From unbalanced optimal transport to the Camassa-Holm equation

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based on recent works with L. Chizat, T. Gallouët, G. Peyré, B. Schmitzer.

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A somewhat surprising result

**Theorem**

Solutions $u(t) \in C^\infty(S_1, \mathbb{R})$ to the Camassa-Holm equation

$$\partial_t u - \frac{1}{4} \partial_{txx} u + 3 \partial_x u u - \frac{1}{2} \partial_{xx} u \partial_x u - \frac{1}{4} \partial_{xxx} u u = 0 \quad (1)$$

are particular solutions of an incompressible Euler equation on $R^2 \setminus \{0\}$ for a density $\rho(r, \theta) = \frac{1}{r^3} \, dr \, d\theta = \frac{1}{r^4} \, \text{Leb}$

$$\begin{cases} \dot{v} + \nabla_v v = -\nabla p, \\ \nabla \cdot (\rho v) = 0. \end{cases} \quad (2)$$
Outline

1. Arnold’s point of view

2. Ebin-Marsden’s approach and Brenier’s

3. The Wasserstein-Fisher-Rao metric

4. The Camassa-Holm equation as an incompressible Euler equation
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Arnold’s point of view
Ebin-Marsden’s approach and Brenier’s
The Wasserstein-Fisher-Rao metric
The Camassa-Holm equation as an incompressible Euler equation

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Arnold’s geometric point of view


**Proposition**

The incompressible Euler equation is the geodesic flow of the right-invariant metric $L^2$ on $SDiff(M)$.

- An extrinsic point of view by Brenier, relaxation of the variational problem, optimal transport, polar factorization.
Arnold’s remark continued

- Let $(M, g)$ be a Riemannian manifold without boundary.
- Treat $\text{SDiff}(M)$ as an infinite dimensional Riemannian submanifold of $\text{Diff}(M)$.
- Consider the metric $L^2$ on $\text{SDiff}(M)$, it is right-invariant.

\section*{Proof.}

\textbf{Notations:} \( \varphi \in \text{SDiff}(M), X \in T_\varphi \text{SDiff}(M). \) Since \( \varphi_*(\text{vol}) = \text{vol}, \)

\[
G(\varphi)(X, X) = \int_M |X|^2 \, d\text{vol} = \int_M |X \circ \varphi^{-1}|^2 \, d\text{vol}
\]
Right-invariant metric on a Lie group

Definition (Right-invariant metric)

Let \( g_1, g_2 \in G \) be two group elements, the distance between \( g_1 \) and \( g_2 \) can be defined by:

\[
d^2(g_1, g_2) = \inf_{g(t)} \left\{ \int_0^1 \|v(t)\|_g^2 dt \mid g(0) = g_0 \text{ and } g(1) = g_1 \right\}
\]

where \( \partial_t g(t)g(t)^{-1} = v(t) \in \mathfrak{g} \), with \( \mathfrak{g} \) the Lie algebra.

Right-invariance simply means:

\[
d^2(g_1 g, g_2 g) = d(g_1, g_2).
\]

It comes from:

\[
\partial_t (g(t)g_0)(g(t)g_0)^{-1} = \partial_t g(t)g_0g_0^{-1}g(t)^{-1} = \partial_t g(t)g(t)^{-1}.
\]
Euler-Arnold-Poincaré equation

Compute the Euler-Lagrange equation of the distance functional:

\[ \frac{\partial L}{\partial g} - \frac{d}{dt} \frac{\partial L}{\partial \dot{g}} = 0 \]

In the case of \( \int_0^1 L(g, \dot{g}) dt = \int_0^1 \|u\|^2 dt \),
Euler-Poincaré-Arnold equation

\[
\begin{align*}
\dot{g} &= u \circ g \\
\dot{u} + {\text{ad}}_u^* u &= 0
\end{align*}
\]

(3)

where \( \text{ad}_u^* \) is the (metric) adjoint of \( \text{ad}_u v = [v, u] \).

Proof.

Compute variations of \( \nu(t) \) in terms of \( u(t) = \delta g(t) g(t)^{-1} \). Find that admissible variations on \( g \) can be written as:

\( \delta \nu(t) = \dot{u} - \text{ad}_\nu u \) for any \( u \) vanishing at 0 and 1.
Euler-Arnold-Poincaré equation

Consider the Hilbert scalar product on vector fields on $\mathbb{R}^d$ 
\[ \langle u, u \rangle = \langle u, Lu \rangle_{L^2} \]. Denoting $m = Lu$,
\[ \partial_t m + Dm.u + Du^T . m + \text{div}(u)m = 0 \] \hspace{1cm} (4)

For example, the $L^2$ metric, $L = \text{Id}$ gives:
\[ \partial_t u + Du . u + Du^T . u + \text{div}(u)u = 0 \] \hspace{1cm} (5)

On the group of volume preserving diffeomorphisms of $(M, \mu)$ with the $L^2$ metric:
Euler’s equation for ideal fluid where $\text{div}(u) = 0$
\[ \partial_t u + \nabla_u u = -\nabla p \],
(\text{use } \text{div}(u) = 0 \text{ and write the term } Du^T . u \text{ as a gradient as } \frac{1}{2} \nabla \langle u, u \rangle)
Fluid dynamics examples of Euler-Arnold equations

