Parameter dependent systems,
Averaged and simultaneous controllability

Jérôme Lohéac,
joint work with E. Zuazua

Institut de Recherche en Communications et Cybernétique de Nantes

Journée Équipe McTAO

The general Problem

Let X and U be two Banach spaces. Consider the parameter dependent Cauchy problems:

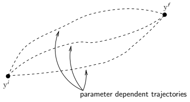
$$\dot{x}_\zeta = A_\zeta x_\zeta + B_\zeta u\,, \qquad x_\zeta(0) = \mathrm{x}^i_\zeta \in X\,, \qquad (\star)$$

with parameter $\zeta \in \Omega$ and $(\Omega, \mathcal{F}, \mu)$ a probability space.

The aim:
given $(\mathrm{x}^i_\zeta)_{\zeta\in\Omega}$, $(\mathrm{x}^f_\zeta)_{\zeta\in\Omega}$ and $T > 0$,
find $u \in L^2([0, T], U)$ such that the solution of $x_\zeta(\cdot; u)$ of $(\star)$ satisfies:

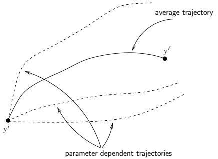
Simultaneous controllability:

$$x_\zeta(T; u) = x^f_\zeta \quad (\zeta \in \Omega \ \ \mu - \text{a.e.})\,. \quad \text{(SC)}$$



Averaged controllability:

$$\int_\Omega x_\zeta(T; u)\,\mathrm{d}\mu_\zeta = \int_\Omega x^f_\zeta\,\mathrm{d}\mu_\zeta\,. \qquad \text{(AC)}$$



State of art

- Averaged controllability:
  - E. Zuazua, *Averaged control*, 2014
  - M. Lazar and E. Zuazua, *Averaged control and observation of parameter depending wave equations*, 2014
- Ensemble controllability:
  - J.-S. Li and N. Khaneja, *Ensemble control of Bloch equations*, 2009
  - J.-S. Li, *Ensemble control of finite dimensional time-varying linear systems*, 2011
- Simultaneous controllability:
  - D. L. Russel, *The Dirichlet-Neumann boundary control problem associated with Maxwell's equations in a cylindrical region*, 1986
  - J. L. Lions, *Contrôlabilité exacte, perturbation et stabilisation de systèmes distribués. Tome 1*, 1988
  - ...

Admissibility conditions I

For $u \in L^2_{loc}(\mathbb{R}_+, U)$, the solution $x_\zeta(\cdot; u)$ of $(\star)$ is:

$$x_\zeta(t,u) = e^{tA_\zeta} \mathrm{x}^i_\zeta + \int_0^t e^{(t-s)A_\zeta} B_\zeta u(s)\,\mathrm{d}s \qquad (\zeta \in \Omega\,,\ t \geqslant 0)\,.$$

Define the Hilbert space:

$$L^2(\Omega, X; \mu) = \left\{ (\mathrm{x}_\zeta)_\zeta \in X^\Omega\,,\ \int_\Omega \|\mathrm{x}_\zeta\|_X^2 \,\mathrm{d}\mu_\zeta < \infty \right\}\,,$$

with the scalar product:

$$(\mathrm{x}_\zeta, \mathrm{y}_\zeta)_{L^2(\Omega,X;\mu)} = \int_\Omega (\mathrm{x}_\zeta, \mathrm{y}_\zeta)_X \,\mathrm{d}\mu_\zeta \qquad ((\mathrm{x}_\zeta)_\zeta, (\mathrm{y}_\zeta)_\zeta \in L^2(\Omega, X; \mu))\,.$$

Lemma

Assume for every $T > 0$ and almost every $\zeta \in \Omega$, there exists $\varsigma_\zeta(T) > 0$ such that:

$$\left\| e^{tA_\zeta^*} e^{tA_\zeta} \mathrm{x} \right\|_X \leqslant \varsigma_\zeta(T) \|\mathrm{x}\|_X \qquad (\mathrm{x} \in X)\,.$$

Then, for every $T > 0$, there exists $\varsigma(T) > 0$ such that:

$$\left\| e^{tA_\zeta} \mathrm{x}_\zeta \right\|_{L^2(\Omega,X;\mu)} \leqslant \varsigma(T) \|\mathrm{x}_\zeta\|_{L^2(\Omega,X;\mu)} \qquad ((\mathrm{x}_\zeta)_\zeta \in L^2(\Omega, X; \mu))\,.$$

Admissibility conditions II

Example

A_ζ skew-adjoint for almost every $\zeta \in \Omega$.

