Multiphase flows in porous media: a variational approach

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Settings

$N+1$ phases with ratio $s_i : \Omega \times (0,T) =: Q \to [0,1]$, immiscible and incompressible.

$$\partial_t s_i + \nabla \cdot (s_i \boldsymbol{v}_i) = 0 \quad \text{in } Q, \quad \forall i \in \{0,\dots,N\},$$

The filtration speed of each phase is given by the Darcy law :

$$\boldsymbol{v}_i = -\frac{1}{\mu_i}\mathbb{K}(\nabla p_i - \rho_i \mathbf{g}) \quad \text{in } Q, \quad \forall i \in \{0,\dots,N\}.$$

- $\mathbf{g}$ is the gravity vector (can be any potentential Ψ_i).
- μ_i is the viscosity.
- p_i, ρ_i the pressure and the density.

Settings II

The pore volume is saturated :

$$\sigma := \sum_{i=0}^{N} s_i = \omega(\boldsymbol{x}) \quad \text{a.e. in } Q.$$

We impose N capillary pressure relations

$$p_i - p_0 = \pi_i(\boldsymbol{s}^*, \boldsymbol{x}) \quad \text{a.e in } Q, \quad \forall i \in \{1, \dots, N\},$$

and no flux boundary

$$\boldsymbol{v}_i \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega \times (0, T), \quad \forall i \in \{0, \dots, N\},$$

- mass conservation $\int_\Omega s_i(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{x} =: m_i$.
- $\boldsymbol{s}^* = (s_1, \dots, s_N)$
- ω : is smooth, $\omega_\star \le \omega(\boldsymbol{x}) \le \omega^\star$.
- The permeability tensor $\mathbb{K}$ is smooth, symmetric.
- $\kappa_\star |\boldsymbol{\xi}|^2 \le \mathbb{K}(\boldsymbol{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \le \kappa^\star |\boldsymbol{\xi}|^2$

Limitations on Π

- Π continuous, differentiable
- $\pi_i(\boldsymbol{s}^*, \boldsymbol{x}) = \frac{\partial \Pi}{\partial s_i}(\boldsymbol{s}^*, \boldsymbol{x})$, Π convex (uniformly on i).
- $\pi(\boldsymbol{s}^*, \boldsymbol{x}) = (\pi_1(\boldsymbol{s}^*, \boldsymbol{x}), \dots, \pi_N(\boldsymbol{s}^*, \boldsymbol{x}))$ is monotone injective
- The inverse is $\boldsymbol{z} \mapsto \boldsymbol{\phi}(\boldsymbol{z}, \boldsymbol{x}) = (\phi_1(\boldsymbol{z}, \boldsymbol{x}), \dots, \phi_N(\boldsymbol{z}, \boldsymbol{x}))$
- $0 < \frac{1}{\varpi^\star} \leq \mathbb{J}_{\boldsymbol{z}} \boldsymbol{\phi}(\boldsymbol{z}, \boldsymbol{x}) \leq \frac{1}{\varpi_\star}$

Weak solution :

- $\boldsymbol{p} = (p_0, \dots, p_N) \in L^2((0, T); H^1(\Omega))^{N+1}$
- For all $\phi \in C_c^\infty(\overline{\Omega} \times [0, T))$ and all $i \in \{0, \dots, N\}$:

$$\iint_Q s_i \partial_t \phi \mathrm{d}\boldsymbol{x} \mathrm{d}t + \int_\Omega s_i^0 \phi(\cdot, 0) \mathrm{d}\boldsymbol{x} - \iint_Q \frac{s_i}{\mu_i} \mathbb{K} \left(\boldsymbol{\nabla} p_i - \rho_i \mathbf{g} \right) \cdot \boldsymbol{\nabla} \phi \mathrm{d}\boldsymbol{x} \mathrm{d}t = 0.$$

