# From unbalanced optimal transport to the Camassa-Holm equation

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based on recent works with L. Chizat, T. Gallouët, G. Peyré, B. Schmitzer.

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# A somewhat surprising result

## Theorem

*Solutions $u(t) \in C^\infty(S_1, \mathbb{R})$ to the Camassa-Holm equation*

$$\partial_t u - \frac{1}{4}\partial_{txx} u + 3\partial_x u\, u - \frac{1}{2}\partial_{xx} u\, \partial_x u - \frac{1}{4}\partial_{xxx} u\, u = 0 \tag{1}$$

*are particular solutions of an incompressible Euler equation on $R^2 \setminus \{0\}$ for a density $\rho(r,\theta) = \frac{1}{r^3}\,\mathrm{d}r\,\mathrm{d}\theta = \frac{1}{r^4}\,\mathrm{Leb}$*

$$\begin{cases} \dot{v} + \nabla_v v = -\nabla p\,, \\ \nabla \cdot (\rho v) = 0\,. \end{cases} \tag{2}$$

# Outline

# Contents

# Arnold's geometric point of view

*Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits*, Ann. Inst. Fourier, 1966.

**Proposition**

*The incompressible Euler equation is the geodesic flow of the right-invariant metric $L^2$ on $\mathrm{SDiff}(M)$.*

- An intrinsic point of view by Ebin and Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. of Math., 1970. Short time existence results for smooth initial conditions.
- An extrinsic point of view by Brenier, relaxation of the variational problem, optimal transport, polar factorization.

# Arnold's remark continued

- Let $(M, g)$ be a Riemannian manifold without boundary.
- Treat $\mathrm{SDiff}(M)$ as an infinite dimensional Riemannian submanifold of $\mathrm{Diff}(M)$.
- Consider the metric $L^2$ on $\mathrm{SDiff}(M)$, it is right-invariant.

**Proof.**

Notations: $\varphi \in \mathrm{SDiff}(M)$, $X \in T_\varphi \mathrm{SDiff}(M)$. Since $\varphi_*(\mathrm{vol}) = \mathrm{vol}$,

$$G(\varphi)(X, X) = \int_M |X|^2 \,\mathrm{dvol} = \int_M |X \circ \varphi^{-1}|^2 \,\mathrm{dvol}$$

□

# Right-invariant metric on a Lie group

## Definition (Right-invariant metric)

*Let $g_1, g_2 \in G$ be two group elements, the distance between $g_1$ and $g_2$ can be defined by:*

$$d^2(g_1, g_2) = \inf_{g(t)} \left\{ \int_0^1 \|v(t)\|_{\mathfrak{g}}^2 \, dt \,|\, g(0) = g_0 \, and \, g(1) = g_1 \right\}$$

*where $\partial_t g(t) g(t)^{-1} = v(t) \in \mathfrak{g}$, with $\mathfrak{g}$ the Lie algebra.*

Right-invariance simply means:

$$d^2(g_1 g, g_2 g) = d(g_1, g_2)\,.$$

It comes from:

$$\partial_t (g(t) g_0)(g(t) g_0)^{-1} = \partial_t g(t) g_0 g_0^{-1} g(t)^{-1} = \partial_t g(t) g(t)^{-1}\,.$$

# Euler-Arnold-Poincaré equation

Compute the Euler-Lagrange equation of the distance functional:

$$\frac{\partial L}{\partial g} - \frac{d}{dt}\frac{\partial L}{\partial \dot{g}} = 0$$

In the case of $\int_0^1 L(g, \dot{g})dt = \int_0^1 \|u\|^2 dt$,
Euler-Poincaré-Arnold equation

$$\begin{cases} \dot{g} = u \circ g \\ \dot{u} + \mathrm{ad}_u^* u = 0 \end{cases} \tag{3}$$

where $\mathrm{ad}_u^*$ is the (metric) adjoint of $\mathrm{ad}_u v = [v, u]$.

**Proof.**

Compute variations of $v(t)$ in terms of $u(t) = \delta g(t) g(t)^{-1}$. Find that admissible variations on $\mathfrak{g}$ can be written as:
$\delta v(t) = \dot{u} - \mathrm{ad}_v u$ for any $u$ vanishing at 0 and 1. □

# Euler-Arnold-Poincaré equation

Consider the Hilbert scalar product on vector fields on $\mathbb{R}^d$ $\langle u, u \rangle = \langle u, Lu \rangle_{L^2}$. Denoting $m = Lu$,

$$\partial_t m + Dm.u + Du^T.m + \mathrm{div}(u)m = 0\,. \qquad (4)$$

For example, the $L^2$ metric, $L = \mathrm{Id}$ gives:

$$\partial_t u + Du.u + Du^T.u + \mathrm{div}(u)u = 0\,. \qquad (5)$$

On the group of volume preserving diffeomorphisms of $(M, \mu)$ with the $L^2$ metric:
Euler's equation for ideal fluid where $\mathrm{div}(u) = 0$

$$\partial_t u + \nabla_u u = -\nabla p\,,$$

(use $\mathrm{div}(u) = 0$ and write the term $Du^T.u$ as a gradient as $\frac{1}{2}\nabla\langle u, u\rangle$)

# Fluid dynamics examples of Euler-Arnold equations

- Incompressible Euler equation.
- Korteweg-de-Vries equation.
- Camassa-Holm equation 1981/1993. *An integrable shallow water equation with peaked solitons*

Consider $\mathrm{Diff}(S_1)$ endowed with the $H^1$ right-invariant metric $\|v\|^2_{L^2} + \frac{1}{4}\|\partial_x v\|^2_{L^2}$. One has

$$\begin{cases} \partial_t u - \frac{1}{4}\partial_{txx} u\, u + 3\partial_x u\, u - \frac{1}{2}\partial_{xx} u\, \partial_x u - \frac{1}{4}\partial_{xxx} u\, u = 0 \\ \partial_t \varphi(t,x) = u(t,\varphi(t,x))\,. \end{cases} \tag{6}$$

- Model for waves in shallow water.
- Completely integrable system (bi-Hamiltonian).
- Exhibits particular solutions named as peakons. (geodesics as collective Hamiltonian).
- Blow-up of solutions which gives a model for wave breaking.

