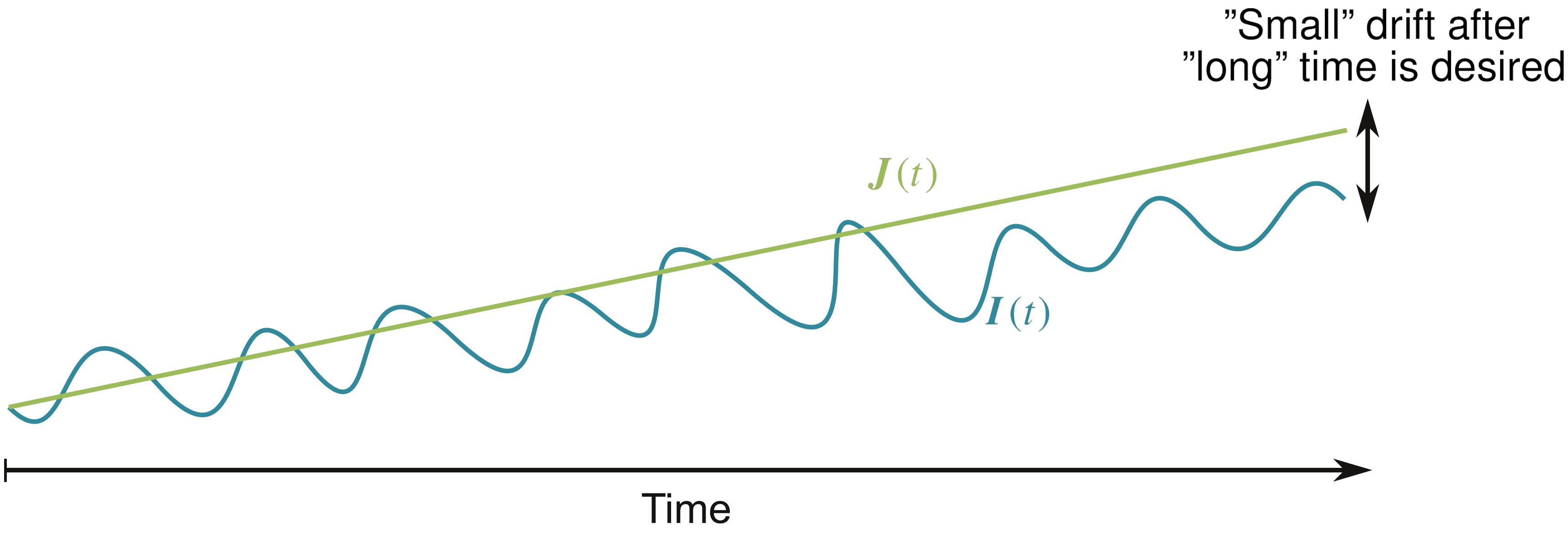
Averaging: an effective way to remove fast dynamics



Original system

$$\dot{I} = \epsilon f(I, \phi, \epsilon)$$
$$\dot{\phi} = \epsilon g(I, \phi, \epsilon) + \omega(I)$$