Lemma

Assume for every $T > 0$ and almost every $\zeta \in \Omega$, there exists a constant $C_\zeta(T) > 0$ such that:

$$\left\| \int_0^T e^{(T-t)A_\zeta} B_\zeta u(t) \, \mathrm{d}t \right\|_X^2 \leqslant C_\zeta(T) \|u\|^2_{L^2([0,T],U)}$$

and assume that:

$$\int_\Omega C_\zeta(T) \, \mathrm{d}\mu_\zeta < \infty \, .$$

Then for every $T > 0$, there exists $\hat{C}(T) > 0$ such that:

$$\int_\Omega \left\| \int_0^T e^{(T-t)A_\zeta} B_\zeta u(t) \, \mathrm{d}t \right\|_X^2 \mathrm{d}\mu_\zeta \leqslant \hat{C}(T) \|u\|^2_{L^2([0,T],U)} \qquad (u \in L^2([0,T],U)) \, .$$

Admissibility conditions III

By application of these two lemmas, we have for instance:

Example

Assume $\Omega \subset \mathbb{R}^d$ is a bounded set and the map $\zeta \in \Omega \mapsto (A_\zeta, B_\zeta) \in \mathcal{L}(X) \times \mathcal{L}(U, X)$ is continuous on $\overline{\mathrm{co}(\Omega)}$.
Then for every $\mathrm{x}^i_\zeta \in L^2(\Omega, X; \mu)$ end every $u \in L^2_{loc}(\mathbb{R}_+, U)$, the solution $x_\zeta(t; u)$ of $(\star)$ belongs to $L^2(\Omega, X; \mu)$ for every $t \geqslant 0$.

In the reminding part of this talk, we assume the assumption of these two lemmas satisfied.

Exact simultaneous controllability and observability

Definition

System $(\star)$ is said *exactly simultaneously controllable* in time $T > 0$ if:

$$\left\{\int_0^T e^{(T-t)A_\zeta} B_\zeta u(t)\,\mathrm{d}t\,, \quad \zeta \in \Omega\,,\ u \in L^2([0,T],U)\right\} = L^2(\Omega, X; \mu)\,.$$

Let us consider for every $(\mathrm{z}_\zeta^f)_\zeta \in L^2(\Omega, X; \mu)$ the adjoint system:

$$-\dot{z}_\zeta = A_\zeta^* z_\zeta\,, \quad z_\zeta(T) = \mathrm{z}_\zeta^f\,. \tag{Adj}$$

Definition

The system (Adj) is said *exactly simultaneously observable* in time $T > 0$ if there exists $c(T) > 0$ such that:

$$c(T)\|\mathrm{z}_\zeta^f\|^2_{L^2(\Omega,X;\mu)} \leqslant \int_0^T \left\|\int_\Omega B_\zeta^* z_\zeta(t)\,\mathrm{d}\mu_\zeta\right\|_X^2 \mathrm{d}t \qquad ((\mathrm{z}_\zeta^f)_\zeta \in L^2(\Omega, X; \mu))\,.$$

Duality results I

Theorem

- *The system* $(\star)$ *is simultaneously admissible if and only if* (Adj) *is, i.e.*

$$\forall T>0\,,\ \exists \hat{C}(T)>0\,,$$
$$\int_0^T \left\| \int_\Omega B_\zeta^* z_\zeta(t)\,\mathrm{d}\mu_\zeta \right\|_X^2 \mathrm{d}t \leqslant \hat{C}(T) \|\mathrm{z}_\zeta^f\|^2_{L^2(\Omega,X;\mu)} \qquad ((\mathrm{z}_\zeta^f)_\zeta \in L^2(\Omega,X;\mu))\,;$$

- *The system* $(\star)$ *is exactly simultaneously controllable in time* $T>0$ *if and only if* (Adj) *is exactly simultaneously observable in time* T*.*