# Contents

# Ebin-Marsden analytical framework

Rewrite the metric in Lagrangian coordinates $\varphi$ and a tangent vector $X_\varphi$ and realize that it is smooth...

• The right-invariant $H^{\mathrm{div}}$ metric:

$$G_\varphi(X_\varphi, X_\varphi) = \int_M a^2 |X_\varphi \circ \varphi^{-1}|^2 + b^2 \operatorname{div}(X_\varphi \circ \varphi^{-1})^2 \, \mathrm{d}\mu \,. \quad (7)$$

can be written

$$G_\varphi(X_\varphi, X_\varphi) = \int_M a^2 |X_\varphi|^2 \operatorname{Jac}(\varphi) + b^2 \left( \operatorname{Tr}(DX_\varphi \cdot [D\varphi]^{-1}) \right)^2 \operatorname{Jac}(\varphi) \, \mathrm{d}\mu \,.$$

Smooth metric on an infinite dimensional Riemannian manifold. Consequences:

- Geodesic equations is a simple ODE (No need for a Riemannian connection)
- Gauss lemma on $H^s$ for $s > d/2 + 2$
- Geodesics are minimizing within $H^s$ topology.

# The case of incompressible Euler

Need to deal with the projection on SDiff($M$)...
Geodesic equation on SDiff($M$) as a submanifold of Diff($M$):

$$\ddot{\varphi} = -\nabla p \circ \varphi \,. \tag{8}$$

where $-\nabla p = A(\varphi, \dot{\varphi})$ is a smooth function of $\varphi, \dot{\varphi}$.
More explicitely,

$$A(\varphi, \dot{\varphi}) = (\nabla \Delta)^{-1}_{\varphi} [\mathrm{div}, \nabla_v]_{\varphi} \dot{\varphi} \,. \tag{9}$$

where, if $L$ is a differential operator on functions,

$$L_{\varphi}(f) := L(f \circ \varphi^{-1}) \circ \varphi \,. \tag{10}$$

If $\varphi(t)$ is a smooth curve in $H^s$ s.t. $\dot{\varphi} = w \circ \varphi$, then
$\frac{\mathrm{d}}{\mathrm{d}s} L_{\varphi} = [L_{\varphi}, \nabla_w]$.

# About Brenier's approach to incompressible Euler

Variational approach to geodesics on SDiff($M$) isometrically embedded in a Hilbert space.

- Smooth solutions of Euler are minimizing (for $t \in [0, 1]$) if $\nabla^2 p$ is bounded in $L^\infty$ (by $\pi$).
- In general, relaxation of the boundary value problem as (infinite) multimarginal optimal transport.
- Polar factorization as a nonlinear extension of the pressure.
- Benamou-Brenier's dynamic formulation.

# A geometric picture: Otto's Riemannian submersion

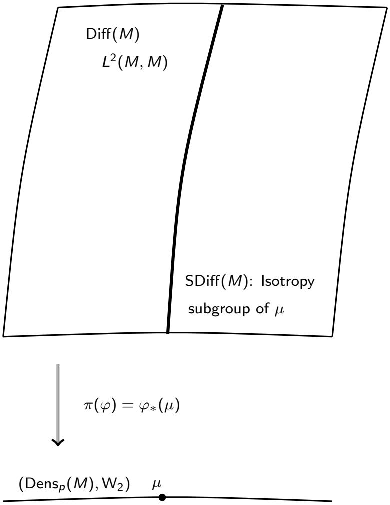



Figure – A Riemannian submersion: SDiff($M$) as a Riemannian submanifold of $L^2(M, M)$: Incompressible Euler equation on SDiff($M$)

# Reminders: Riemannian submersion

Let $(M, g_M)$ and $(N, g_N)$ be two Riemannian manifolds and $f : M \mapsto N$ a differentiable mapping.

## Definition

The map $f$ is a Riemannian submersion if $f$ is a submersion and for any $x \in M$, the map $df_x : \mathrm{Ker}(df_x)^\perp \mapsto T_{f(x)}N$ is an isometry.

- $\mathrm{Vert}_{f(x)} := \mathrm{Ker}(df(x))$ is the vertical space.
- $\mathrm{Hor}_{f(x)} \stackrel{\text{def.}}{=} \mathrm{Ker}(df(x))^\perp$ is the horizontal space.
- Geodesics on $N$ can be lifted "horizontally" to geodesics on $M$.

## Theorem (O'Neill's formula)

*Let $f$ be a Riemannian submersion and $X, Y$ be two orthonormal vector fields on $M$ with horizontal lifts $\tilde{X}$ and $\tilde{Y}$, then*

$$K_N(X,Y) = K_M(\tilde{X},\tilde{Y}) + \frac{3}{4}\|\operatorname{vert}([\tilde{X},\tilde{Y}])\|_M^2\,, \tag{11}$$

*where $K$ denotes the sectional curvature and* vert *the orthogonal projection on the vertical space.*

# A pre-formulation of the polar factorization

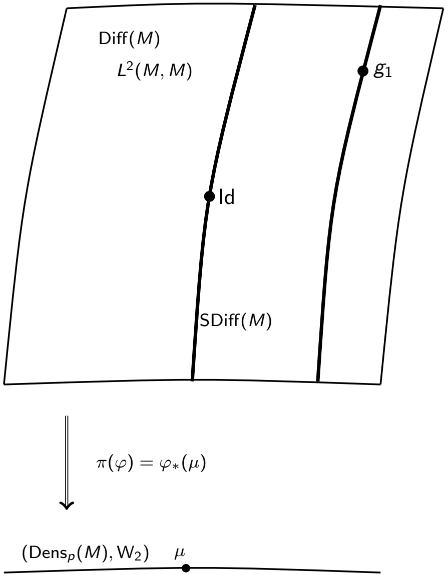



Figure – A Riemannian submersion: SDiff($M$) as a Riemannian submanifold of $L^2(M, M)$: Incompressible Euler equation on SDiff($M$)

# Reminders: Static Formulation

## Monge formulation (1781)

Let $\mu, \nu \in \mathcal{P}_+(M)$,

$$\textbf{Minimize} \int_\Omega c(x, \varphi(x)) d\mu \qquad (12)$$

among the map s.t. $\varphi_*(\mu) = \nu$.