- Incompressible Euler equation.
- Korteweg-de-Vries equation.
- Camassa-Holm equation 1981/1993. *An integrable shallow water equation with peaked solitons*

Consider $\text{Diff}(S_1)$ endowed with the $H^1$ right-invariant metric $\|v\|_{L^2}^2 + \frac{1}{4}\|\partial_x v\|_{L^2}^2$. One has

\[
\begin{cases}
\partial_t u - \frac{1}{4} \partial_{txx} u u + 3 \partial_x u u - \frac{1}{2} \partial_{xx} u \partial_x u - \frac{1}{4} \partial_{xxx} u u = 0 \\
\partial_t \varphi(t, x) = u(t, \varphi(t, x))
\end{cases}
\]  

(6)

- Model for waves in shallow water.
- Completely integrable system (bi-Hamiltonian).
- Exhibits particular solutions named as peakons. (geodesics as collective Hamiltonian).
- Blow-up of solutions which gives a model for wave breaking.
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Ebin-Marsden analytical framework

Rewrite the metric in Lagrangian coordinates $\varphi$ and a tangent vector $X_\varphi$ and realize that it is smooth...

- The right-invariant $H^{\text{div}}$ metric:

$$G_\varphi(X_\varphi, X_\varphi) = \int_M a^2 |X_\varphi \circ \varphi^{-1}|^2 + b^2 \text{div}(X_\varphi \circ \varphi^{-1})^2 \ d\mu. \quad (7)$$

can be written

$$G_\varphi(X_\varphi, X_\varphi) = \int_M a^2 |X_\varphi|^2 \text{Jac}(\varphi) + b^2 \left( \text{Tr}(DX_\varphi \cdot [D\varphi]^{-1}) \right)^2 \text{Jac}(\varphi) \ d\mu. $$

Smooth metric on an infinite dimensional Riemannian manifold.

Consequences:

- Geodesic equations is a simple ODE (No need for a Riemannian connection)
- Gauss lemma on $H^s$ for $s > d/2 + 2$
- Geodesics are minimizing within $H^s$ topology.
The case of incompressible Euler

Need to deal with the projection on $\text{SDiff}(M)$...

Geodesic equation on $\text{SDiff}(M)$ as a submanifold of $\text{Diff}(M)$:

$$\ddot{\varphi} = -\nabla p \circ \varphi.$$  \hfill (8)

where $-\nabla p = A(\varphi, \dot{\varphi})$ is a smooth function of $\varphi, \dot{\varphi}$.

More explicitly,

$$A(\varphi, \dot{\varphi}) = (\nabla \Delta)^{-1}_\varphi [\text{div}, \nabla_v] \varphi \dot{\varphi}. \hfill (9)$$

where, if $L$ is a differential operator on functions,

$$L_\varphi(f) := L(f \circ \varphi^{-1}) \circ \varphi.$$  \hfill (10)

If $\varphi(t)$ is a smooth curve in $H^s$ s.t. $\dot{\varphi} = w \circ \varphi$, then

$$\frac{d}{ds} L_\varphi = [L_\varphi, \nabla w].$$
About Brenier’s approach to incompressible Euler

Variational approach to geodesics on $\text{SDiff}(M)$ isometrically embedded in a Hilbert space.

- Smooth solutions of Euler are minimizing (for $t \in [0, 1]$) if $\nabla^2 p$ is bounded in $L^\infty$ (by $\pi$).
- In general, relaxation of the boundary value problem as (infinite) multimarginal optimal transport.
- Polar factorization as a nonlinear extension of the pressure.
- Benamou-Brenier’s dynamic formulation.
A geometric picture: Otto’s Riemannian submersion

Figure – A Riemannian submersion: SDiff($M$) as a Riemannian submanifold of $L^2(M, M)$: Incompressible Euler equation on SDiff($M$)
Reminders: Riemannian submersion

Let \((M, g_M)\) and \((N, g_N)\) be two Riemannian manifolds and \(f : M \to N\) a differentiable mapping.

**Definition**

The map \(f\) is a Riemannian submersion if \(f\) is a submersion and for any \(x \in M\), the map \(df_x : \text{Ker}(df_x) \to T_{f(x)}N\) is an isometry.

- \(\text{Vert}_{f(x)} := \text{Ker}(df(x))\) is the vertical space.
- \(\text{Hor}_{f(x)} \overset{\text{def.}}{=} \text{Ker}(df(x))^{\perp}\) is the horizontal space.
- Geodesics on \(N\) can be lifted "horizontally" to geodesics on \(M\).

**Theorem (O’Neill’s formula)**

Let \(f\) be a Riemannian submersion and \(X, Y\) be two orthonormal vector fields on \(M\) with horizontal lifts \(\tilde{X}\) and \(\tilde{Y}\), then

\[
K_N(X, Y) = K_M(\tilde{X}, \tilde{Y}) + \frac{3}{4} \| \text{vert}([\tilde{X}, \tilde{Y}]) \|^2_M, \tag{11}
\]

where \(K\) denotes the sectional curvature and \(\text{vert}\) the orthogonal projection on the vertical space.
A pre-formulation of the polar factorization

Figure – A Riemannian submersion: SDiff($M$) as a Riemannian submanifold of $L^2(M, M)$: Incompressible Euler equation on SDiff($M$)
Reminders: Static Formulation

Monge formulation (1781)

Let \( \mu, \nu \in \mathcal{P}_+(M) \),

\[
\text{Minimize } \int_{\Omega} c(x, \varphi(x)) \, d\mu
\]

among the map s.t. \( \varphi_*(\mu) = \nu \).