➡

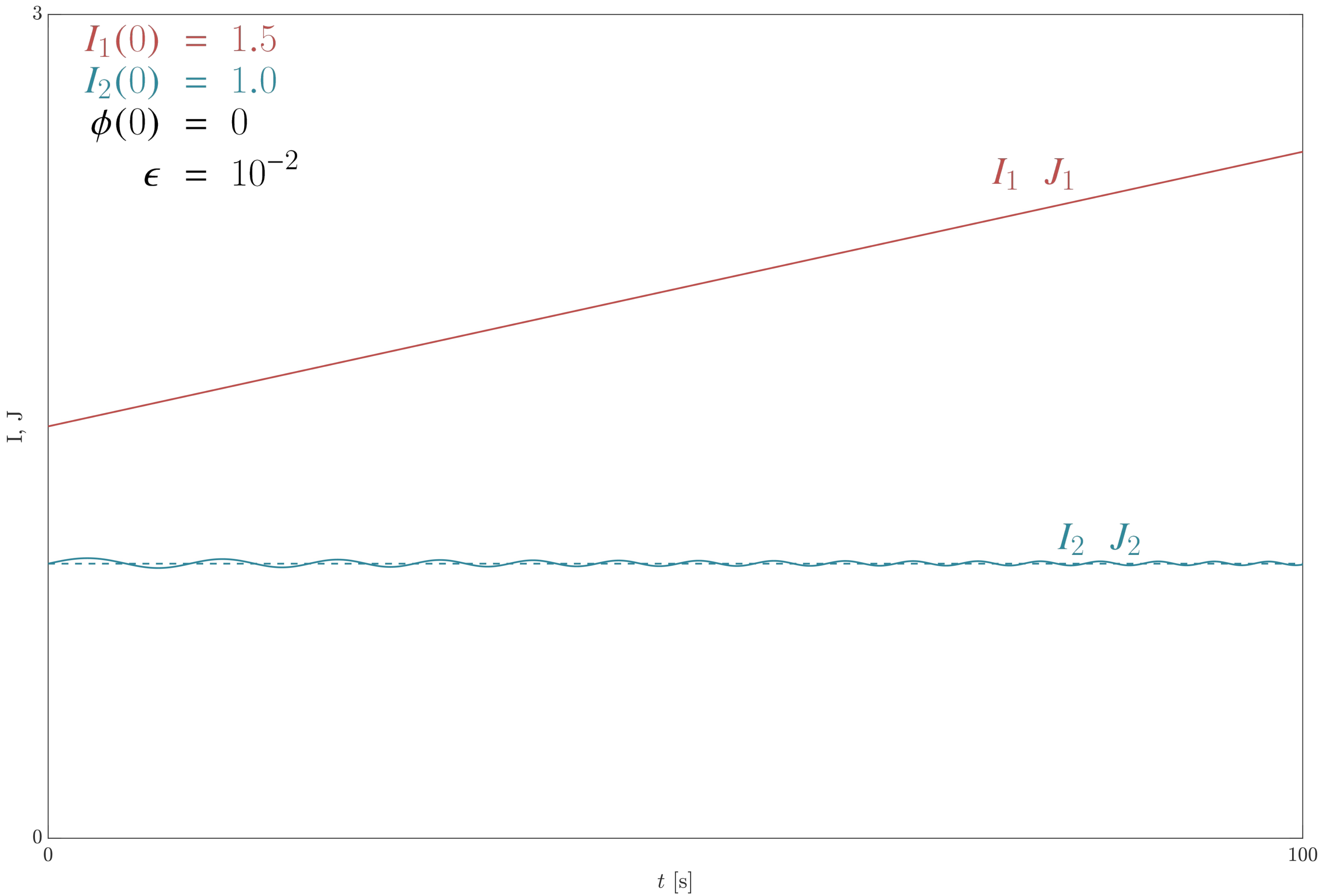
Averaged system

$$\dot{J} = \epsilon \bar{f}(J)$$
$$\bar{f}(J) := \int_{\mathbb{T}^r} f(J, \phi, 0)\, d\phi$$

f, g periodic in ϕ, $\epsilon \ll 1$ Slow variables: $I \subset \mathbb{R}^n$ Fast variables: $\phi \subset \mathbb{T}^r$

Is averaging compatible with resonance crossing?

$$\dot{I}_1 = \epsilon,\ \dot{I}_2 = \epsilon\cos(\phi_2 - \phi_1),\ \dot{\phi}_1 = I_1,\ \dot{\phi}_2 = I_2$$



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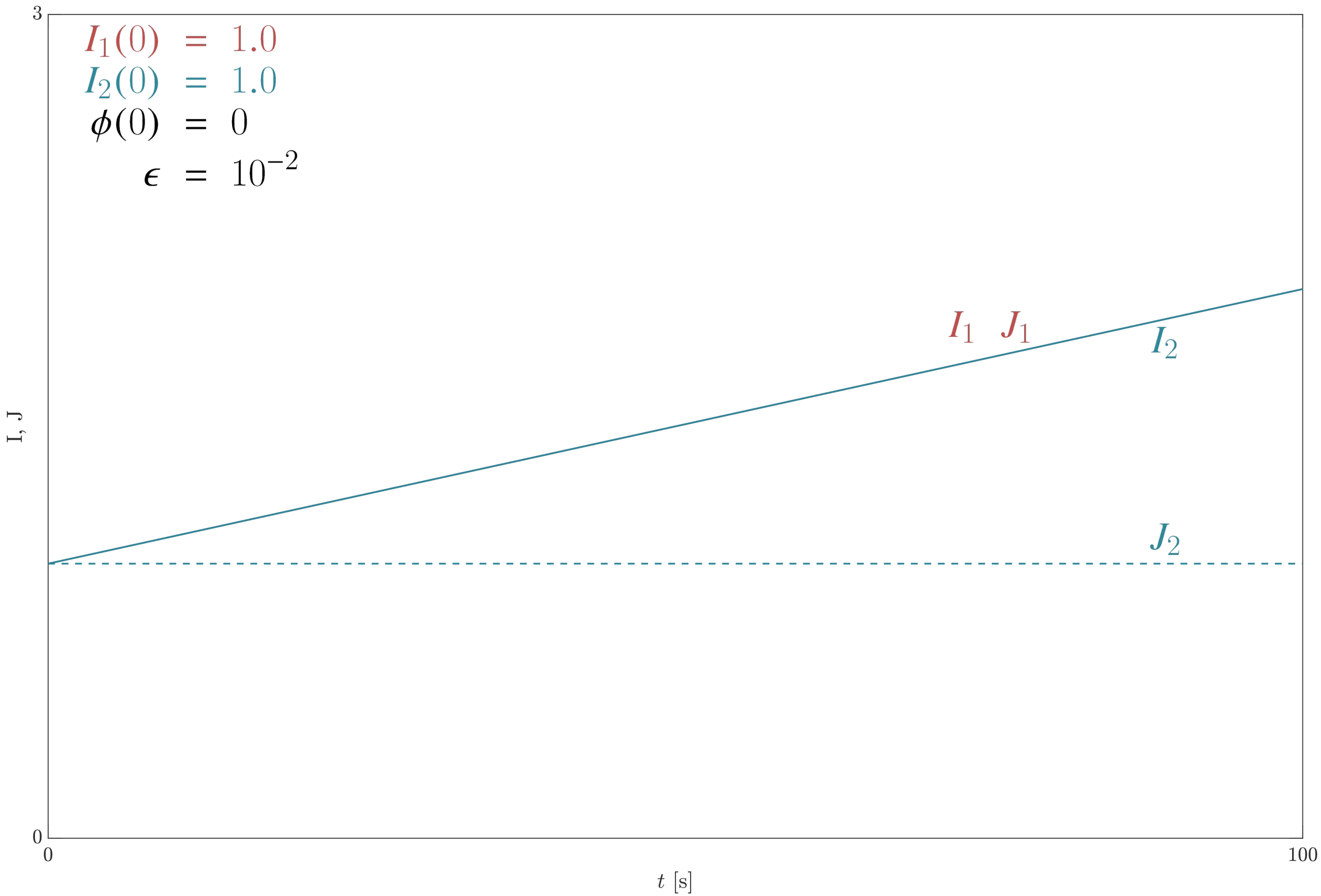