1. ill posed problem, the constraint may not be satisfied.
2. the constraint can hardly be made weakly closed.

$\rightarrow$ Relaxation of the Monge problem.

# Reminders: Static Formulation

## Kantorovich formulation (1942)

Let $\mu, \nu \in \mathcal{P}_+(\Omega)$, define $D$ by

$$D(\mu,\nu) = \inf_{\gamma \in \mathcal{P}(\Omega^2)} \left\{ \int_{\Omega^2} c(x,y)\, \mathrm{d}\gamma(x,y) : \pi^1_* \gamma = \mu \text{ and } \pi^2_* \gamma = \nu \right\}$$

1. Existence result: $c$ lower semi-continuous and bounded from below.
2. Also valid in Polish spaces.
3. If $c(x,y) = \frac{1}{p}|x-y|^p$, $D^{1/p}$ is the Wasserstein distance denoted by $W_p$.

Linear optimization problem and associated numerical methods. Recently introduced, entropic regularization. (C. Léonard, M. Cuturi)

# Reminders: Dynamic formulation (Benamou-Brenier)

**For geodesic costs, for instance** $c(x,y) = \frac{1}{2}|x-y|^2$

$$\inf \mathcal{E}(v) = \frac{1}{2}\int_0^1 \int_\Omega |v(x)|^2 \rho(x)\, \mathrm{d}x\, \mathrm{d}t \ , \tag{13}$$

s.t.

$$\begin{cases} \dot\rho + \nabla \cdot (v\rho) = 0 \\ \rho(0) = \mu_0 \text{ and } \rho(1) = \mu_1 \, . \end{cases} \tag{14}$$

**Convex reformulation:** Change of variable: momentum $m = \rho v$,

$$\inf \mathcal{E}(m) = \frac{1}{2}\int_0^1 \int_\Omega \frac{|m(x)|^2}{\rho(x)}\, \mathrm{d}x\, \mathrm{d}t \, , \tag{15}$$

s.t.

$$\begin{cases} \dot\rho + \nabla \cdot m = 0 \\ \rho(0) = \mu_0 \text{ and } \rho(1) = \mu_1 \, . \end{cases} \tag{16}$$

where $(\rho, m) \in \mathcal{M}([0,1] \times \Omega, \mathbb{R} \times \mathbb{R}^d)$.

Existence of minimizers: Fenchel-Rockafellar.
Numerics: First-order splitting algorithm: Douglas-Rachford.

# Starting point and initial motivation

- Extend the Wasserstein $L^2$ distance to positive Radon measures.
- Develop associated numerical algorithms.

Possible applications: Imaging, machine learning, gradient flows, ...

# Bibliography before (june) 2015

Taking into account locally the change of mass:

Two directions: Static and dynamic.

- Static, Partial Optimal Transport [FG10]
- Static, Hanin 1992, Benamou and Brenier 2001.
- Dynamic, Numerics, Metamorphoses [MRSS15]
- Dynamic, Numerics, Growth model [LM13]
- Dynamic and static, [PR13, PR14]
- ...

# Bibliography after june 2015

**More than 300 pages on the same model!**

Starting point: Dynamic formulation

- Dynamic, Numerics, Imaging [CSPV15] (40 pages)
- Dynamic, Geometry and Static [CSPV15] (40 pages)
- Dynamic, Gradient flow [KMV15] (50 pages)
- Dynamic, Gradient flow [LMS15b] (40 pages)
- Static and more [LMS15a] (100 pages)
- Static relaxation of OT, Numerics [FZM+15] (20 pages)

# Contents

# An extension of Benamou-Brenier formulation

Add a source term in the constraint: (weak sense)

$$\dot{\rho} = -\nabla \cdot (\rho v) + \alpha \rho \,,$$

where $\alpha$ can be understood as the growth rate.

$$\mathrm{WF}(m, \alpha)^2 = \frac{1}{2} \int_0^1 \int_\Omega |v(x,t)|^2 \rho(x,t) \, \mathrm{d}x \, \mathrm{d}t + \frac{\delta^2}{2} \int_0^1 \int_\Omega \alpha(x,t)^2 \rho(x,t) \, \mathrm{d}x \, \mathrm{d}t \,.$$

Remark: very natural and not studied before...

# Convex reformulation

Add a source term in the constraint: (weak sense)

$$\dot{\rho} = -\nabla \cdot m + \mu \,.$$

$$\mathrm{WF}(m,\mu)^2 = \frac{1}{2}\int_0^1 \int_\Omega \frac{|m(x,t)|^2}{\rho(x,t)} \,\mathrm{d}x\,\mathrm{d}t + \frac{\delta^2}{2}\int_0^1 \int_\Omega \frac{\mu(x,t)^2}{\rho(x,t)} \,\mathrm{d}x\,\mathrm{d}t \,.$$

- Fisher-Rao metric: Hessian of the Boltzmann entropy/ Kullback-Leibler divergence and reparametrization invariant. Wasserstein metric on the space of variances in 1D.
- Convex and 1-homogeneous: convex analysis (existence and more)
- Numerics: First-order splitting algorithm: Douglas-Rachford.
- Code available at `https://github.com/lchizat/optimal-transport/`

# A general framework

## Definition (Infinitesimal cost)

An infinitesimal cost is $f : \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}_+ \cup \{+\infty\}$ such that for all $x \in \Omega$, $f(x, \cdot, \cdot, \cdot)$ is convex, positively 1-homogeneous, lower semicontinuous and satisfies

$$f(x, \rho, m, \mu) \begin{cases} = 0 & \text{if } (m, \mu) = (0, 0) \text{ and } \rho \geq 0 \\ > 0 & \text{if } |m| \text{ or } |\mu| > 0 \\ = +\infty & \text{if } \rho < 0 \,. \end{cases}$$