Gradient flow reformulation

The Riemmanian space is

$$\boldsymbol{\mathcal{X}} = \left\{ \boldsymbol{s} \in L^1(\Omega; \mathbb{R}_+^{N+1}) \;\middle|\; \sum_{i=0}^{N} s_i = 1 \right\} \subset \mathbb{P}(\Omega)^{N+1}.$$

with the induced Monge Kantorovich metric $(MK^2)^{N+1}$.
The energy is

$$\mathcal{E}(\boldsymbol{s}) = \int_\Omega (\Pi(\boldsymbol{s}, \boldsymbol{x}) + \boldsymbol{s} \cdot \boldsymbol{\Psi})\, \mathrm{d}\boldsymbol{x} \geq 0, \qquad \forall \boldsymbol{s} \in L^1(\Omega; \mathbb{R}^{N+1}).$$

where

$$\Pi(\boldsymbol{s}, \boldsymbol{x}) = \begin{cases} \Pi\left(\omega \frac{\boldsymbol{s}^*}{\sigma}, \boldsymbol{x}\right) = \Pi\left(\omega \frac{s_1}{\sigma}, \dots, \omega \frac{s_N}{\sigma}, \boldsymbol{x}\right) & \boldsymbol{s} \in \mathbb{R}_+^{N+1} \text{ and } \sigma \leq \omega(\boldsymbol{x}), \\ +\infty & \text{otherwise,} \end{cases}$$

and $\mathcal{A}_i = \left\{ s_i \in L^1(\Omega; \mathbb{R}_+) \,\middle|\, \int_\Omega s_i \mathrm{d}\boldsymbol{x} = m_i \right\}$.

JKO-scheme

JKO-scheme

$$\boldsymbol{s}^n = \underset{\boldsymbol{s} \in \boldsymbol{\mathcal{A}}}{\operatorname{argmin}} \left(\frac{\boldsymbol{W}(\boldsymbol{s}, \boldsymbol{s}^{n-1})^2}{2\tau} + \mathcal{E}(\boldsymbol{s}) \right),$$

Theorem

Let $(\tau_k)_{k \geq 1}$ be a sequence of time steps tending to 0, then there exists one weak solution $\boldsymbol{s}$ such that, up to an unlabeled subsequence, $(\boldsymbol{s}^{\tau_k})_{k \geq 1}$ converges a.e. in Q towards s as k tends to ∞.

Remark

a posteriori $\partial_t s_i \in L^2((0,T); H^1(\Omega)')$, $s_i \in L^2((0,T); H^1(\Omega))$, and thus $s_i \in \mathcal{C}([0,T]; L^2(\Omega))$. It yields *Energy-Dissipation-Equality*

$$\frac{d}{dt}\mathcal{E}(\boldsymbol{s}(t)) = -\sum_{i=0}^{N} \int_\Omega \mathbb{K} \frac{s_i(t)}{\mu_i} \nabla(p_i(t) + \Psi_i) \cdot \nabla(p_i(t) + \Psi_i) \mathrm{d}\boldsymbol{x} \leq 0 \quad \text{a.e.},$$

Preliminary

Proposition

Let $n \geq 1$ and $\boldsymbol{s}^{n-1} \in \boldsymbol{\mathcal{X}} \cap \boldsymbol{\mathcal{A}}$, then there exists a unique solution $\boldsymbol{s}^n$ to the JKO scheme. Moreover, one has $\boldsymbol{s}^n \in \boldsymbol{\mathcal{X}} \cap \boldsymbol{\mathcal{A}}$.

Existence/uniqueness of the minimizer ok.

$$\frac{\boldsymbol{W}(\boldsymbol{s}^n, \boldsymbol{s}^{n-1})^2}{2\tau} + \mathcal{E}(\boldsymbol{s}^n) \leq \mathcal{E}(\boldsymbol{s}^{n-1}),$$

Classical estimates

- $\mathcal{E}(\boldsymbol{s}^n) \leq \mathcal{E}(\boldsymbol{s}^{n-1}) < \infty$.
- $\sum_{n \geq 0} \boldsymbol{W}^2(\boldsymbol{s}^{n+1}, \boldsymbol{s}^n) \leq 2\tau\mathcal{E}(\boldsymbol{s}^0)$,
- 1/2-Hölder estimate $\boldsymbol{W}(\boldsymbol{s}^{n_1}, \boldsymbol{s}^{n_2}) \leq C\sqrt{|n_2 - n_1|\tau + 1)\tau}$.