1. Ill posed problem, the constraint may not be satisfied.
2. The constraint can hardly be made weakly closed.

\( \rightarrow \) Relaxation of the Monge problem.
Reminders: Static Formulation

Kantorovich formulation (1942)

Let $\mu, \nu \in \mathcal{P}_+(\Omega)$, define $D$ by

$$D(\mu, \nu) = \inf_{\gamma \in \mathcal{P}(\Omega^2)} \left\{ \int_{\Omega^2} c(x, y) \, d\gamma(x, y) : \pi_1^* \gamma = \mu \text{ and } \pi_2^* \gamma = \nu \right\}$$

1. Existence result: $c$ lower semi-continuous and bounded from below.
2. Also valid in Polish spaces.
3. If $c(x, y) = \frac{1}{p} |x - y|^p$, $D^{1/p}$ is the Wasserstein distance denoted by $\mathcal{W}_p$.

Linear optimization problem and associated numerical methods. Recently introduced, entropic regularization. (C. Léonard, M. Cuturi)
Reminders: Dynamic formulation (Benamou-Brenier)

For geodesic costs, for instance \( c(x, y) = \frac{1}{2}|x - y|^2 \)

\[
\inf E(\nu) = \frac{1}{2} \int_0^1 \int_\Omega |\nu(x)|^2 \rho(x) \, dx \, dt ,
\] (13)

\[
\begin{cases}
\dot{\rho} + \nabla \cdot (\nu \rho) = 0 \\
\rho(0) = \mu_0 \text{ and } \rho(1) = \mu_1 .
\end{cases}
\] (14)

Convex reformulation: Change of variable: momentum \( m = \rho \nu \),

\[
\inf E(m) = \frac{1}{2} \int_0^1 \int_\Omega \frac{|m(x)|^2}{\rho(x)} \, dx \, dt ,
\] (15)

\[
\begin{cases}
\dot{\rho} + \nabla \cdot m = 0 \\
\rho(0) = \mu_0 \text{ and } \rho(1) = \mu_1 .
\end{cases}
\] (16)

where \((\rho, m) \in \mathcal{M}([0, 1] \times \Omega, \mathbb{R} \times \mathbb{R}^d)\).

Existence of minimizers: Fenchel-Rockafellar.
Starting point and initial motivation

- Extend the Wasserstein $L^2$ distance to positive Radon measures.
- Develop associated numerical algorithms.

Possible applications: Imaging, machine learning, gradient flows, ...
Taking into account locally the change of mass:

Two directions: Static and dynamic.

- Static, Partial Optimal Transport [FG10]
- Dynamic, Numerics, Metamorphoses [MRSS15]
- Dynamic, Numerics, Growth model [LM13]
- Dynamic and static, [PR13, PR14]
- ...
Bibliography after june 2015

More than 300 pages on the same model!

Starting point: Dynamic formulation

- Dynamic, Numerics, Imaging [CSPV15] (40 pages)
- Dynamic, Geometry and Static [CSPV15] (40 pages)
- Dynamic, Gradient flow [KMV15] (50 pages)
- Dynamic, Gradient flow [LMS15b] (40 pages)
- Static and more [LMS15a] (100 pages)
- Static relaxation of OT, Numerics [FZM+15] (20 pages)
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An extension of Benamou-Brenier formulation

Add a source term in the constraint: (weak sense)

\[ \dot{\rho} = -\nabla \cdot (\rho \mathbf{v}) + \alpha \rho, \]

where \( \alpha \) can be understood as the growth rate.

\[
WF(m, \alpha)^2 = \frac{1}{2} \int_0^1 \int_{\Omega} |\mathbf{v}(x, t)|^2 \rho(x, t) \, dx \, dt
\]
\[ + \frac{\delta^2}{2} \int_0^1 \int_{\Omega} \alpha(x, t)^2 \rho(x, t) \, dx \, dt. \]

Remark: very natural and not studied before...
Convex reformulation

Add a source term in the constraint: (weak sense)

\[ \dot{\rho} = -\nabla \cdot m + \mu . \]

\[
\WF(m, \mu)^2 = \frac{1}{2} \int_0^1 \int_\Omega \frac{|m(x, t)|^2}{\rho(x, t)} \, dx \, dt + \frac{\delta^2}{2} \int_0^1 \int_\Omega \frac{\mu(x, t)^2}{\rho(x, t)} \, dx \, dt .
\]

- Fisher-Rao metric: Hessian of the Boltzmann entropy/Kullback-Leibler divergence and reparametrization invariant. Wasserstein metric on the space of variances in 1D.
- Convex and 1-homogeneous: convex analysis (existence and more)
- Code available at https://github.com/lchizat/optimal-transport/
A general framework

Definition (Infinitesimal cost)