## Definition (Dynamic problem)

For $(\rho, m, \mu) \in \mathcal{M}([0,1] \times \Omega)^{1+d+1}$, let

$$J(\rho, m, \mu) \stackrel{\text{def.}}{=} \int_0^1 \int_\Omega f\left(x, \frac{\mathrm{d}\rho}{\mathrm{d}\lambda}, \frac{\mathrm{d}m}{\mathrm{d}\lambda}, \frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\right) \mathrm{d}\lambda(t, x) \tag{17}$$

The dynamic problem is, for $\rho_0, \rho_1 \in \mathcal{M}_+(\Omega)$,

$$C(\rho_0, \rho_1) \stackrel{\text{def.}}{=} \inf_{(\rho, \omega, \zeta) \in \mathcal{CE}_0^1(\rho_0, \rho_1)} J(\rho, \omega, \zeta) \,. \tag{18}$$

# Existence of minimizers

## Proposition (Fenchel-Rockafellar)

*Let $B(x)$ be the polar set of $f(x,\cdot,\cdot,\cdot)$ for all $x \in \Omega$ and assume it is a lower semicontinuous set-valued function. Then the minimum of (18) is attained and it holds*

$$C_D(\rho_0,\rho_1) = \sup_{\varphi \in K} \int_\Omega \varphi(1,\cdot)\,\mathrm{d}\rho_1 - \int_\Omega \varphi(0,\cdot)\,\mathrm{d}\rho_0 \tag{19}$$

*with* $K \stackrel{\text{def.}}{=} \left\{\varphi \in C^1([0,1]\times\Omega) : (\partial_t\varphi, \nabla\varphi, \varphi) \in B(x),\ \forall (t,x) \in [0,1]\times\Omega\right\}$ .

$$\mathrm{WF}(x,y,z) = \begin{cases} \frac{|y|^2+\delta^2 z^2}{2x} & \text{if } x > 0, \\ 0 & \text{if } (x,|y|,z) = (0,0,0) \\ +\infty & \text{otherwise} \end{cases}$$

and the corresponding Hamilton-Jacobi equation is

$$\partial_t\varphi + \frac{1}{2}\left(|\nabla\varphi|^2 + \frac{\varphi^2}{\delta^2}\right) \leq 0\,.$$

# Numerical simulations

Initial and final densities: gray and blue curves.

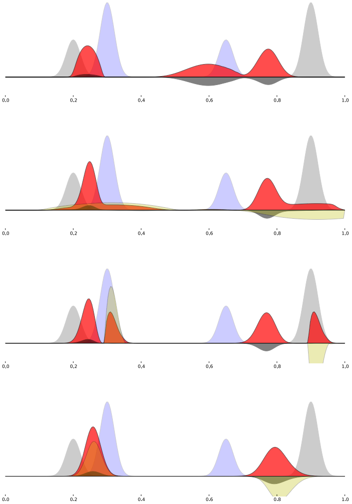



Figure – $1^{st}$ row: Standard $W_2$, $2^{nd}$ row: Non-homogeneous $L^2$, $3^{rd}$ row: Partial OT, $4^{th}$ row: Wasserstein-Fisher-Rao

# Numerical simulations

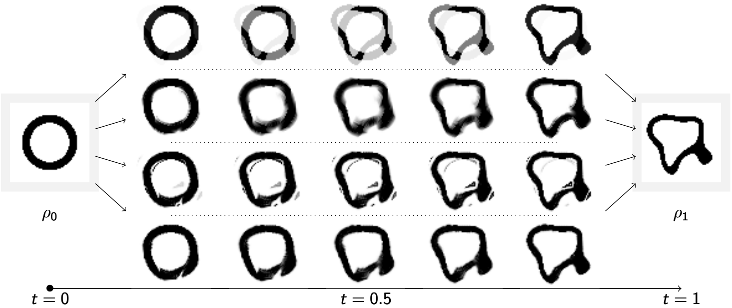



Figure – Geodesics between $\rho_0$ and $\rho_1$ for (1st row) Hellinger, (2nd row) $W_2$, (3rd row) partial OT, (4th row) WF.

*An Interpolating Distance between Optimal Transport and Fisher-Rao*, L. Chizat, B. Schmitzer, G. Peyré, and F.-X. Vialard, FoCM, 2016.

# From dynamic to static

## Group action

Mass can be moved and changed: consider $m(t)\delta_{x(t)}$.

## Infinitesimal action

$$\dot{\rho} = -\nabla \cdot (v\rho) + \mu \Leftrightarrow \begin{cases} \dot{x}(t) = v(x(t)) \\ \dot{m}(t) = \mu(x(t)) \end{cases}$$

## A cone metric

$$\mathrm{WF}^2(x,m) = \frac{1}{2}(m\,\mathrm{d}x^2 + \frac{\mathrm{d}m^2}{m}),$$

# Riemannian cone

## Definition

*Let $(M, g)$ be a Riemannian manifold. The cone over $(M, g)$ is the Riemannian manifold $(M \times \mathbb{R}_+^*, r^2 g + \mathrm{d}r^2)$.*

- Change of variable: $\mathrm{WF}^2 = \frac{1}{2} r^2 g + 2\,\mathrm{d}r^2$.
- Non complete metric space: add the vertex $M \times \{0\}$.
- The distance:

$$d((x_1, m_1), (x_2, m_2))^2 = m_2 + m_1 - 2\sqrt{m_1 m_2} \cos\left(\frac{1}{2} d_M(x_1, x_2) \wedge \pi\right) . \quad (20)$$

- Curvature tensor: $R(\tilde{X}, e) = 0$ and $R(\tilde{X}, \tilde{Y})\tilde{Z} = (R_g(X, Y)Z - g(Y, Z)X + g(X, Z)Y, 0)$.
- $M = \mathbb{R}$ then $(x, m) \mapsto \sqrt{m} e^{ix/2} \in \mathbb{C}$ local isometry.