Flow-interchange

Lemma

There exists C depending only on $\Omega, \Pi, (\mu_i)_i$, and $\boldsymbol{\Psi}$ such that, for all $n \geq 1$ and all $\tau > 0$, one has

$$\sum_{i=0}^{N} \|\nabla \pi_i^n\|_{L^2(\Omega)}^2 \leq C \left(1 + \frac{\boldsymbol{W}^2(\boldsymbol{s}^n, \boldsymbol{s}^{n-1})}{\tau} + \sum_{i=0}^{N} \frac{\mathcal{H}(s_i^{n-1}) - \mathcal{H}(s_i^n)}{\tau} \right).$$

Auxiliary flow

$$\begin{cases} \partial_t \check{s}_i = \mathrm{div}(\nabla \check{s}_i), & t > 0,\ \boldsymbol{x} \in \Omega, \\ (\nabla \check{s}_i) \cdot \nu = 0, & t > 0,\ \boldsymbol{x} \in \partial\Omega, \\ \check{s}_i\big|_{t=0} = s_i^n, & \boldsymbol{x} \in \Omega. \end{cases}$$

Linearized problem

Lemma

For $n \geq 1$ and $i = 0, \ldots, N$ there exist some (backwards, optimal) Kantorovich potentials φ_i^n from s_i^n to s_i^{n-1} such that, setting

$$\begin{cases} F_0^n = 0, \\ F_i^n = \frac{\varphi_i^n}{\tau} - \frac{\varphi_0^n}{\tau} + \pi_i^n + \Psi_i - \Psi_0, \quad \forall i \in \{1, \ldots, N\}, \end{cases}$$

and $\mathbf{F}^n = (F_i^n)_{0 \leq i \leq N}$, there holds

$$\boldsymbol{s}^n = \underset{\boldsymbol{s} \in \boldsymbol{\mathcal{X}} \cap \boldsymbol{\mathcal{A}}}{\operatorname{Argmin}} \int_\Omega \mathbf{F}^n(\boldsymbol{x}) \cdot \boldsymbol{s}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}.$$

Moreover, $F_i^n \in L^\infty \cap H^1(\Omega)$ for all $i \in \{0, \ldots, N\}$.

Linearized problem, proof

- $s_i^\varepsilon := (1-\varepsilon)s_i^n + \varepsilon s_i \quad$ for $i \in \{1, \dots, N\}$ and $s_0^\varepsilon := 1 - \sum_{i=1}^N s_i^\varepsilon$.
- $(\varphi_i^\varepsilon, \psi_i^\varepsilon)$ is unique Kantorovich potentials from s_i^ε to s_i^{n-1}
- $\frac{1}{2\tau}\left(W_i^2(s_i^\varepsilon, s_i^{n-1}) - W_i^2(s_i^n, s_i^{n-1})\right) \leq \frac{\varepsilon}{\tau}\int_\Omega \varphi_i^\varepsilon(s_i - s_i^n)\mathrm{d}\boldsymbol{x}$,
- $\frac{1}{2\tau}\left(W_0^2(s_0^\varepsilon, s_0^{n-1}) - W_0^2(s_0^n, s_0^{n-1})\right) \leq -\frac{\varepsilon}{\tau}\sum_{i=1}^N \int_\Omega \varphi_0^\varepsilon(s_i - s_i^n)\mathrm{d}\boldsymbol{x}$.
- Convexity of Π.
- $0 \leq \frac{\mathcal{F}_\tau^n(\boldsymbol{s}^\varepsilon) - \mathcal{F}_\tau^n(\boldsymbol{s}^n)}{\varepsilon} \leq$
  $\sum_{i=1}^N \int_\Omega \left(\frac{\varphi_i^\varepsilon}{\tau} - \frac{\varphi_0^\varepsilon}{\tau} + \pi_i^\varepsilon + \Psi_i - \Psi_0\right)(s_i - s_i^n)\mathrm{d}\boldsymbol{x}$