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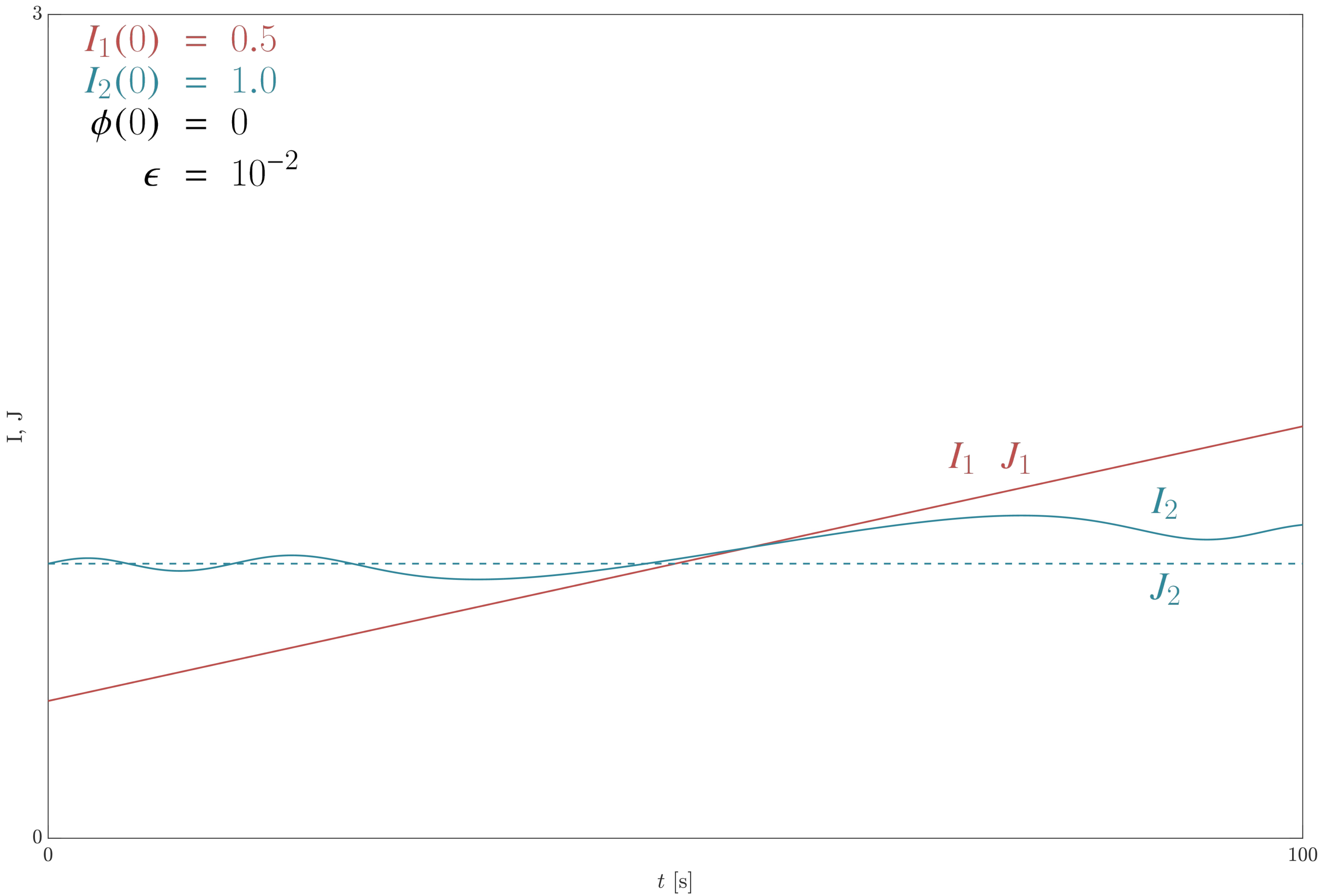
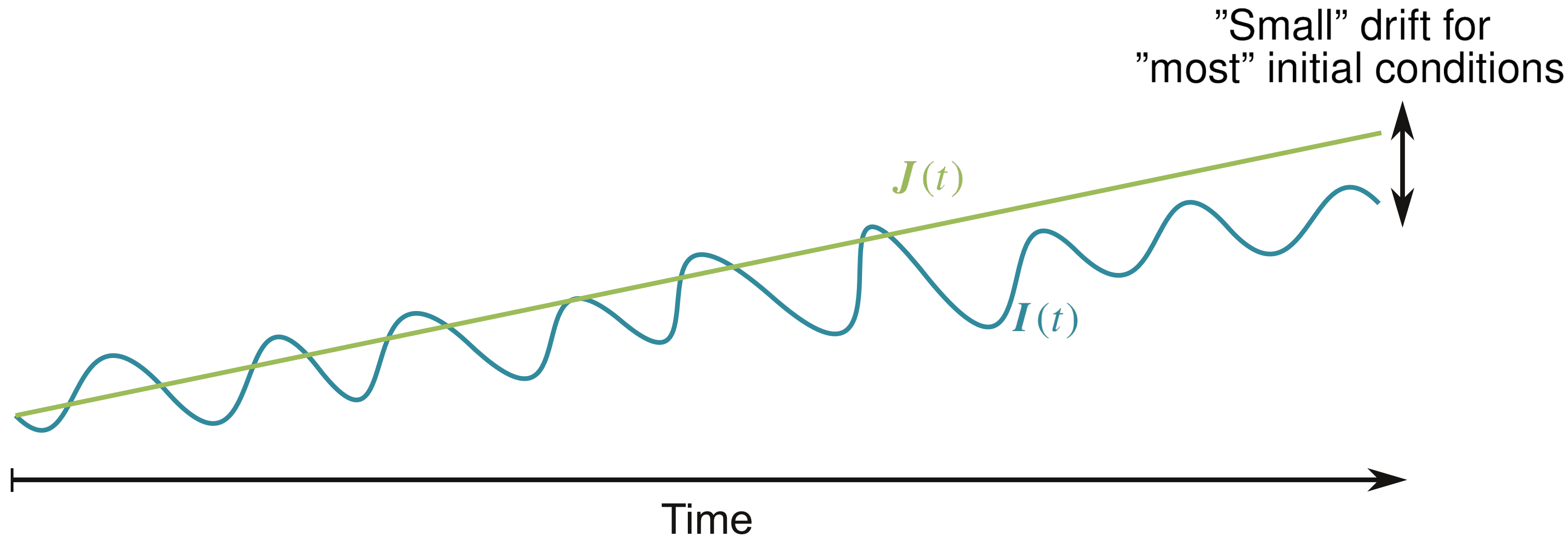