# Geometry of a cone

## Corollary

*If* $(\Omega, g)$ *has sectional curvature greater than* $1$*, then* $(\Omega \times \mathbb{R}_+^*, m\, g + \frac{1}{4m}\,\mathrm{d}m^2)$ *has non-negative sectional curvature. For* $X, Y$ *two orthornormal vector fields on* $\Omega$*,*

$$K(\tilde{X}, \tilde{Y}) = (K_g(X, Y) - 1) \tag{21}$$

*where* $K$ *and* $K_g$ *denote respectively the sectional curvatures of* $\Omega \times \mathbb{R}_+^*$ *and* $\Omega$*.*

# Generalization of Otto's Riemannian submersion

Idea of a left group action:

$$\pi : \left(\mathrm{Diff}(M) \ltimes C^\infty(M,\mathbb{R}_+^*)\right) \times \mathrm{Dens}(M) \mapsto \mathrm{Dens}(M)$$
$$\pi\left((\varphi,\lambda),\rho\right) := \varphi_*(\lambda^2\rho)$$

Group law:

$$(\varphi_1,\lambda_1)\cdot(\varphi_2,\lambda_2) = (\varphi_1\circ\varphi_2,(\lambda_1\circ\varphi_2)\lambda_2) \tag{22}$$

**Theorem**

*Let $\rho_0 \in \mathrm{Dens}(M)$ and $\pi_0 : \mathrm{Diff}(M) \ltimes C^\infty(M,\mathbb{R}_+^*) \mapsto \mathrm{Dens}(M)$ defined by $\pi_0(\varphi,\lambda) := \varphi_*(\lambda^2\rho_0)$. It is a Riemannian submersion*

$$(\mathrm{Diff}(M) \ltimes C^\infty(M,\mathbb{R}_+^*), L^2(M, M\times\mathbb{R}_+^*)) \xrightarrow{\pi_0} (\mathrm{Dens}(M),\mathrm{WF})$$

*(where $M\times\mathbb{R}_+^*$ is endowed with the cone metric).*

O'Neill's formula: sectional curvature of $(\mathrm{Dens}(\Omega),\mathrm{WF})$.

# Horizontal lift

## Proposition (Horizontal lift)

*Let* $\rho \in \mathrm{Dens}^s(\Omega)$ *be a smooth density and* $X_\rho \in H^s(\Omega, \mathbb{R})$ *be a tangent vector at the density* $\rho$*. The horizontal lift at* $(Id, 1)$ *of* $X_\rho$ *is given by* $(\frac{1}{2}\nabla\Phi, \Phi)$ *where* $\Phi$ *is the solution to the elliptic partial differential equation:*

$$-\operatorname{div}(\rho\nabla\Phi) + 2\Phi\rho = X_\rho \,. \tag{23}$$

*By elliptic regularity, the unique solution* $\Phi$ *belongs to* $H^{s+1}(M)$*.*

# Geometric consequence

The sectional curvature of $\mathrm{Dens}(\Omega)$ at point $\rho$ is:

$$K(\rho)(X_1, X_2) = \int_\Omega k(x,1)(Z_1(x), Z_2(x)) w(Z_1(x), Z_2(x)) \rho(x) \,\mathrm{d}\nu(x) + \frac{3}{4} \left\| [Z_1, Z_2]^V \right\|^2 \quad (24)$$

where

$$w(Z_1(x), Z_2(x)) = g(x)(Z_1(x), Z_1(x)) g(x)(Z_2(x), Z_2(x)) - g(x)(Z_1(x), Z_2(x))^2$$

and $[Z_1, Z_2]^V$ denotes the vertical projection of $[Z_1, Z_2]$ at identity and $\| \cdot \|$ denotes the norm at identity.

## Corollary

*Let $(\Omega, g)$ be a compact Riemannian manifold of sectional curvature bounded below by 1, then the sectional curvature of $(\mathrm{Dens}(\Omega), \mathrm{WF})$ is non-negative.*

# Consequences

## Monge formulation

$$WF(\rho_0, \rho_1) = \inf_{(\varphi,\lambda)} \left\{ \|(\varphi, \lambda) - (Id, 1)\|_{L^2(\rho_0)} \, : \, \varphi_*(\lambda^2 \rho_0) = \rho_1 \right\} \tag{25}$$

Under existence and smoothness of the minimizer, there exists a function $p \in C^\infty(M, \mathbb{R})$ such that

$$(\varphi(x), \lambda(x)) = \exp_x^{\mathcal{C}(M)} \left( \frac{1}{2} \nabla p(x), p(x) \right) , \tag{26}$$

## Equivalent to Monge-Ampère equation

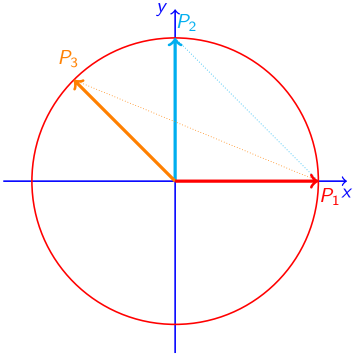
With $z \overset{\text{def.}}{=} \log(1 + p)$ one has

$$(1 + |\nabla z|^2) e^{2z} \rho_0 = \det(D\varphi) \rho_1 \circ \varphi \tag{27}$$

and

$$\varphi(x) = \exp_x^M \left( \arctan \left( \frac{1}{2} |\nabla z| \right) \frac{\nabla z(x)}{|\nabla z(x)|} \right) .$$

# Distance between Diracs




$$\frac{1}{4} WF(m_1\delta_{x_1}, m_2\delta_{x_2})^2 = m_2 + m_1 - 2\sqrt{m_1 m_2}\cos\left(\frac{1}{2} d_\Omega(x_1,x_2) \wedge \pi/2\right) .$$

Proof: prove that an explicit geodesic is a critical point of the convex functional.

Properties: positively 1-homogeneous and convex in $(m_1, m_2)$.

# General Kantorovich formulation

## Definition (Cost function)

A *cost function* is

$$c : \begin{array}{ccc} (\Omega \times [0,+\infty[)^2 & \to & [0,+\infty] \\ (x_1,m_1),(x_2,m_2) & \mapsto & c(x_1,m_1,x_2,m_2) \end{array}$$

which is l.s.c. and positively 1-homogeneous and convex in $(m_0, m_1)$.

Example: $c_d(x_1,m_1,x_2,m_2) \stackrel{\text{def.}}{=} C_D(m_1\delta_{x_1}, m_2\delta_{x_2})$ (if l.s.c.)