Duality results II

Thus,

Theorem

If the system $(\star)$ is admissible and exactly simultaneously controllable in time $T > 0$, then for every $(\mathrm{x}^f_\zeta)_\zeta \in L^2(\Omega, X; \mu)$, the cost function:

$$\begin{aligned} \hat{\mathfrak{J}} : L^2(\Omega, X; \mu) &\to \mathbb{R} \\ (\mathrm{z}^f_\zeta)_\zeta &\mapsto \frac{1}{2}\int_0^T \left\| \int_\Omega B^*_\zeta z_\zeta(t)\,\mathrm{d}\mu_\zeta \right\|_X^2 \mathrm{d}t \\ &\qquad - \int_\Omega (\mathrm{x}^f_\zeta, \mathrm{z}^f_\zeta)_X\,\mathrm{d}\mu_\zeta + \int_\Omega (\mathrm{x}^i_\zeta, \mathrm{z}_\zeta(0))_X\,\mathrm{d}\mu_\zeta \end{aligned}$$

is convex and coercive.
In addition, let $(\hat{\mathrm{z}}^f_\zeta)_\zeta \in L^2(\Omega, X; \mu)$ be the minimizer of $\hat{\mathfrak{J}}$, then, the control:

$$\hat{u}(t) = \int_\Omega B^*_\zeta \hat{z}_\zeta(t)\,\mathrm{d}\mu_\zeta \qquad (t \in [0, t] \text{ a.e.})$$

is the control of minimal $L^2([0, T], U)$-norm steering $(\mathrm{x}^i_\zeta)_\zeta$ to $(\mathrm{x}^f_\zeta)_\zeta$ in time T.

When simultaneous controllability is easy to check...

Set $(\Omega, \mathcal{F}, \mu)$, a probability space with $\Omega = \{\zeta_1, \dots, \zeta_N\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and μ such that $\mu(\{\zeta_n\}) > 0$ for every $n \in \{1, \dots, N\}$.
Then the simultaneous controllability of $(\star)$ means controllability of the augmented system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u\,,$$

with:

$$\mathbf{x} = \begin{pmatrix} x_{\zeta_1} \\ \vdots \\ x_{\zeta_N} \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} A_{\zeta_1} & & 0 \\ & \ddots & \\ 0 & & A_{\zeta_N} \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} B_{\zeta_1} \\ \vdots \\ B_{\zeta_N} \end{pmatrix}.$$

In particular, if $\dim X < \infty$, the simultaneous controllability of $(\star)$ means that the Kalman rank, $\operatorname{rank}\begin{pmatrix}\mathbf{B} & \mathbf{A}\mathbf{B} & \dots & \mathbf{A}^{N\dim X - 1}\mathbf{B}\end{pmatrix} = N\dim X$.

When simultaneous controllability cannot holds...

Proposition

If $L^2(\Omega, \mathbb{R}; \mu)$ is an infinite dimensional space, then the system $(\star)$ cannot be exactly simultaneously controllable.

Example

Consider the system $\dot{y}_\zeta = (\zeta + 1)y_\zeta + u$ with $\zeta \in \left\{\frac{1}{n},\ n \in \mathbb{N}^*\right\} = \Omega$ with a probability measure μ such that $\mu(\{\zeta\}) > 0$ for every $\zeta \in \Omega$.
This system is not exactly simultaneously controllable although the truncated system in which we consider $\zeta \in \left\{\frac{1}{n},\ n \in \{1, \cdots, N\}\right\}$ with probability measure μ_N given by $\mu_N(\{\zeta\}) = \frac{\mu(\{\zeta\})}{\mu(\{\frac{1}{N}, \frac{1}{N-1}, \cdots, 1\})}$, for $\zeta \in \left\{\frac{1}{n},\ n \in \{1, \cdots, N\}\right\}$ is simultaneously controllable, whatever $N \in \mathbb{N}^*$ is.

Exact averaged controllability and observability

Definition

System $(\star)$ is said *exactly controllable in average* in time $T>0$ if:

$$\left\{\int_\Omega \int_0^T e^{(T-t)A_\zeta} B_\zeta u(t)\,\mathrm{d}t\mathrm{d}\mu_\zeta\,,\ u\in L^2([0,T],U)\right\} = X\,.$$

Let us consider for every $\mathrm{z}^f \in X$ the adjoint system:

$$-\dot{z}_\zeta = A_\zeta^* z_\zeta\,, \quad z_\zeta(T) = \mathrm{z}^f\,. \tag{$\overline{\mathrm{Adj}}$}$$