Rich literature quantifying "small" and "most"



Results for 2 phases:

Arnold theorem	very restrictive assumptions
Neistadt theorem	restrictive assumptions, optimal estimate

P. Lochak, C. Meunier *Multiphase Averaging for Classical Systems*, Springer, 1988.

Optimal estimate for two-phase problem (Neistadt)

Strong assumption

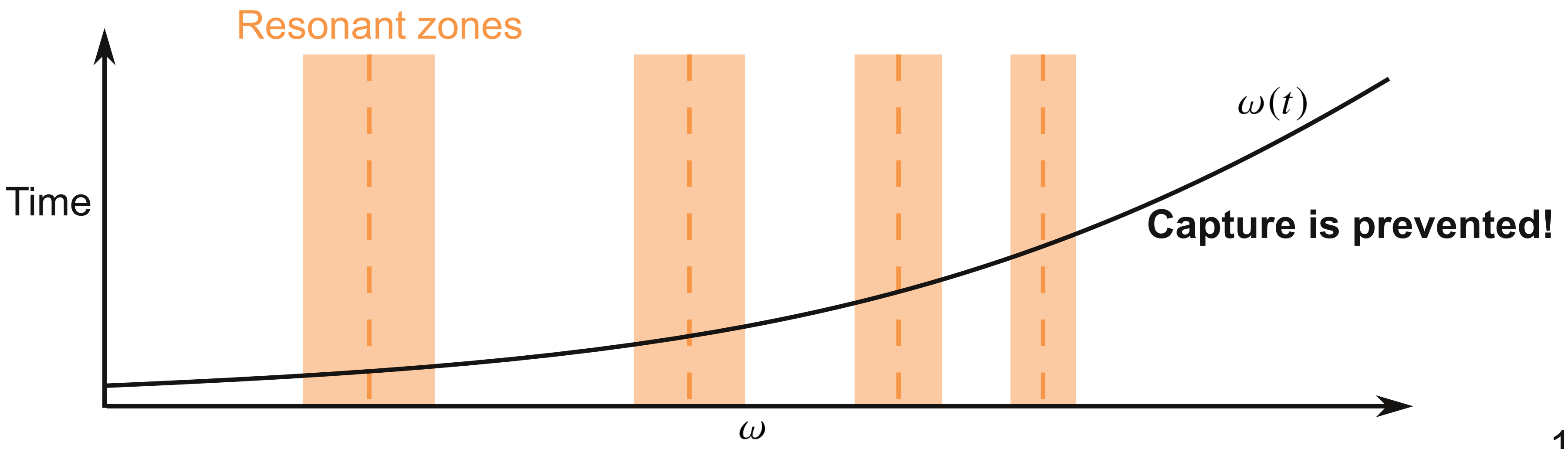
$$\omega_2(\boldsymbol{J}) \geq 0 \qquad \forall\, \boldsymbol{J} \in K$$

$$\left|\frac{\mathrm{d}\omega}{\mathrm{d}t}\left(\boldsymbol{J}\left(\boldsymbol{J}_0, \boldsymbol{\phi}_0, t, \epsilon\right)\right)\right| \geq c_1\epsilon \qquad \forall\ \left(\boldsymbol{J}_0, \boldsymbol{\phi}_0\right) \in K' \times \mathbb{T}^2,\ t \leq \frac{1}{\epsilon} \qquad \text{where } \omega := \frac{\omega_1(\boldsymbol{J})}{\omega_2(\boldsymbol{J})}$$

Main result

for $\epsilon \to 0,\ \exists\ K'' \subseteq K',\ \mu\left(K'' - K'\right) < c\sqrt{\epsilon},$ such that $\forall\ \left(\boldsymbol{J}_0, \boldsymbol{\phi}_0, t\right) \in K'' \times \mathbb{T}^2$

$$\sup_{t \in [0, \epsilon^{-1}]} ||\boldsymbol{I}(t) - \boldsymbol{J}(t)|| < c\sqrt{\epsilon}\log\frac{1}{\epsilon}$$