## Definition (Semi-couplings)

Let $\rho_1, \rho_2 \in \mathcal{M}_+(\Omega)$, the set of semi-couplings is

$$\Gamma(\rho_1,\rho_2) := \left\{(\gamma_1,\gamma_2) \in \left(\mathcal{M}_+(M^2)\right)^2 : (\mathrm{Proj}_1)_*\gamma_1 = \rho_1,\, (\mathrm{Proj}_2)_*\gamma_2 = \rho_2\right\},$$

# Kantorovich formulation

The functional is

$$J_K(\gamma_0, \gamma_1) \stackrel{\text{def.}}{=} \int_{\Omega^2} c\left(x, \frac{\mathrm{d}\gamma_0}{\mathrm{d}\gamma}, y, \frac{\mathrm{d}\gamma_1}{\mathrm{d}\gamma}\right) \mathrm{d}\gamma(x, y)\,, \tag{28}$$

The new Kantorovich problem is

$$C_K(\rho_0, \rho_1) \stackrel{\text{def.}}{=} \inf_{(\gamma_0,\gamma_1)\in\Gamma(\rho_0,\rho_1)} J_K(\gamma_0, \gamma_1)\,. \tag{29}$$

1. If $c$ is a *cost function* then a minimizer for $C_K(\rho_0, \rho_1)$ exists.
2. If $c^{1/p}$ is a metric on the cone, $C_K^{1/p}$ is a metric on $\mathcal{M}_+(\Omega)$.
3. If, in addition, $c(x, 1, y, 1)$ is continuous in the space variable and $c(x, 1, x, 0) < \infty$ for some $x$, $C_K$ weak* continuous on $\mathcal{M}_+(\Omega)^2$.

# Kantorovich formulation

Recall

$$\frac{1}{4}c_d^2(x_1,m_1,x_2,m_2) = m_2 + m_1 - 2\sqrt{m_1 m_2}\cos\left(\frac{1}{2}d_\Omega(x_1,x_2)\wedge \pi/2\right).$$

then

$$WF(\rho_1,\rho_2)^2 = \inf_{(\gamma_1,\gamma_2)\in\Gamma(\rho_1,\rho_2)} \int_{M^2} c_d^2\left((x,\frac{\mathrm{d}\gamma_1}{\mathrm{d}\gamma}),(y,\frac{\mathrm{d}\gamma_2}{\mathrm{d}\gamma})\right)\,\mathrm{d}\gamma(x,y)\,,$$

## Theorem (Dual formulation)

$$WF^2(\rho_0,\rho_1) = \sup_{(\phi,\psi)\in C(M)^2} \int_M \phi(x)\,\mathrm{d}\rho_0 + \int_M \psi(y)\,\mathrm{d}\rho_1$$

*subject to* $\forall (x,y)\in M^2$,

$$\begin{cases} \phi(x)\leq 1\,, \quad \psi(y)\leq 1\,,\\ (1-\phi(x))(1-\psi(y)) \geq \cos^2\left(|x-y|/2\wedge \pi/2\right) \end{cases}$$

# A relaxed static OT formulation

Define

$$KL(\gamma,\nu) = \int \frac{\mathrm{d}\gamma}{\mathrm{d}\nu} \log\left(\frac{\mathrm{d}\gamma}{\mathrm{d}\nu}\right) \mathrm{d}\nu + |\nu| - |\gamma|$$

## The corresponding primal formulation

$$\begin{aligned} WF^2(\rho_1,\rho_2) = {} & KL(\mathrm{Proj}^1_* \gamma, \rho_1) + KL(\mathrm{Proj}^2_* \gamma, \rho_2) \\ & - \int_{\Omega^2} \gamma(x,y) \log(\cos^2(d(x,y)/2 \wedge \pi/2))\, \mathrm{d}x\, \mathrm{d}y \end{aligned}$$

# New algorithm

*Scaling Algorithms for Unbalanced Transport Problems*, L. Chizat, G. Peyré, B. Schmitzer, F.-X. Vialard.

- Use of entropic regularization.

$$
WF^2(\rho_1, \rho_2) = KL(\mathrm{Proj}^1_* \gamma, \rho_1) + KL(\mathrm{Proj}^2_* \gamma, \rho_2)
$$
$$
- \int_{\Omega^2} \gamma(x,y) \log(\cos^2(d(x,y)/2 \wedge \pi/2))\, \mathrm{d}x\, \mathrm{d}y + \varepsilon KL(\gamma, \mu_0)\,.
$$

- Alternate projection algorithm (contraction for a Hilbert type metric).
- Applications to color transfer, Fréchet-Karcher mean (barycenters).
- Simulations for gradient flows.

# Contents

# The Riemannian submersion

The Riemannian submersion is given by the push forward

$$\pi : \mathrm{Diff}(M) \to \mathrm{Dens}_p(M)$$
$$\pi(\varphi) = \varphi_*(\rho_0)$$

between $(\mathrm{Diff}(M), L^2(M, M))$ and $(Dens_p(M), W_2)$. The vertical space is

$$\mathrm{Vert}_\varphi = \{v \circ \varphi \,;\, v \in \mathrm{Vect}(M) \text{ s.t. } \mathrm{div}(\rho v) = 0\} \,, \tag{30}$$

and the horizontal space

$$\mathrm{Hor}_\varphi = \{\nabla p \circ \varphi \,;\, p \in C^\infty(M, \mathbb{R})\} \,. \tag{31}$$

# The incompressible Euler equation

The usual Eulerian formulation of the equation:

$$\begin{cases} \partial_t v(t,x) + v(t,x) \cdot \nabla v(t,x) = -\nabla p(t,x), \quad t > 0,\ x \in M\,, \\ v(0,x) = v_0(x)\,, \end{cases} \tag{32}$$

As a geodesic equation of the Riemannian submanifold $\mathrm{SDiff}(M)$, a Lagrangian formulation

$$\ddot{\phi} = -\nabla p \circ \phi\,. \tag{33}$$

and under the condition $\phi \in \mathrm{SDiff}(M)$.

