On abnormal length minimizers in sub-Riemannian groups

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Journée McTAO
Dijon, December 4-5, 2017

Plan

From collaborations with
Hakavuori [Inventiones, 2016],
Montgomery, Ottazzi, Pansu, Vittone [Annales de l'IHP, 2016].

- geodesics in SR groups
  normal & abnormal curves.
- Sard-type problems
  abnormal varieties.
- Limits of geodesics
  blow-ups & blow-downs.

Sub-Riemannian geometry

M manifold
Δ bracket-generating subbundle of the tangent bundle of M
$\|\cdot\|$ norm on Δ coming from a smoothly varying scalar product

The **SR distance** is

$$d_{SR}(p,q) = \inf\left\{\int \|\dot\gamma\| \;\middle|\; \gamma : [0,1] \stackrel{AC}{\to} M, \qquad p \rightsquigarrow q, \qquad \dot\gamma \in \Delta\right\}$$

In this talk, the structure will be invariant under left translations with respect to a group structure.

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In this talk, the structure will be invariant under left translations with respect to a group structure.

Polarized groups

G connected Lie group with Lie algebra $\mathfrak{g}$.

V $\subseteq \mathfrak{g}$ linear subspace, called *polarization*.

For $u \in L^2([0,1], V)$, let γ_u be the curve in G that solves

$$\frac{\mathrm{d}\,\gamma}{\mathrm{d}\,t}(t) = \left(\mathrm{d}\,L_{\gamma(t)}\right)_{1_G} u(t), \tag{ODE}$$

with initial condition $\gamma(0) = 1_G$.

If $\gamma : [0,1] \to G$ is an AC curve that solves the ODE for some $u \in L^2([0,1], V)$, then γ is said *horizontal* with respect to V and $u = u_\gamma$ is its *control*.

γ is horizontal $\iff$ the derivatives of γ lie in the left-invariant subbundle Δ that coincides with V at the origin 1_G.

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Endpoint map

The *endpoint map starting at* 1_G *with controls in* V is the map

$$\begin{array}{rccc} \mathrm{End}: & L^2([0,1],V) & \to & G \\ & u & \mapsto & \gamma_u(1). \end{array}$$

Differential of Endpoint map and its image

Theorem

The endpoint map $\mathrm{End} : u \mapsto \gamma_u(1)$ *is smooth and*

$$\mathrm{d}\,\mathrm{End}_u\, v = (\mathrm{d}\, R_{\gamma_u(1)})_{1_G} \int_0^1 \mathrm{Ad}_{\gamma_u(t)}\, v(t)\, \mathrm{d}\, t, \quad \forall u, v \in L^2([0,1], V),$$

where $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ *is* $\mathrm{Ad}_g = (C_g)_*$ *with* $C_g h = ghg^{-1}$.

Corollary

$$\mathrm{Im}(\mathrm{d}\,\mathrm{End}_u) = (\mathrm{d}\, R_{\gamma_u(1)})_{1_G}(\mathrm{span}\{\mathrm{Ad}_{\gamma_u(t)}\, V \ :\ t \in [0,1]\}).$$

Sketch of the proof

Easier in matrix groups, so $\mathrm{Ad}_B(A) = BAB^{-1}$.

$\sigma_1(t) := \frac{\mathrm{d}}{\mathrm{d}\,\epsilon}\gamma_{u+\epsilon v}(t)|_{\epsilon=0}$

$\sigma_2(t) := \int_0^t \mathrm{Ad}_{\gamma(s)}(v(s))\,\mathrm{d}\,s \cdot \gamma(t)$

$\Longrightarrow$ σ_1 and σ_2 satisfy the ODE

$$\frac{\mathrm{d}\,\sigma_i}{\mathrm{d}\,t}(t) = \gamma(t) \cdot v(t) + \sigma_i(t) \cdot u(t),$$

with the same initial condition. $\square$

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Extended Endpoint map

Fix a Euclidean norm $\|\cdot\|$ on V.
Then $\Omega := L^2([0,1]; V)$ is normed:

$$\|u\| := \left(\int_0^1 \|u(t)\|^2 dt \right)^{\frac{1}{2}}.$$

The *extended endpoint map* is

$$\begin{aligned} \widetilde{\mathrm{End}} : \Omega &\to G \times \mathbb{R} \\ u &\mapsto \left(\gamma_u(1), \tfrac{1}{2}\|u\|^2 \right). \end{aligned}$$

We are interested in those curves that start from 1_G and reach a given point $p = \gamma_u(1)$ minimizing the *energy* $\frac{1}{2}\|u\|^2$.