An infinitesimal cost is $f: \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}_+ \cup \{+\infty\}$ such that for all $x \in \Omega$, $f(x, \cdot, \cdot, \cdot)$ is convex, positively 1-homogeneous, lower semicontinuous and satisfies

$$f(x, \rho, m, \mu) \begin{cases} = 0 & \text{if } (m, \mu) = (0, 0) \text{ and } \rho \geq 0 \\ > 0 & \text{if } |m| \text{ or } |\mu| > 0 \\ = +\infty & \text{if } \rho < 0. \end{cases}$$

Definition (Dynamic problem)

For $(\rho, m, \mu) \in \mathcal{M}(\mathbb{R})^{1+d+1}$, let

$$J(\rho, m, \mu) \overset{\text{def.}}{=} \int_0^1 \int_{\Omega} f(x, \frac{d\rho}{d\lambda}, \frac{dm}{d\lambda}, \frac{d\mu}{d\lambda}) \, d\lambda(t,x) \quad (17)$$

The dynamic problem is, for $\rho_0, \rho_1 \in \mathcal{M}_+(\Omega)$,

$$C(\rho_0, \rho_1) \overset{\text{def.}}{=} \inf_{(\rho, \omega, \zeta) \in \mathcal{C}E_0^1(\rho_0, \rho_1)} J(\rho, \omega, \zeta). \quad (18)$$
Existence of minimizers

**Proposition (Fenchel-Rockafellar)**

Let $B(x)$ be the polar set of $f(x, \cdot, \cdot, \cdot)$ for all $x \in \Omega$ and assume it is a lower semicontinuous set-valued function. Then the minimum of (18) is attained and it holds

$$ C_D(\rho_0, \rho_1) = \sup_{\varphi \in K} \int_{\Omega} \varphi(1, \cdot) \, d\rho_1 - \int_{\Omega} \varphi(0, \cdot) \, d\rho_0 $$

(19)

with $K \overset{\text{def.}}{=} \{ \varphi \in C^1([0, 1] \times \Omega) : (\partial_t \varphi, \nabla \varphi, \varphi) \in B(x), \forall (t, x) \in [0, 1] \times \Omega \}$.

$$WF(x, y, z) = \begin{cases} \frac{|y|^2 + \delta^2 z^2}{2x} & \text{if } x > 0, \\ 0 & \text{if } (x, |y|, z) = (0, 0, 0) \\ +\infty & \text{otherwise} \end{cases}$$

and the corresponding Hamilton-Jacobi equation is

$$ \partial_t \varphi + \frac{1}{2} \left( |\nabla \varphi|^2 + \frac{\varphi^2}{\delta^2} \right) \leq 0.$$
Numerical simulations
Initial and final densities: gray and blue curves.

Figure – 1st row: Standard $W_2$, 2nd row: Non-homogeneous $L^2$, 3rd row: Partial OT, 4th row: Wasserstein-Fisher-Rao
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Numerical simulations

Figure – Geodesics between $\rho_0$ and $\rho_1$ for (1st row) Hellinger, (2nd row) $W_2$, (3rd row) partial OT, (4th row) WF.

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From dynamic to static

Group action

Mass can be moved and changed: consider $m(t)\delta_x(t)$.

Infinitesimal action

$$\dot{\rho} = -\nabla \cdot (v\rho) + \mu \Leftrightarrow \begin{cases} \dot{x}(t) = v(x(t)) \\ \dot{m}(t) = \mu(x(t)) \end{cases}$$

A cone metric

$$WF^2(x, m) = \frac{1}{2} \left( m \, dx^2 + \frac{dm^2}{m} \right),$$
Riemannian cone

**Definition**

Let \((M, g)\) be a Riemannian manifold. The cone over \((M, g)\) is the Riemannian manifold \((M \times \mathbb{R}^*_+, r^2 g + dr^2)\).

- Change of variable: \(WF^2 = \frac{1}{2}r^2 g + 2 dr^2\).
- Non complete metric space: add the vertex \(M \times \{0\}\).
- The distance:

\[
d((x_1, m_1), (x_2, m_2))^2 = m_2 + m_1 - 2\sqrt{m_1 m_2} \cos \left(\frac{1}{2} d_M(x_1, x_2) \wedge \pi\right). \tag{20}
\]

- Curvature tensor: \(R(\tilde{X}, e) = 0\) and \(R(\tilde{X}, \tilde{Y})\tilde{Z} = (R_g(X, Y)Z - g(Y, Z)X + g(X, Z)Y, 0)\).
- \(M = \mathbb{R}\) then \((x, m) \mapsto \sqrt{me^{ix/2}} \in \mathbb{C}\) local isometry.
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Corollary

If \((\Omega, g)\) has sectional curvature greater than 1, then \((\Omega \times \mathbb{R}^*, m g + \frac{1}{4m} \, \text{d}m^2)\) has non-negative sectional curvature.