# The Riemannian submersion for WFR

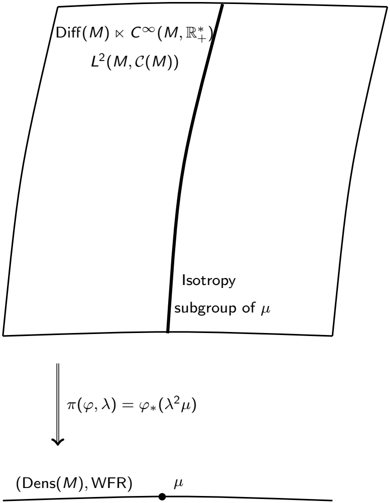



Figure – The same picture in our case: what is the corresponding equation to Euler?

# The isotropy subgroup for unbalanced optimal transport

Recall that

$$\pi_0^{-1}(\{\rho_0\}) = \{(\varphi, \lambda) \in \mathrm{Diff}(M) \ltimes C^\infty(M, \mathbb{R}_+^*) \; : \; \varphi_*(\lambda^2 \rho_0) = \rho_0\}$$

$$\pi_0^{-1}(\{\rho_0\}) = \{(\varphi, \sqrt{\mathrm{Jac}(\varphi)}) \in \mathrm{Diff}(M) \ltimes C^\infty(M, \mathbb{R}_+^*) \; : \; \varphi \in \mathrm{Diff}(M)\} .$$

The vertical space is

$$\mathrm{Vert}_{(\varphi,\lambda)} = \{(v, \alpha) \circ (\varphi, \lambda) \, ; \, \mathrm{div}(\rho v) = 2\alpha\rho\} \, , \tag{34}$$

where $(v, \alpha) \in \mathrm{Vect}(M) \times C^\infty(M, \mathbb{R})$. The horizontal space is

$$\mathrm{Hor}_{(\varphi,\lambda)} = \left\{ \left( \frac{1}{2} \nabla p, p \right) \circ (\varphi, \lambda) \, ; \, p \in C^\infty(M, \mathbb{R}) \right\} . \tag{35}$$

The induced metric is

$$G(v, \mathrm{div}\, v) = \int_M |v|^2 \, \mathrm{d}\mu + \frac{1}{4} \int_M |\, \mathrm{div}\, v|^2 \, \mathrm{d}\mu \, . \tag{36}$$

The $H^{\mathrm{div}}$ right-invariant metric on the group of diffeomorphisms.

# An isometric embedding

We have

$$\mathrm{inj} : (\mathrm{Diff}(M), H^{\mathrm{div}}) \hookrightarrow L^2(M, \mathcal{C}(M)) \tag{37}$$

The geodesic equations can be written as

$$\begin{cases} \frac{D}{Dt}\dot{\varphi} + 2\frac{\dot{\lambda}}{\lambda}\dot{\varphi} = -\nabla^g P \circ \varphi \\ \ddot{\lambda} r - \lambda r g(\dot{\varphi}, \dot{\varphi}) = -2\lambda r P \circ \varphi \,. \end{cases} \tag{38}$$

## Corollary (Michor and Mumford)

*The distance on* Diff$(M)$ *with the right-invariant metric* $H^{\mathrm{Div}}$ *is non degenerate.*

## Proof.

Segments are length minimizing in the ambient space. □

# Consequences

- Smooth geodesics are length minimizing for a short enough time under mild conditions (generalization of Brenier's proof).
- Using Gauss-Codazzi formula, it generalizes a curvature formula by Khesin et al. obtained on $\mathrm{Diff}(S_1)$.

# Generalisation of Brenier's proof

## Theorem

*Let $(\varphi(t), r(t))$ be a smooth solution to the geodesic equations on the time interval $[t_0, t_1]$. If $(t_1 - t_0)^2 \langle w, \nabla^2 \Psi_{P(t)}(x,r) w \rangle < \pi^2 \|w\|^2$ holds for all $t \in [t_0, t_1]$ and $(x, r) \in \mathcal{C}(M)$ and $w \in T_{(x,r)}\mathcal{C}(M)$, then for every smooth curve $(\varphi_0(t), r_0(t)) \in \pi_0^{-1}(\{\mu\})$ satisfying $(\varphi_0(t_i), r_0(t_i)) = (\varphi(t_i), r_0(t_i))$ for $i = 0, 1$ and the condition $(*)$, one has*

$$\int_{t_0}^{t_1} \|(\dot\varphi, \dot r)\|^2 \, \mathrm{d}t \leq \int_{t_0}^{t_1} \|(\dot\varphi_0, \dot r_0)\|^2 \, \mathrm{d}t \,, \tag{39}$$

*with equality if and only if the two paths coincide on $[t_0, t_1]$.*
*The condition $(*)$ is:*

1. *If the sectional curvature of $\mathcal{C}(M)$ can assume both signs, there exists $\delta > 0$ such that the curve $(\varphi_0(t), r_0(t))$ has to belong to a $\delta$-neighborhood of $(\varphi(t), r(t))$, namely*

   $$d_{\mathcal{C}(M)}\left((\varphi_0(t,x), r_0(t,x)), (\varphi_0(t,x), r(t,x))\right)) \leq \delta$$

   *for all $(x, t) \in M \times [t_0, t_1]$ where $d_{\mathcal{C}(M)}$ is the distance on the cone.*
2. *If $\mathcal{C}(M)$ has non positive sectional curvature, then, for every $\delta$ as above, there exists a short enough time interval on which the geodesic will be length minimizing.*
3. *If $M = S_n(r)$ the Euclidean sphere in $\mathbb{R}^{n+1}$ of radius $r \leq 1$, the result is valid for every path $(\dot\varphi, \dot r)$.*

# Consequences

## Corollary

*When $M = S_1$ and $a \leq 2b$, smooth solutions to the Camassa-Holm equation (here given in 1D)*

$$a^2 \partial_t u - b^2 \partial_{txx} u + 3a^2 \partial_x u \, u - 2b^2 \partial_{xx} u \, \partial_x u - b^2 \partial_{xxx} u \, u = 0 \,. \quad (40)$$

*are length minimizing for short times.*

# Toward the incompressible Euler equation

Why? Liero, Mielke, Savaré derived the new metric using a minimization problem using probability densities on the cone.