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Consequence of minimizing energy

u is a minimizer for the energy
$\iff$ $\widetilde{\mathrm{End}}$ not open at u
$\implies$ [Open Mapping Theorem] $\mathrm{d}\,\widetilde{\mathrm{End}}$ is not surjective at u
$\iff$ $\exists(\xi, \xi_0) \in \left(T_{\mathrm{End}(u)}G\right)^* \times \mathbb{R} \setminus \{(0,0)\}$:

$$\langle(\xi, \xi_0), \mathrm{d}\,\widetilde{\mathrm{End}}_u(v)\rangle = 0, \quad \forall v \in \Omega.$$

Formula:

$$\mathrm{d}\,\widetilde{\mathrm{End}}_u : \Omega \to T_{\widetilde{\mathrm{End}}(u)}(G \times \mathbb{R}) = T_{\mathrm{End}(u)}G \times \mathbb{R} = \left(dR_{\gamma_u(1)}\right)_{1_G}\mathfrak{g} \times \mathbb{R}$$

$$v \mapsto \left(\left(dR_{\gamma_u(1)}\right)_{1_G}\int_0^1 \mathrm{Ad}_{\gamma_u(t)}(v(t))dt, \langle u, v\rangle\right).$$

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u is a minimizer for the energy

$\Longrightarrow$

$\exists \lambda \in \mathfrak{g}^*$ and $\xi_0 \in \mathbb{R}$ such that $(\lambda, \xi_0) \neq (0,0)$ and

$$\lambda\left(\int_0^1 \mathrm{Ad}_{\gamma_u(t)} v(t)dt\right) = \xi_0\langle u, v\rangle, \quad \forall v \in \Omega. \tag{1}$$

Two cases

(1) $\xi_0 \neq 0$. *normal curve*

(2) $\xi_0 = 0$. *abnormal curve*

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Normal curves

First case: May assume $\xi_0 = 1$.

$$\langle u, v\rangle = \lambda\left(\int_0^1 \mathrm{Ad}_{\gamma_u(t)}\, v(t)dt\right).$$

$(e_1, \dots, e_r)$ o.n. basis for $(V, \|\cdot\|)$.
In this basis, the controls are

$$\begin{aligned} u_i(t) &= \langle u, \delta_t e_i\rangle \\ &= \lambda\int_0^1 \mathrm{Ad}_{\gamma_u(s)}(\delta_t e_i)ds \\ &= \lambda(\mathrm{Ad}_{\gamma_u(t)}(e_i)). \end{aligned}$$

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Normal curves

γ is a normal curve
$\iff$ γ satisfies the *normal equation* (or geodesic equation):
$\exists \lambda \in \mathfrak{g}^*$:

$$\dot{\gamma}(t) = \sum_{i=1}^{r} \lambda \left(\mathrm{Ad}_{\gamma_u(t)}(e_i) \right) X_i(\gamma_u(t)), \tag{2}$$

for the left-invariant vector fields $X_i(g) := \left(dL_g\right) e_i$.

Facts:
* Every normal curve is analytic & constant-speed param.
* Every normal curve is locally energy minimizing.

The converse is not true.

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Expectations

Folk-conjectures:

1. $\exists$ full-measure set $A \subset G$:
   $\forall p \in A$ $\exists$ normal energy-minimizing curve from 1_G to p.
2. Every energy-minimizing curve is differentiable
   (or even analytic!)

Abnormal curves

Second case: $\xi_0 = 0$.
$\iff \gamma$ satisfies the *abnormal equation*:
$\exists \lambda \in \mathfrak{g}^* \setminus \{0\}$:

$$\lambda \left(\int_0^1 \mathrm{Ad}_{\gamma_u(t)} v(t) dt \right) = 0, \quad \forall v \in \Omega.$$