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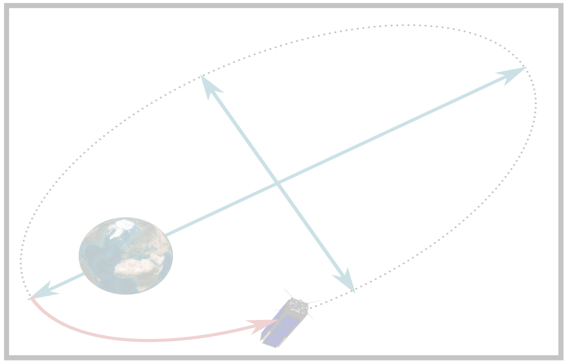

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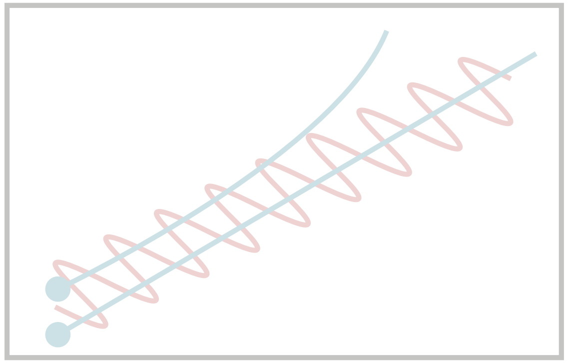

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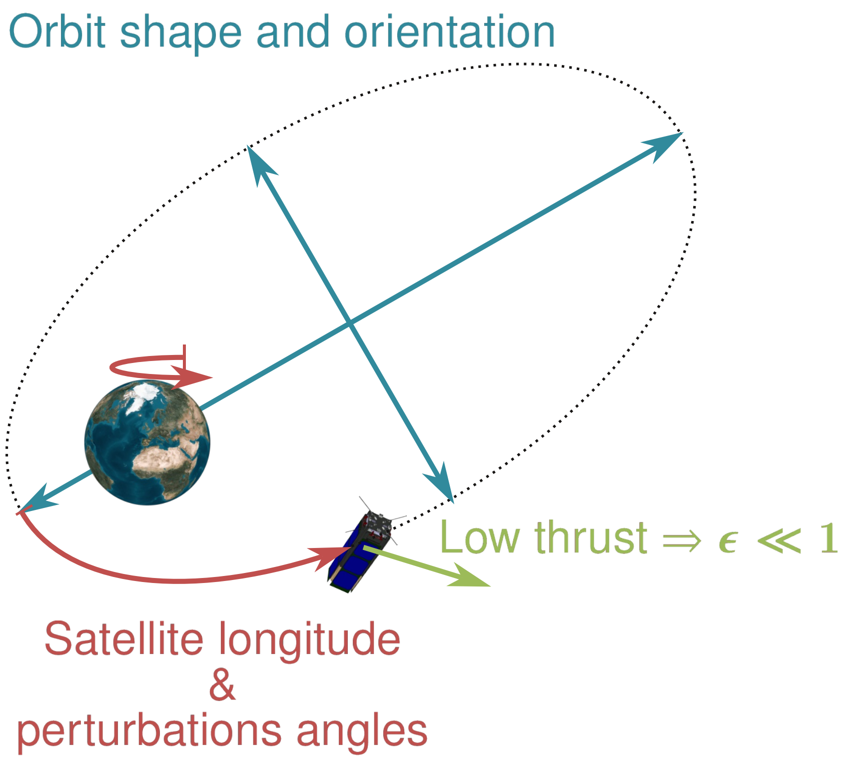
Minimum time low-thrust transfer



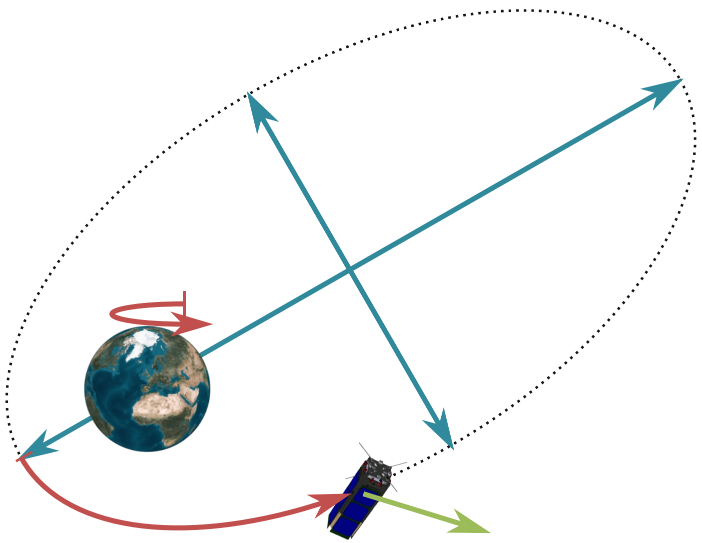
$$\min_{\|\boldsymbol{u}\| \leq 1} t_f \text{ subject to:}$$

$$\begin{aligned}
\dot{\boldsymbol{I}} &= \epsilon\left(\boldsymbol{f}_0(\boldsymbol{I}, \boldsymbol{\phi}, \epsilon) + \sum_{i=1}^{m} \boldsymbol{f}_i(\boldsymbol{I}, \boldsymbol{\phi}, \epsilon)\boldsymbol{u}_i\right) \\
\dot{\boldsymbol{\phi}} &= \underbrace{\epsilon\left(\boldsymbol{g}_0(\boldsymbol{I}, \boldsymbol{\phi}, \epsilon) + \sum_{i=1}^{m} \boldsymbol{g}_i(\boldsymbol{I}, \boldsymbol{\phi}, \epsilon)\boldsymbol{u}_i\right)}_{\text{Slow}} + \underbrace{\boldsymbol{\omega}(\boldsymbol{I})}_{\text{Fast}}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{I}(0) &= \boldsymbol{I}_0 \\
\boldsymbol{I}(t_f) &= \boldsymbol{I}_f
\end{aligned}$$

What about the dynamics of the control?