Definition

The system $(\overline{\mathrm{Adj}})$ is said *exactly observable in average* in time $T>0$ if there exists $\bar{c}(T)>0$ such that:

$$\bar{c}(T)\|\mathrm{z}^f\|_X^2 \leqslant \int_0^T \left\|\int_\Omega B_\zeta^* z_\zeta(t)\,\mathrm{d}\mu_\zeta\right\|_X^2 \mathrm{d}t \qquad (\mathrm{z}^f\in X)\,.$$

Exact averaged observability property is exact simultaneous observability one restricted to the subspace $\{(\mathrm{z}^f)_\zeta\,,\ \mathrm{z}^f\in X\}$ of $L^2(\Omega, X;\mu)$.
Consequently, it is obvious that:

exact simultaneous observability $\Longrightarrow$ exact averaged observability.

Duality results I

Theorem

- *The system* $(\star)$ *is admissible in average if and only if* $(\overline{\mathrm{Adj}})$ *is, i.e.*

$$\forall T > 0\,,\ \exists \bar{C}(T) > 0\,,$$
$$\int_0^T \left\| \int_\Omega B_\zeta^* z_\zeta(t)\,\mathrm{d}\mu_\zeta \right\|_X^2 \mathrm{d}t \leqslant \bar{C}(T) \|\mathrm{z}^f\|_X^2 \qquad (\mathrm{z}^f \in X)\,;$$

- *The system* $(\star)$ *is exactly controllable in average in time* $T > 0$ *if and only if* $(\overline{\mathrm{Adj}})$ *is exactly observable in average in time* T*.*

Duality results II

Thus,

Theorem

If the system $(\star)$ *is admissible and exactly controllable in average in time* $T > 0$*, then for every* $\mathrm{x}^f \in X$*, the cost function:*

$$\begin{aligned} \bar{\mathfrak{J}} : X &\to \mathbb{R} \\ \mathrm{z}^f &\mapsto \frac{1}{2}\int_0^T \left\| \int_\Omega B_\zeta^* z_\zeta(t)\,\mathrm{d}\mu_\zeta \right\|_X^2 \mathrm{d}t \\ &\qquad - \int_\Omega (\mathrm{x}_\zeta^f, \mathrm{z}^f)_X \,\mathrm{d}\mu_\zeta + \int_\Omega (\mathrm{x}_\zeta^i, \mathrm{z}_\zeta(0))_X \,\mathrm{d}\mu_\zeta \end{aligned}$$

is convex and coercive.
In addition, let $\bar{\mathrm{z}}^f \in X$ *be the minimizer of* $\bar{\mathfrak{J}}$*, then, the control:*

$$\bar{u}(t) = \int_\Omega B_\zeta^* \bar{z}_\zeta(t)\,\mathrm{d}\mu_\zeta \qquad (t \in [0, t] \text{ a.e.})$$

is the control of minimal $L^2([0, T], U)$*-norm steering* $(\mathrm{x}_\zeta^i)_\zeta$ *to* $(\mathrm{x}_\zeta^f)_\zeta$ *in average in time* T*.*

Kalman rank condition

Theorem (Zuazua 2014)

Assume $\dim X < \infty$, *then the system* $(\star)$ *is controllable in average if and only if the rank condition:*

$$\operatorname{rank}\left[\int_\Omega A_\zeta^j B_\zeta \,\mathrm{d}\mu_\zeta\,, \quad j \in \mathbb{N}\right] = \dim X$$

is satisfied.

Perturbation argument I

Theorem

Set $T > 0$, $(\Omega, \mathcal{F}, \tilde{\mu})$ *a probability space and* $\zeta_0 \in \Omega$.
Assume:

- $\{\zeta_0\} \in \mathcal{F}$ *and there exists* $C_{\zeta_0}(T), c_{\zeta_0}(T) > 0$ *such that:*
$$c_{\zeta_0}(T)\|\mathrm{z}^f\|_X^2 \leqslant \int_0^T \left\|B_{\zeta_0}^* z_{\zeta_0}(t)\right\|_U^2 \,\mathrm{d}t \leqslant C_{\zeta_0}(T)\|\mathrm{z}^f\|_X^2 \qquad (\mathrm{z}^f \in X);$$
- *For almost every* $\zeta \in \Omega$, *there exists* $C_\zeta(T) > 0$ *such that:*
$$\int_0^T \left\|B_\zeta^* z_\zeta(t)\right\|_U^2 \,\mathrm{d}t \leqslant C_\zeta(T)\|\mathrm{z}^f\|_X^2 \qquad (\mathrm{z}^f \in X);$$
and
$$\int_\Omega \sqrt{C_\zeta(T)}\,\mathrm{d}\tilde{\mu}_\zeta < \infty\,.$$