For \(X, Y\) two orthonormal vector fields on \(\Omega\),

\[ K(\tilde{X}, \tilde{Y}) = (K_g(X, Y) - 1) \tag{21} \]

where \(K\) and \(K_g\) denote respectively the sectional curvatures of \(\Omega \times \mathbb{R}^*\) and \(\Omega\).
Generalization of Otto’s Riemannian submersion

Idea of a left group action:

\[ \pi : \left( \text{Diff}(M) \times \text{C}^\infty(M, \mathbb{R}^*_+) \right) \times \text{Dens}(M) \hookrightarrow \text{Dens}(M) \]
\[ \pi \left( (\varphi, \lambda), \rho \right) := \varphi_*(\lambda^2 \rho) \]

Group law:

\[ (\varphi_1, \lambda_1) \cdot (\varphi_2, \lambda_2) = (\varphi_1 \circ \varphi_2, (\lambda_1 \circ \varphi_2)\lambda_2) \quad (22) \]

**Theorem**

Let \( \rho_0 \in \text{Dens}(M) \) and \( \pi_0 : \text{Diff}(M) \times \text{C}^\infty(M, \mathbb{R}^*_+) \hookrightarrow \text{Dens}(M) \)
defined by \( \pi_0(\varphi, \lambda) := \varphi_*(\lambda^2 \rho_0) \). It is a Riemannian submersion

\[ (\text{Diff}(M) \times \text{C}^\infty(M, \mathbb{R}^*_+), L^2(M, M \times \mathbb{R}^*_+)) \xrightarrow{\pi_0} (\text{Dens}(M), \text{WF}) \]

(where \( M \times \mathbb{R}^*_+ \) is endowed with the cone metric).

O’Neill’s formula: sectional curvature of (Dens(\( \Omega \)), WF).
Horizontal lift

Proposition (Horizontal lift)

\[ \text{Let } \rho \in \text{Dens}^s(\Omega) \text{ be a smooth density and } X_\rho \in H^s(\Omega, \mathbb{R}) \text{ be a tangent vector at the density } \rho. \text{ The horizontal lift at } (\text{Id}, 1) \text{ of } X_\rho \text{ is given by } (\frac{1}{2} \nabla \Phi, \Phi) \text{ where } \Phi \text{ is the solution to the elliptic partial differential equation:} \]

\[ - \text{div}(\rho \nabla \Phi) + 2\Phi \rho = X_\rho. \quad (23) \]

By elliptic regularity, the unique solution \( \Phi \) belongs to \( H^{s+1}(M) \).
Geometric consequence

The sectional curvature of $\text{Dens}(\Omega)$ at point $\rho$ is:

$$K(\rho)(X_1, X_2) = \int_{\Omega} k(x, 1)(Z_1(x), Z_2(x)) w(Z_1(x), Z_2(x)) \rho(x) \, d\nu(x)$$

$$+ \frac{3}{4} \left\| [Z_1, Z_2]^V \right\|^2 \tag{24}$$

where

$$w(Z_1(x), Z_2(x)) = g(x)(Z_1(x), Z_1(x)) g(x)(Z_2(x), Z_2(x))$$

$$- g(x)(Z_1(x), Z_2(x))^2$$

and $[Z_1, Z_2]^V$ denotes the vertical projection of $[Z_1, Z_2]$ at identity and $\| \cdot \|$ denotes the norm at identity.

Corollary

Let $(\Omega, g)$ be a compact Riemannian manifold of sectional curvature bounded below by $1$, then the sectional curvature of $(\text{Dens}(\Omega), \text{WF})$ is non-negative.
Consequences

Monge formulation

\[ \text{WF}(\rho_0, \rho_1) = \inf_{(\varphi, \lambda)} \left\{ \| (\varphi, \lambda) - (\text{Id}, 1) \|_{L^2(\rho_0)} : \varphi_\ast (\lambda^2 \rho_0) = \rho_1 \right\} \]

(25)

Under existence and smoothness of the minimizer, there exists a function \( p \in C^\infty(M, \mathbb{R}) \) such that

\[ (\varphi(x), \lambda(x)) = \exp_x^{C(M)} \left( \frac{1}{2} \nabla p(x), p(x) \right), \]

(26)

Equivalent to Monge-Ampère equation

With \( z \overset{\text{def.}}{=} \log(1 + p) \) one has

\[ (1 + |\nabla z|^2) e^{2z} \rho_0 = \det(D\varphi) \rho_1 \circ \varphi \]

(27)

and

\[ \varphi(x) = \exp_x^M \left( \arctan \left( \frac{1}{2} |\nabla z| \right) \frac{\nabla z(x)}{|\nabla z(x)|} \right). \]
Distance between Diracs

\[ \frac{1}{4} WF(m_1 \delta_{x_1}, m_2 \delta_{x_2})^2 = m_2 + m_1 - 2\sqrt{m_1 m_2} \cos \left( \frac{1}{2} d_\Omega(x_1, x_2) \wedge \pi / 2 \right) . \]

Proof: prove that an explicit geodesic is a critical point of the convex functional.

Properties: positively 1-homogeneous and convex in \((m_1, m_2)\).
General Kantorovich formulation

**Definition (Cost function)**

A *cost function* is

\[
c : (\Omega \times [0, +\infty[^2) \rightarrow [0, +\infty] \\
(x_1, m_1), (x_2, m_2) \mapsto c(x_1, m_1, x_2, m_2)
\]

which is l.s.c. and positively 1-homogeneous and convex in \((m_0, m_1)\).

Example: \(c_d(x_1, m_1, x_2, m_2) \overset{\text{def.}}{=} C_D(m_1 \delta_{x_1}, m_2 \delta_{x_2})\) (if l.s.c.)