Necessary conditions: are adjoints slow or fast?



$$\mathcal{H} = \boldsymbol{p_\phi} \cdot \omega(\boldsymbol{I}) + \epsilon K\left(\boldsymbol{I}, \boldsymbol{\phi}, \boldsymbol{p_I}, \boldsymbol{p_\phi}, \epsilon\right)$$

$$\dot{\boldsymbol{I}} = \epsilon \frac{\partial K}{\partial \boldsymbol{p_I}} \qquad \dot{\boldsymbol{p}}_{\boldsymbol{I}} = -\epsilon \frac{\partial K}{\partial \boldsymbol{I}} - \underbrace{\frac{\partial \omega}{\partial \boldsymbol{I}} \boldsymbol{p_\phi}}_{?}$$

$$\dot{\boldsymbol{\phi}} = \epsilon \frac{\partial K}{\partial \boldsymbol{p_\phi}} + \omega(\boldsymbol{I}) \qquad \dot{\boldsymbol{p}}_{\boldsymbol{\phi}} = -\epsilon \frac{\partial K}{\partial \phi}$$

$$\boldsymbol{I}(0) = \boldsymbol{I_0} \qquad \boldsymbol{p_\phi}(0) = \boldsymbol{0}$$

$$\boldsymbol{I}(t_f) = \boldsymbol{I_f} \qquad \boldsymbol{p_\phi}(t_f) = \boldsymbol{0}$$

In the single-phase problem, $\boldsymbol{p_I}$ is slow. See:

A. Bombrun, A.; J.B. Pomet, *The averaged control system of fast oscillating control systems* SIAM J. Control Optim., 2013.
J.B. Caillau, J.B. Pomet, J. Rouot, J. CNES contract report, 2015.

Adjoints remain slow in the multi-phase case

Main idea:

$$\boldsymbol{\omega}(\boldsymbol{I}^*) \cdot \boldsymbol{p_\phi}^* = \mathcal{O}(\epsilon) \quad \blacktriangleright \quad \dot{\boldsymbol{p}}_{\boldsymbol{I}} = -\epsilon \frac{\partial K}{\partial \boldsymbol{I}} - \underbrace{\frac{\partial \boldsymbol{\omega}}{\partial \boldsymbol{I}} \boldsymbol{p_\phi}}_{\mathcal{O}(\epsilon)} = \mathcal{O}(\epsilon)$$

Sketch of the proof:

Change of variables: $\boldsymbol{I}, \boldsymbol{\phi} \to L, \begin{pmatrix} \alpha \\ \boldsymbol{\beta} \end{pmatrix}$ s.t. $\boldsymbol{L} = \boldsymbol{I}$, $\alpha = \dfrac{\boldsymbol{\omega} \cdot \boldsymbol{\phi}}{||\boldsymbol{\omega}||}$ and $\boldsymbol{\beta} = \dfrac{\boldsymbol{\omega}^\perp \boldsymbol{\phi}}{||\boldsymbol{\omega}||}$

The Hamiltonian becomes: $\mathcal{H} = ||\boldsymbol{\omega}(\boldsymbol{L})|| p_\alpha + \epsilon K\left(\boldsymbol{L}, \alpha, \boldsymbol{\beta}, \boldsymbol{p_L}, p_\alpha, \boldsymbol{p_\beta}, \epsilon\right)$