## Question

*Understand* $\mathrm{Diff}(M) \ltimes C^\infty(M, \mathbb{R}_+^*)$ *as a subgroup of* $\mathrm{Diff}(\mathcal{C}(M))$*?*

## Proposition

*The space of half-densities on $M$ is a trivial principal fibre bundle over $M$ which can be written $M \times \mathbb{R}_+^*$ once a reference density is chosen.*
*The automorphism group* $\mathrm{Aut}(\mathcal{C}(M))$ *can be identified with* $\mathrm{Diff}(M) \ltimes C^\infty(M, \mathbb{R}_+^*)$*. One has* $(\varphi, \lambda) : (x, r) \mapsto (\varphi(x), \lambda(x) r)$*.*

Recall that $\psi \in \mathrm{Aut}(\mathcal{C}(M))$ if $\psi \in \mathrm{Diff}(\mathcal{C}(M))$ and $\forall \lambda \in \mathbb{R}_+^*$ one has $\psi(\lambda \cdot (x, r)) = \lambda \cdot \psi(x, r)$ where $\lambda \cdot (x, r) \overset{\text{def.}}{=} (x, \lambda r)$.

# CH as an incompressible Euler equation

The geodesic equation on $\mathrm{Diff}(M) \ltimes C^\infty(M, \mathbb{R}_+^*)$ can be extended to $\mathrm{Aut}(\mathcal{C}(M))$ as

$$\frac{D}{Dt}(\dot{\varphi}, \dot{\lambda r}) = -\nabla \Psi_P \circ (\varphi, \lambda r)\,, \tag{41}$$

where $\Psi_P(x, r) \stackrel{\text{def.}}{=} r^2 P(x)$.

**Question**

*Does there exist a density $\tilde{\mu}$ on the cone such that $\mathrm{inj}(\mathrm{Diff}(M)) \subset \mathrm{SDiff}_{\tilde{\mu}}(\mathcal{C}(M))$? (answer: yes)*

**Proof.**

The measure $\tilde{\mu} \stackrel{\text{def.}}{=} r^{-3}\,\mathrm{d}r\,\mathrm{d}\mu$ where $\mu$ denotes the volume form on $M$. □

# Results

**Theorem**

*The solutions of the Camassa-Holm equation can be lifted to solutions to the incompressible Euler equation on the cone for a density which is non integrable at the cone point.*
*In Lagrangian coordinates, the correspondence is given by* $\mathcal{M} : \varphi \mapsto (\varphi, \sqrt{\mathrm{Jac}(\varphi)})$.

Case when $M = S_1$, $\mathcal{M}(\varphi) = \sqrt{\varphi'} e^{i\varphi}$ then the CH equation is

$$\begin{cases} \partial_t u - \frac{1}{4}\partial_{txx} u\, u + 3\partial_x u\, u - \frac{1}{2}\partial_{xx} u\, \partial_x u - \frac{1}{4}\partial_{xxx} u\, u = 0 \\ \partial_t \varphi(t,x) = u(t, \varphi(t,x))\,. \end{cases} \tag{42}$$

The cone is $\mathcal{C}(M) = \mathbb{R}^2 \setminus \{0\}$, the density is $\frac{1}{r^4}$ Leb.

# Results

A corresponding polar factorization:

## Proposition

*Let* $(\phi, \lambda) \in \mathrm{Aut}(\mathcal{C}(M))$ *be an element of the automorphism group of the half-densities bundle and* $\rho_0$*. Denote by* $C^\infty(\mathcal{C}(M)))^{\mathbb{R}_+^*}$ *functions* $f : \mathcal{C}(M) \mapsto \mathbb{R}$ *of the form* $f(x,r) = r^2 p(x)$*.*
*There exists a couple* $(\varphi, \Psi) \in \mathrm{Diff}(M) \times (C^\infty(\mathcal{C}(M)))^{\mathbb{R}_+^*}$ *such that* $\log(1+p)$ *is* $-\log(\cos^2(d(x,y) \wedge \frac{\pi}{2}))$*-convex and*

$$(\phi, \lambda) = \exp^{\mathcal{C}(M)}(\nabla \Psi) \circ (\varphi, \sqrt{\mathrm{Jac}(\varphi)})\,, \tag{43}$$

*where* $(\varphi, \sqrt{\mathrm{Jac}(\varphi)})$ *is the natural lift of* $\varphi$ *in* $\mathrm{Aut}(\mathcal{C}(M))$*.*

# A new geometric picture

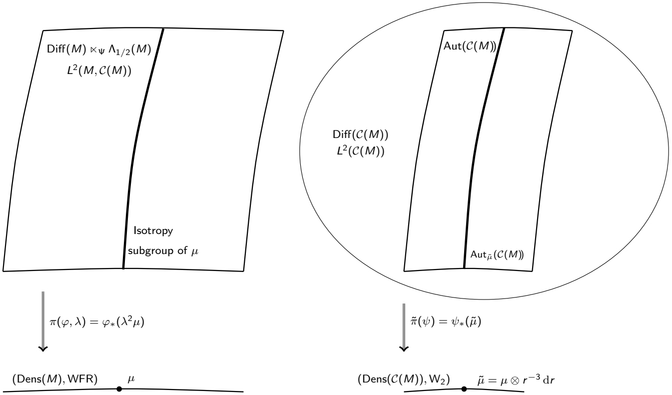



Figure – The group $\mathrm{Aut}(\mathcal{C}(M)) = \mathrm{Diff}(M) \ltimes C^\infty(M, \mathbb{R}_+^*)$ is totally geodesic in $(\mathrm{Diff}(\mathcal{C}(M)), L^2(\mathcal{C}(M))$ and there is a Riemannian submersion of the automorphism group of the cone endowed with the $L^2$ metric to the space of positive densities on $M$ endowed with the Wasserstein-Fisher-Rao metric.

# Perspectives

- Study the relaxation of geodesics for CH (uniqueness of the pressure, how the angle of the cone affects the results...)
- Develop numerical approaches following Mérigot et al.
- Treat other fluid dynamic equations ?

Figure – CH equation after the Madelung transform

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