**Definition (Semi-couplings)**

Let \(\rho_1, \rho_2 \in \mathcal{M}_+(\Omega)\), the set of semi-couplings is

\[
\Gamma(\rho_1, \rho_2) := \\
\left\{ (\gamma_1, \gamma_2) \in (\mathcal{M}_+(M^2))^2 : (\text{Proj}_1)_* \gamma_1 = \rho_1, (\text{Proj}_2)_* \gamma_2 = \rho_2 \right\}
\]
The functional is

$$J_K(\gamma_0, \gamma_1) \overset{\text{def.}}{=} \int_{\Omega^2} c \left( x, \frac{d\gamma_0}{d\gamma}, y, \frac{d\gamma_1}{d\gamma} \right) d\gamma(x, y), \quad (28)$$

The new Kantorovich problem is

$$C_K(\rho_0, \rho_1) \overset{\text{def.}}{=} \inf_{(\gamma_0, \gamma_1) \in \Gamma(\rho_0, \rho_1)} J_K(\gamma_0, \gamma_1). \quad (29)$$

1. If $c$ is a cost function then a minimizer for $C_K(\rho_0, \rho_1)$ exists.
2. If $c^{1/p}$ is a metric on the cone, $C_K^{1/p}$ is a metric on $\mathcal{M}_+(\Omega)$.
3. If, in addition, $c(x, 1, y, 1)$ is continuous in the space variable and $c(x, 1, x, 0) < \infty$ for some $x$, $C_K$ weak* continuous on $\mathcal{M}_+(\Omega)^2$. 
Kantorovich formulation

Recall

\[
\frac{1}{4} c_d^2(x_1, m_1, x_2, m_2) = m_2 + m_1
- 2 \sqrt{m_1 m_2} \cos \left( \frac{1}{2} d_\Omega(x_1, x_2) \wedge \pi / 2 \right).
\]

then

\[
WF(\rho_1, \rho_2)^2 = \inf_{(\gamma_1, \gamma_2) \in \Gamma(\rho_1, \rho_2)} \int_{M^2} c_d^2 \left( (x, \frac{d\gamma_1}{d\gamma}), (y, \frac{d\gamma_2}{d\gamma}) \right) d\gamma(x, y),
\]

Theorem (Dual formulation)

\[
WF^2(\rho_0, \rho_1) = \sup_{(\phi, \psi) \in C(M)^2} \int_M \phi(x) d\rho_0 + \int_M \psi(y) d\rho_1
\]

subject to \( \forall (x, y) \in M^2, \)

\[
\begin{cases}
\phi(x) \leq 1, & \psi(y) \leq 1, \\
(1 - \phi(x))(1 - \psi(y)) \geq \cos^2 (|x - y| / 2 \wedge \pi / 2)
\end{cases}
\]
A relaxed static OT formulation

Define

\[ KL(\gamma, \nu) = \int \frac{d\gamma}{d\nu} \log \left( \frac{d\gamma}{d\nu} \right) d\nu + |\nu| - |\gamma| \]

The corresponding primal formulation

\[ WF^2(\rho_1, \rho_2) = KL(\text{Proj}^1_\star \gamma, \rho_1) + KL(\text{Proj}^2_\star \gamma, \rho_2) \]

\[ - \int_{\Omega^2} \gamma(x, y) \log(\cos^2(d(x, y)/2 \wedge \pi/2)) \, dx \, dy \]
New algorithm

- Use of entropic regularization.

\[
WF^2(\rho_1, \rho_2) = KL(\text{Proj}^1_\ast \gamma, \rho_1) + KL(\text{Proj}^2_\ast \gamma, \rho_2) \\
- \int \int_{\Omega^2} \gamma(x, y) \log(\cos^2(d(x, y)/2 \land \pi/2)) \, dx \, dy + \varepsilon KL(\gamma, \mu_0).
\]

- Alternate projection algorithm (contraction for a Hilbert type metric).
- Applications to color transfer, Fréchet-Karcher mean (barycenters).
- Simulations for gradient flows.
Contents

1 Arnold’s point of view

2 Ebin-Marsden’s approach and Brenier’s

3 The Wasserstein-Fisher-Rao metric

4 The Camassa-Holm equation as an incompressible Euler equation
The Riemannian submersion

The Riemannian submersion is given by the push forward

$$\pi : \text{Diff}(M) \rightarrow \text{Dens}_p(M)$$

$$\pi(\varphi) = \varphi_*(\rho_0)$$

between (Diff($M$), $L^2(M, M)$) and (Dens$_p(M)$, $W_2$). The vertical space is

$$\text{Vert}_\varphi = \{v \circ \varphi ; \ v \in \text{Vect}(M) \text{ s.t. } \text{div}(\rho v) = 0\} \ , \quad (30)$$

and the horizontal space

$$\text{Hor}_\varphi = \{\nabla p \circ \varphi ; \ p \in C^\infty(M, \mathbb{R})\} \ . \quad (31)$$
The incompressible Euler equation

The usual Eulerian formulation of the equation:

\[
\begin{cases}
\partial_t v(t, x) + v(t, x) \cdot \nabla v(t, x) = -\nabla p(t, x), & t > 0, \ x \in M, \\
v(0, x) = v_0(x),
\end{cases}
\]  