Equivalently,

$$\lambda \left(\mathrm{Ad}_{\gamma_u(t)} V \right) = \{0\}. \tag{3}$$

Corollary

(G, V) polarized group, $\gamma : [0,1] \to G$ horizontal curve. TFAE

1. *γ is abnormal;*
2. *$\exists \lambda \in \mathfrak{g}^* \setminus \{0\}$: $\lambda(\mathrm{Ad}_{\gamma(t)} V) = \{0\}$, $\forall t \in [0,1]$;*
3. *$\exists$ right-invariant 1-form α on G: $\alpha(\Delta_{\gamma(t)}) = \{0\}$, $\forall t \in [0,1]$, where Δ is the left-invariant distribution induced by V.*

Given $\lambda \in \mathfrak{g}^* \setminus \{0\}$, set $Z^\lambda := \{g \in G : ((\mathrm{Ad}_g)^*\lambda)_{|V} = 0\}$.
Z^λ is a proper real analytic variety.
If G is nilpotent, then Z^λ is a proper real algebraic variety.

Proposition

A horizontal curve γ is abnormal $\iff$ γ is contained in Z^λ for some nonzero $\lambda \in \mathfrak{g}^*$.

Abnormal curves & abnormal varieties

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How many abnormal curves are there?

Size of abnormal set

The *abnormal set* of (G, V) is the subset $\mathrm{Abn} \subset G$ of all singular values of the endpoint map. Equivalently, Abn is the union of all *abnormal curves* passing through 1_G.

Sub-Riemannian Sard Conjecture:

Abn has measure zero.

Theorem (LMOPV 2016)

In the following polarized groups Abn is contained in a proper algebraic subvariety:

1. *Carnot groups of step 2;*
2. *The free-nilpotent group of rank 3 and step 3;*
3. *The free-nilpotent group of rank 2 and step 4.*

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3. *The free-nilpotent group of rank* 2 *and step* 4*.*

Theorem (LMOPV 2016)

In every sub-Riemannian Carnot group G of step 3, the union of all locally length-minimizing abnormal curves passing through 1_G is contained in a proper algebraic subvariety.

Regularity of geodesics

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What is the regularity of SR energy minimizers?

(1) They are Lipschitz / AC.
[Sussmann, 2014] They are analytic on an open dense set.

(2) All known examples are analytic.
But, even corners were not excluded until 2016.

Recent results

1. In every (constant-rank, smooth) SR manifold, geodesics cannot have corners [HL2016].
2. ... and among tangents we always have some line [Monti-Pigati-Vittone 2017].
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Setting & tangents

$(M, \Delta, \|\cdot\|)$ SR manifold (equiregular), $\quad p \in M$.

$\implies \exists G$ SR group:

$$\lim_{\epsilon \to 0}{}^{\mathrm{GH}}(M, \Delta, \tfrac{1}{\epsilon}\|\cdot\|) = G.$$

Moreover, $\exists$ maps $\delta_\lambda : M \to M$ that are isometries from $(M, \Delta, \frac{1}{\epsilon}\|\cdot\|, p)$ to $(M, \Delta, \frac{\lambda}{\epsilon}\|\cdot\|, p)$.
In the limit, $\delta_\lambda : G \to G$ are dilations by λ.

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In the limit, $\delta_\lambda : G \to G$ are dilations by λ.

A group as blow-up space

We call G the *tangent space* (or *blow-up*) at p.

Extra property – G is a Carnot group:

$$\mathrm{Lie}(G) = V_1 \oplus \ldots \oplus V_s$$

with

$$[V_1, V_j] = V_{j+1},$$

and

$$\delta_\lambda(v) = \lambda^j v, \qquad v \in V_j.$$

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Blow-ups and blow-downs of curves

G Carnot group, $\gamma : I \to G$ Lipschitz curve, $t \in I$

Blow-ups: **tangents**
$\mathrm{Tang}(\gamma, t_0)$ – set of all curves limits of $\delta_{1/h}(\gamma(t_0 + ht))$, as $h \to 0$.

Blow-downs: **asymptotes**
$\mathrm{Asymp}(\gamma)$ – set of all curves limits of $\delta_{1/h}(\gamma(t_0 + ht))$, as $h \to \infty$.

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Results

Theorem (Hakavuori-L 2018)

G SR Carnot group. $\gamma : \mathbb{R} \to G$ *energy-minimizer.*
$\implies \exists H < G$ *proper Carnot subgroup such that* $\mathrm{Asymp}(\gamma) \subseteq H$.