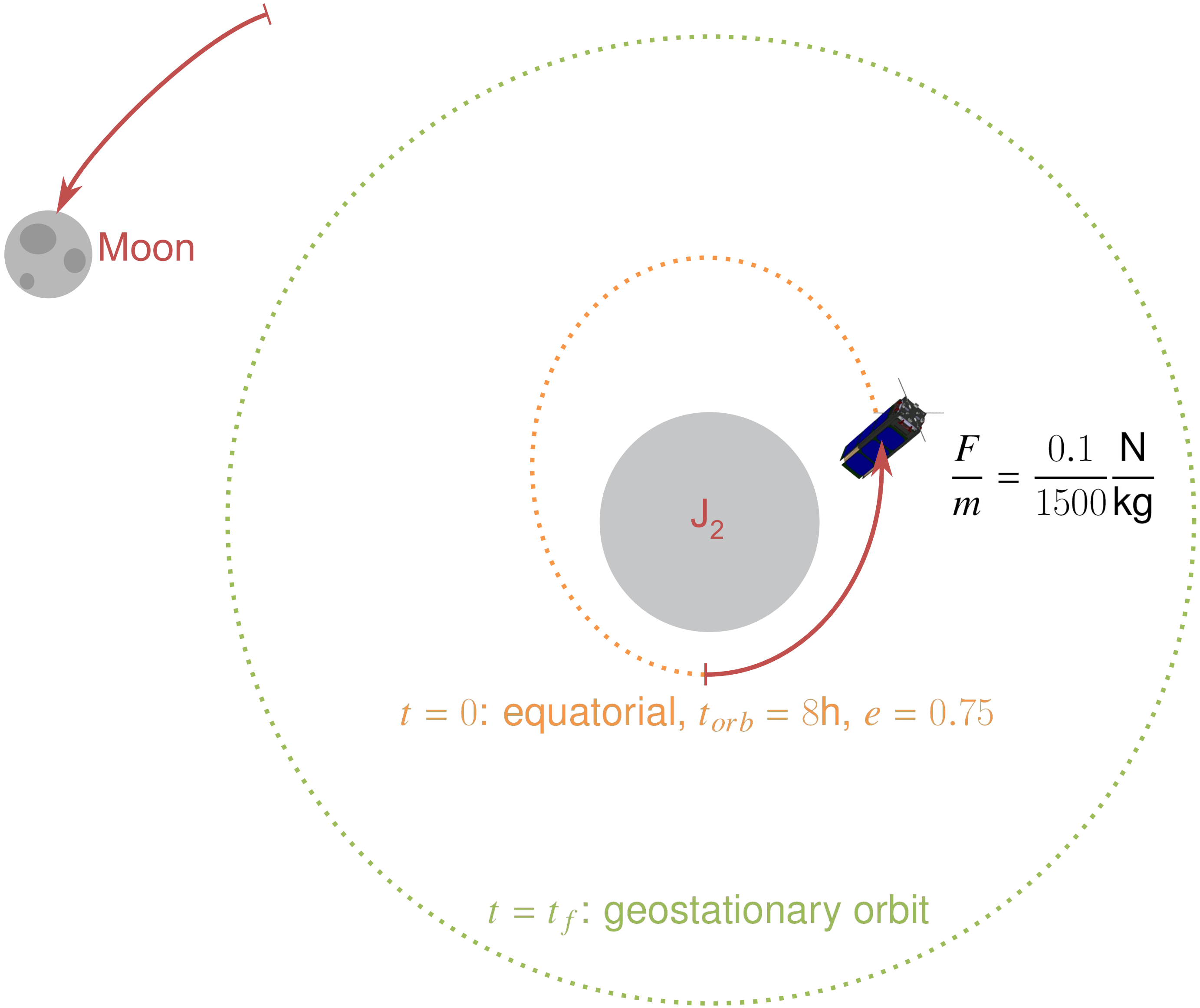
By noting that $p_\alpha(0) = \boldsymbol{p_\beta}(0) = 0$ and normalizing $||\boldsymbol{p_L}(0)|| = 1$, we have $\mathcal{H} = \epsilon c$

Hence, p_α can be implicitly expressed as: $p_\alpha = \epsilon \dfrac{c - K(\boldsymbol{L}, \alpha, \boldsymbol{\beta}, \boldsymbol{p_L}, p_\alpha, \boldsymbol{p_\beta}, \epsilon)}{||\boldsymbol{\omega}(\boldsymbol{L})||}$

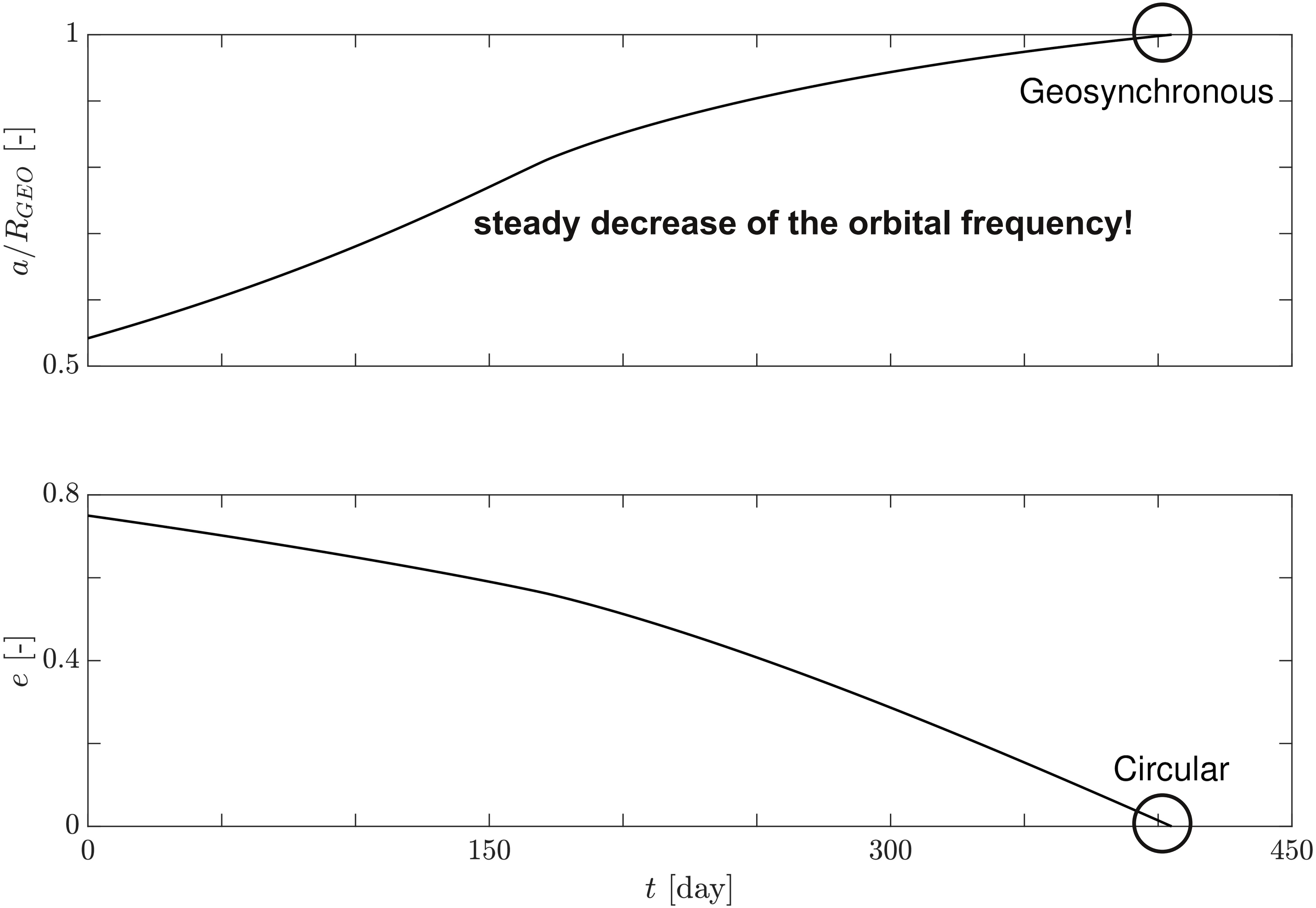
Because $\dfrac{\partial \mathcal{H}}{\partial p_\alpha} = ||\boldsymbol{\omega}(\boldsymbol{L})|| + \mathcal{O}(\epsilon) > 0$, we have: $p_\alpha = -\epsilon h(\boldsymbol{L}, \alpha, \boldsymbol{\beta}, \boldsymbol{p_L}, \boldsymbol{p_\beta}, \epsilon)$