Decomposition Lemma

Lemma

There exists $\boldsymbol{\alpha}^n \in \mathbb{R}^{N+1}$ and $\boldsymbol{\lambda}^n \in L^\infty \cap H^1(\Omega;\mathbb{R}^{N+1})$ such that

$$F_i^n = \lambda_i^n - \lambda_0^n - \alpha_i^n \text{ a.e. in } \Omega, \qquad \forall i \in \{0,\dots,N\}, \tag{1}$$

$$\boldsymbol{\nabla}\lambda_i^n = 0 \text{ in } \{s_i > 0\}, \qquad \forall i \in \{0,\dots,N\}. \tag{2}$$

Proof

$$\underline{I} = \inf_{\boldsymbol{s}\in\overline{\boldsymbol{\mathcal{X}}}\cap\overline{\boldsymbol{\mathcal{A}}}} \mathrm{I}(\boldsymbol{s}) = \min_{\boldsymbol{s}\in\boldsymbol{\mathcal{X}}\cap\boldsymbol{\mathcal{A}}} \mathrm{I}(\boldsymbol{s}).$$

Let $\boldsymbol{\alpha} \in \mathbb{R}^{N+1}$ and define $\mu_{\boldsymbol{\alpha}} \in E$ by

$$\mu_{\boldsymbol{\alpha}} = \sup_{i\in\{0,\dots,N\}} \left(-F_i^n - \alpha_i\right).$$

Decomposition Lemma II

Let $\mathcal{J} : \mathbb{R}^{N+1} \to \mathbb{R}$ by

$$\mathcal{J}(\boldsymbol{\alpha}) = -\int_\Omega \mu_{\boldsymbol{\alpha}} \omega \mathrm{d}\boldsymbol{x} - \sum_{i=0}^{N} \alpha_i m_i, \qquad \forall \boldsymbol{\alpha} \in \mathbb{R}^{N+1},$$

then

$$\underline{I} = \overline{J} = \sup_{\boldsymbol{\alpha} \in \mathbb{R}^{N+1}} \mathcal{J}(\boldsymbol{\alpha}).$$

- $\boldsymbol{\alpha}$ are the Lagrange multiplier of the mass contraint.
- $\mu_{\boldsymbol{\alpha}}$ is the Lagrange multiplier of the saturation constraint.

Decomposition Lemma III

Optimality conditions

$$\underline{I} = \mathrm{I}(\boldsymbol{s}^n) = \int_\Omega \mathbf{F}^n \cdot \boldsymbol{s}^n \mathrm{d}\boldsymbol{x} = \overline{J} = \mathcal{J}(\boldsymbol{\alpha}^n)$$

$$= -\int_\Omega \mu^n \omega \mathrm{d}\boldsymbol{x} - \sum_{i=0}^{N} \alpha_i^n m_i = -\sum_{i=0}^{N} \int_\Omega (\mu^n + \alpha_i^n) s_i \mathrm{d}\boldsymbol{x}$$

which leads to

$$\sum_{i=0}^{N} \int_\Omega (F_i^n(\boldsymbol{x}) + \mu^n(\boldsymbol{x}) + \alpha_i^n) s_i(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = 0.$$

Define

$$\lambda_i^n(\boldsymbol{x}) := F_i^n(\boldsymbol{x}) + \mu^n(\boldsymbol{x}) + \alpha_i^n, \qquad i \in \{0, \dots, N\},$$

Decomposition Lemma IIII

Regularity:

- $\lambda_i^n(\boldsymbol{x}) \geq 0$
- $\mu^n \in H^1(\Omega)$
- $\boldsymbol{\nabla}\mu^n = -\chi_{\{\mu^n = -F_i^n - \alpha_i^n\}} \boldsymbol{\nabla}(F_i^n + \alpha_i^n)$
- $\lambda_i^n(\boldsymbol{x}) \in H^1(\Omega)$