Corollary

$\gamma : \mathbb{R} \to G$ *energy-minimizer in SR Carnot group.*
$\implies$ *some element in* $\mathrm{Asymp}(\gamma)$ *is a line.*

Corollary (already proved by MPV following HL)

$\gamma : I \to G$ *energy-minimizer in SR mfd,* $t_0 \in I$
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Simpler version – For you today

Simpler version

G rank 2 SR Carnot group $\gamma : I \to G$ energy-minimizer
$\implies \operatorname{Asymp}(\gamma)$ consists of a line.

Stronger claim

$\pi_1 \circ \gamma : \mathbb{R} \to G/G^2 \simeq \mathbb{R}^2$
is at bounded distance from a line.

Sketch of the proof

#. If not, $\exists$ triples of points on $\pi_1 \circ \gamma$ forming triangles with every height arbitrarily big.

#. $\pi_1 \circ \gamma$ is not a (Euclidean) geodesic. We may assume:
γ is a geodesic, but
$\sigma := \pi_{s-1} \circ \gamma$ is not a geodesic.
Here, $\pi_j : G \to G/G^{j+1} \simeq V_1 \oplus \ldots \oplus V_j$.

#. Shorten σ in an interval $[a, b]$, and lift it to $\tilde{\gamma}$ on G.

$\Longrightarrow$ $\tilde{\gamma}(t) = \gamma(t)$, for $t < a$, and
$\tilde{\gamma}(t) = \exp(Z)\gamma(t)$, for $t > b$, for some $Z \in V_s$.

#. Take $t_0, t_1, t_2 > b$ s.t. $\pi_1 \circ \gamma(t_i)$ form a big triangle (in terms of Z).
Find small $Y_1, Y_2 \in V_{s-1}$ s.t.

$$[Y_1, \log((\gamma(t_0))^{-1}\gamma(t_1))] + [Y_2, \log(\gamma(t_1))^{-1}\gamma(t_2))] = Z.$$

#. Let α_i geodesics from 1 to $\exp(Y_i)$. Construct a curve shorter than γ with same endpoints. Contradiction. $\square$

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Conclusions

With the same methods we expect that one might prove:

1. Every tangent is a line
2. Differentiability of geodesics

We don't expect to go beyond $C^{1,\alpha}$


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Merci

Thanks

Definition of Carnot group

$\mathfrak{g}$ stratified Lie algebra $\qquad \mathfrak{g} = V_1 \oplus \ldots \oplus V_s$.
G simpl. conn. Lie group with $\mathrm{Lie}(G) = \mathfrak{g}$.
$\| \cdot \|$ norm in V_1.

The CC distance is $d_{cc}(p, q) = \inf \left\{ \int \|\dot\gamma\| \ : \ \begin{array}{c} \gamma : [0,1] \xrightarrow{AC} G, \\ \dot\gamma \in V_1 \end{array} \quad x \rightsquigarrow y \right\}$

(G, d_{cc}) is a (subFinsler) Carnot group.

Theorem

Carnot groups are the only metric spaces that are

1. *locally compact,*
2. *geodesic,*
3. *homogeneous,*
4. *admit a dilation.*

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SubRiemannian Heisenberg group

In $\mathbb{R}^3$, $\quad \begin{array}{l} X = \partial_1 - \frac{y}{2}\partial_3 = (1, 0, -\frac{y}{2}) \\ Y = \partial_2 + \frac{x}{2}\partial_3 = (0, 1, \frac{x}{2}) \end{array} \quad$ vector fields

Consider PWC^∞/AC curves $\gamma : [0,1] \to \mathbb{R}^3$ s.t. $\forall$ almost t
$\dot\gamma(t) = a(t) X \circ \gamma(t) + b(t) Y \circ \gamma(t)$.
Call these curves *horizontal*, and set $\|\dot\gamma\| = \sqrt{a^2 + b^2}$.
Set $d(p,q) = \inf\{\int \|\dot\gamma\| \; : \; \gamma \,\text{horizontal}, x \rightsquigarrow y\}$

$$= \inf \left\{ \int \|(\dot\gamma_1, \dot\gamma_2)\|_{\ell^2} \; \underset{\gamma:[0,1]\overset{AC}{\to}\mathbb{R}^3,}{:} \; \dot\gamma_3 = \tfrac{1}{2}(\gamma_1\dot\gamma_2 - \underset{x \rightsquigarrow y}{\gamma_2}\dot\gamma_1) \right\}$$