So that: $\dot{\boldsymbol{p}}_{\boldsymbol{L}} = -\epsilon \dfrac{\partial H}{\partial p_\alpha} \dfrac{\partial h}{\partial \boldsymbol{L}} = \mathcal{O}(\epsilon)$

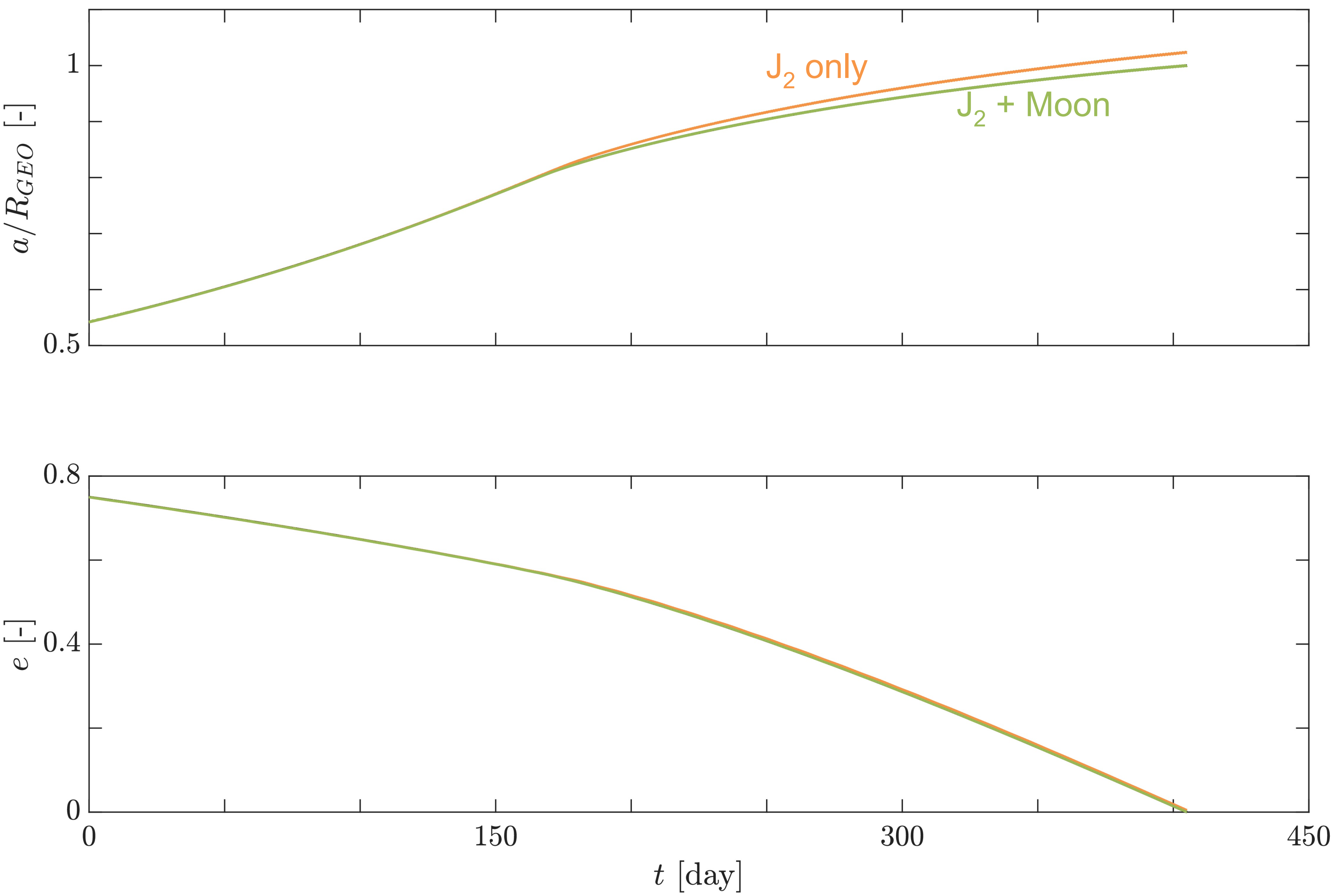
Transfer to geostationary orbit



Averaged solution satisfies Neistadt's requirements



Averaged solution plugged in the original system



Enhanced precision by averaging Moon gravity