(32)

As a geodesic equation of the Riemannian submanifold SDiff(M), a Lagrangian formulation

\[
\ddot{\phi} = -\nabla p \circ \phi.
\]

(33)

and under the condition \( \phi \in \text{SDiff}(M) \).
The Riemannian submersion for WFR

\[ \pi(\varphi, \lambda) = \varphi^*(\lambda^2 \mu) \]

\[ \mu \]

**Figure** – The same picture in our case: what is the corresponding equation to Euler?
The isotropy subgroup for unbalanced optimal transport

Recall that

\[ \pi_0^{-1}(\{\rho_0\}) = \{ (\varphi, \lambda) \in \text{Diff}(M) \times C^\infty(M, \mathbb{R}_+^*) : \varphi_*(\lambda^2 \rho_0) = \rho_0 \} \]

\[ \pi_0^{-1}(\{\rho_0\}) = \{ (\varphi, \sqrt{\text{Jac}(\varphi)}) \in \text{Diff}(M) \times C^\infty(M, \mathbb{R}_+^*) : \varphi \in \text{Diff}(M) \} \]

The vertical space is

\[ \text{Vert}_{(\varphi, \lambda)} = \{ (v, \alpha) \circ (\varphi, \lambda) ; \text{div}(\rho v) = 2\alpha \rho \} , \quad (34) \]

where \( (v, \alpha) \in \text{Vect}(M) \times C^\infty(M, \mathbb{R}) \). The horizontal space is

\[ \text{Hor}_{(\varphi, \lambda)} = \left\{ \left( \frac{1}{2} \nabla p, p \right) \circ (\varphi, \lambda) ; p \in C^\infty(M, \mathbb{R}) \right\} . \quad (35) \]

The induced metric is

\[ G(v, \text{div} v) = \int_M |v|^2 \, d\mu + \frac{1}{4} \int_M |\text{div} v|^2 \, d\mu . \quad (36) \]

The \( H^{\text{div}} \) right-invariant metric on the group of diffeomorphisms.
An isometric embedding

We have

\[ \text{inj} : (\text{Diff}(M), H^{\text{div}}) \leftrightarrow L^2(M, C(M)) \]  \hspace{1cm} (37)

The geodesic equations can be written as

\[
\begin{cases}
\frac{D}{Dt} \tilde{\phi} + 2 \frac{\lambda}{\lambda} \tilde{\phi} = -\nabla g P \circ \phi \\
\lambda r - \lambda \text{rg}(\dot{\phi}, \dot{\phi}) = -2\lambda r P \circ \varphi.
\end{cases}
\]  \hspace{1cm} (38)

**Corollary (Michor and Mumford)**

The distance on \(\text{Diff}(M)\) with the right-invariant metric \(H^{\text{Div}}\) is non degenerate.

**Proof.**

Segments are length minimizing in the ambient space.
Consequences

- Smooth geodesics are length minimizing for a short enough time under mild conditions (generalization of Brenier’s proof).
- Using Gauss-Codazzi formula, it generalizes a curvature formula by Khesin et al. obtained on Diff($S_1$).
Generalisation of Brenier’s proof

Theorem

Let \((\varphi(t), r(t))\) be a smooth solution to the geodesic equations on the time interval \([t_0, t_1]\). If \((t_1 - t_0)^2 \langle w, \nabla^2 \Psi_{P(t)}(x, r)w \rangle < \pi^2 \|w\|^2\) holds for all \(t \in [t_0, t_1]\) and \((x, r) \in \mathcal{C}(M)\) and \(w \in \mathcal{T}_{(x, r)} \mathcal{C}(M)\), then for every smooth curve \((\varphi_0(t), r_0(t))\) satisfying \((\varphi_0(t_i), r_0(t_i)) = (\varphi(t_i), r(t_i))\) for \(i = 0, 1\) and the condition (*)\, one has

\[
\int_{t_0}^{t_1} \| (\dot{\varphi}, \dot{r}) \|^2 \, dt \leq \int_{t_0}^{t_1} \| (\dot{\varphi}_0, \dot{r}_0) \|^2 \, dt,
\]

with equality if and only if the two paths coincide on \([t_0, t_1]\).

The condition (*) is:

1. If the sectional curvature of \(\mathcal{C}(M)\) can assume both signs, there exists \(\delta > 0\) such that the curve \((\varphi_0(t), r_0(t))\) has to belong to a \(\delta\)-neighborhood of \((\varphi(t), r(t))\), namely

   \[
d_{\mathcal{C}(M)}((\varphi(t, x), r(t, x)), (\varphi(t, x), r(t, x))) \leq \delta
\]

   for all \((x, t) \in M \times [t_0, t_1]\) where \(d_{\mathcal{C}(M)}\) is the distance on the cone.

2. If \(\mathcal{C}(M)\) has non positive sectional curvature, then, for every \(\delta\) as above, there exists a short enough time interval on which the geodesic will be length minimizing.