There holds

$$s_i^n(\boldsymbol{x}) > 0 \quad \Rightarrow \quad \lambda_i^n(\boldsymbol{x}) = 0 \quad \Rightarrow \quad \mu^n(\boldsymbol{x}) = -F_i^n(\boldsymbol{x}) - \alpha_i^n.$$

$$\text{in } \{s_i^n > 0\} : \quad \begin{aligned} \boldsymbol{\nabla}\lambda_i^n &= \boldsymbol{\nabla}\left(F_i^n + \mu^n + \alpha_i^n\right) = \boldsymbol{\nabla} F_i^n + \boldsymbol{\nabla}\mu^n \\ &= \boldsymbol{\nabla} F_i^n - \chi_{\{\mu^n = -F_i^n - \alpha_i^n\}} \boldsymbol{\nabla}(F_i^n + \alpha_i^n) \\ &= \boldsymbol{\nabla} F_i^n - \boldsymbol{\nabla} F_i^n = 0, \end{aligned}$$

The $H^1(\Omega)$ estimates

Proposition

For $n \geq 1$ let $\varphi_i^n : s_i^n \to s_i^{n-1}$ be the (backward) Kantorovich potentials, then there exists $\boldsymbol{h} = (h_0^n, \ldots, h_N^n) \in H^1(\Omega)^{N+1}$ such that

(i) $\boldsymbol{\nabla} h_i^n = -\frac{\boldsymbol{\nabla}\varphi_i^n}{\tau}$ for $\mathrm{d}s_i^n$-a.e. $\boldsymbol{x} \in \Omega$

(ii) $h_i^n(\boldsymbol{x}) - h_0^n(\boldsymbol{x}) = \pi_i^n(\boldsymbol{x}) + \Psi_i(\boldsymbol{x}) - \Psi_0(\boldsymbol{x})$ for $\mathrm{d}\boldsymbol{x}$-a.e. $\boldsymbol{x} \in \Omega$, $i \in \{0, \ldots, N\}$

(iii) there exists C depending only on Ω, Π and $\boldsymbol{\Psi}$ such that, for all $n \geq 1$ and all $\tau > 0$, one has

$$\|\boldsymbol{h}^n\|^2_{H^1(\Omega)^{N+1}} \leq C \left(1 + \frac{\boldsymbol{W}^2(\boldsymbol{s}^n, \boldsymbol{s}^{n-1})}{\tau^2} + \sum_{i=0}^{N} \frac{\mathcal{H}(s_i^{n-1}) - \mathcal{H}(s_i^n)}{\tau} \right).$$

$$h_i^n := -\frac{\varphi_i^n}{\tau} + \lambda_i^n - \alpha_i^n, \qquad \forall i \in \{0, \ldots, N\},$$

Convergence

We define $p_i = h_i^n - \Psi_i(\boldsymbol{x})$ then

- $p_i^n - p_0^n = \pi_i^n, \qquad \forall i \in \{1, \dots, N\}.$
- $\|\boldsymbol{p}^n\|^2_{H^1(\Omega)} \leq C\left(1 + \frac{\boldsymbol{W}^2(\boldsymbol{s}^n, \boldsymbol{s}^{n-1})}{\tau^2} + \sum_{i=0}^{N} \frac{\mathcal{H}_\omega(s_i^{n-1}) - \mathcal{H}_\omega(s_i^n)}{\tau}\right)$
- Convergence forte des π_i^n.
- Convergence forte des s_i^n.

Extension

- mobility s_i changed by $\mu_i(s_i)$ convex (concave ok).
- Add some reactions terms/source terms (collect the estimates in the KFR metric).
- Lagrangian Numerical scheme (ALG2-JKO) (Maxime Laborde), Entropic relaxation, ...
- Comparaison with finite volume scheme.