3. If \(M = S_n(r)\) the Euclidean sphere in \(\mathbb{R}^{n+1}\) of radius \(r \leq 1\), the result is valid for every path \((\dot{\varphi}, \dot{r})\).
Consequences

Corollary

When $M = S_1$ and $a \leq 2b$, smooth solutions to the Camassa-Holm equation (here given in 1D)

$$a^2 \partial_t u - b^2 \partial_{txx} u + 3a^2 \partial_x uu - 2b^2 \partial_{xx} u \partial_x u - b^2 \partial_{xxx} uu = 0.$$ (40)

are length minimizing for short times.
Toward the incompressible Euler equation

Why? Liero, Mielke, Savaré derived the new metric using a minimization problem using probability densities on the cone.

**Question**
*Understand $\text{Diff}(M) \ltimes C^\infty(M, \mathbb{R}^*_+)\text{ as a subgroup of } \text{Diff}(\mathcal{C}(M))$?*

**Proposition**
*The space of half-densities on $M$ is a trivial principal fibre bundle over $M$ which can be written $M \times \mathbb{R}^*_+$ once a reference density is chosen. The automorphism group $\text{Aut}(\mathcal{C}(M))$ can be identified with $\text{Diff}(M) \ltimes C^\infty(M, \mathbb{R}^*_+)$. One has $(\varphi, \lambda) : (x, r) \mapsto (\varphi(x), \lambda(x)r)$. Recall that $\psi \in \text{Aut}(\mathcal{C}(M))$ if $\psi \in \text{Diff}(\mathcal{C}(M))$ and $\forall \lambda \in \mathbb{R}^*_+$ one has $\psi(\lambda \cdot (x, r)) = \lambda \cdot \psi(x, r)$ where $\lambda \cdot (x, r) \overset{\text{def.}}{=} (x, \lambda r)$.***
CH as an incompressible Euler equation

The geodesic equation on $\text{Diff}(M) \times C^\infty(M, \mathbb{R}^*_+) \subset \text{Aut}(\mathcal{C}(M))$ can be extended to $\text{Aut}(\mathcal{C}(M))$ as

$$\frac{D}{Dt}(\dot{\varphi}, \lambda r) = -\nabla \psi_P \circ (\varphi, \lambda r), \quad (41)$$

where $\psi_P(x, r) \overset{\text{def.}}{=} r^2 P(x)$.

**Question**

Does there exist a density $\tilde{\mu}$ on the cone such that $\text{inj}(\text{Diff}(M)) \subset \text{SDiff}_{\tilde{\mu}}(\mathcal{C}(M))$? (answer: yes)

**Proof.**

The measure $\tilde{\mu} \overset{\text{def.}}{=} r^{-3} \, dr \, d\mu$ where $\mu$ denotes the volume form on $M$. 


Results

**Theorem**

The solutions of the Camassa-Holm equation can be lifted to solutions to the incompressible Euler equation on the cone for a density which is non integrable at the cone point.

In Lagrangian coordinates, the correspondence is given by $\mathcal{M} : \varphi \mapsto (\varphi, \sqrt{\text{Jac}(\varphi)})$.

Case when $M = S_1$, $\mathcal{M}(\varphi) = \sqrt{\varphi'} e^{i\varphi}$ then the CH equation is

$$\begin{cases} 
\partial_t u - \frac{1}{4} \partial_{txx} u u + 3 \partial_x u u - \frac{1}{2} \partial_{xx} u \partial_x u - \frac{1}{4} \partial_{xxx} u uu = 0 \\
\partial_t \varphi(t, x) = u(t, \varphi(t, x)).
\end{cases} \quad (42)$$

The cone is $C(M) = \mathbb{R}^2 \setminus \{0\}$, the density is $\frac{1}{r^4}$ Leb.
Results

A corresponding polar factorization:

**Proposition**

Let \((\phi, \lambda) \in \text{Aut}(C(M))\) be an element of the automorphism group of the half-densities bundle and \(\rho_0\). Denote by \(C^\infty(C(M)))^{R^*_+}\) functions \(f : C(M) \mapsto \mathbb{R}\) of the form \(f(x, r) = r^2 \rho(x)\).

There exists a couple \((\varphi, \Psi) \in \text{Diff}(M) \times (C^\infty(C(M)))^{R^*_+}\) such that \(\log(1 + p)\) is \(-\log(\cos^2(d(x, y) \wedge \frac{\pi}{2}))\)-convex and

\[
(\phi, \lambda) = \exp^{C(M)}(\nabla \Psi) \circ (\varphi, \sqrt{\text{Jac}(\varphi)}), \tag{43}
\]

where \((\varphi, \sqrt{\text{Jac}(\varphi)})\) is the natural lift of \(\varphi\) in \(\text{Aut}(C(M))\).
A new geometric picture

\[ \text{Figure} \quad \text{The group } \text{Aut}(\mathcal{C}(M)) = \text{Diff}(M) \times C^\infty(M, \mathbb{R}^*_+) \text{ is totally geodesic in } (\text{Diff}(\mathcal{C}(M)), L^2(\mathcal{C}(M))) \text{ and there is a Riemannian submersion of the automorphism group of the cone endowed with the } L^2 \text{ metric to the space of positive densities on } M \text{ endowed with the Wasserstein-Fisher-Rao metric.} \]
Perspectives

- Study the relaxation of geodesics for CH (uniqueness of the pressure, how the angle of the cone affects the results...)
- Develop numerical approaches following Mérigot et al.
- Treat other fluid dynamic equations?

Figure – CH equation after the Madelung transform
References


References II