Then for every $\theta \in \left[0, \left(1 + \int_\Omega \sqrt{\frac{C_\zeta(T)}{c_{\zeta_0}(T)}}\,\mathrm{d}\tilde{\mu}_\zeta\right)^{-1}\right)$, *the system* $(\star)$ *is exactly controllable in average for the probability measure:*
$$\mu = \theta\tilde{\mu} + (1-\theta)\delta_{\zeta_0}\,.$$

Perturbation argument II

Proof: The proof is based on Minkowski inequality,

$$\left(\int_0^T \left\|\int_\Omega B_\zeta^* z_\zeta(t)\,\mathrm{d}\mu_\zeta\right\|_U^2 \mathrm{d}t\right)^{\frac{1}{2}}$$

$$= \left(\int_0^T \left\|(1-\theta)B_{\zeta_0}^* z_{\zeta_0}(t) + \theta\int_\Omega B_\zeta^* z_\zeta(t)\,\mathrm{d}\tilde{\mu}_\zeta\right\|_U^2 \mathrm{d}t\right)^{\frac{1}{2}}$$

$$\geqslant (1-\theta)\left(\int_0^T \left\|B_{\zeta_0}^* z_{\zeta_0}(t)\right\|_U^2 \mathrm{d}t\right)^{\frac{1}{2}} - \theta\left(\int_0^T \left\|\int_\Omega B_\zeta^* z_\zeta(t)\,\mathrm{d}\tilde{\mu}_\zeta\right\|_U^2 \mathrm{d}t\right)^{\frac{1}{2}}$$

$$\geqslant (1-\theta)\sqrt{c_{\zeta_0}(T)}\|\mathrm{z}^f\|_X - \theta\int_\Omega \sqrt{C_\zeta(T)}\,\mathrm{d}\tilde{\mu}_\zeta\|\mathrm{z}\|_X$$

Ingham inequalities I

Consider $X = \ell^2(\mathbb{N}^*)$, $U = \mathbb{C}$ and $(\lambda_n)_{n\in\mathbb{N}^*} \in \mathbb{R}^{\mathbb{N}^*}$.

Set the operator A of domain $\mathcal{D}(A) = \left\{(a_n)_n \in \ell^2(\mathbb{N}^*)\,,\ \sum_{n\in\mathbb{N}^*} |\lambda_n||a_n|^2 < \infty\right\} := X_1$

defined by:

$$Ae_n = 2i\pi\lambda_n e_n \qquad (n \in \mathbb{N}^*)$$

and the operator $B \in \mathcal{L}(U, X_{-1})$ defined by :

$$[Bv]_n = v \qquad (v \in \mathbb{C})\,.$$

Consider the system:

$$\dot{x} = Ax + Bu \qquad \text{i.e.} \qquad \dot{x}_n = 2i\pi\lambda_n x_n + u \quad (n \in \mathbb{N}^*)\,.$$

The adjoint system is:

$$\dot{z} = -Az \qquad \text{i.e.} \qquad \dot{z}_n = 2i\pi\lambda_n z_n \quad (n \in \mathbb{N}^*) \qquad \text{thus} \quad z_n(t) = e^{2i\pi\lambda_n t} z_n(0)$$

and the observation operator:

$$\begin{array}{rcl} X_1 & \to & L^2([0,T],\mathbb{C}) \\ (a_n)_n & \mapsto & \sum_{n\in\mathbb{N}^*} a_n e^{2i\pi\lambda_n t} \end{array}.$$

Ingham inequalities II

Theorem (Ingham inequalities, Ingham 1936)

Assume $\inf_{\substack{m,n\in\mathbb{N}^* \\ m\neq n}} |\lambda_m - \lambda_n| := \gamma > 0.$

Then for every $T > 0$*, there exists* $C(T) > 0$ *such that:*

$$\int_0^T \left| \sum_{n\in\mathbb{N}^*} a_n e^{2i\pi\lambda_n t} \right|^2 \mathrm{d}t \leqslant C(T) \sum_{n\in\mathbb{N}^*} |a_n|^2$$

and for every $T > \frac{1}{\gamma}$*, there exists* $c(T) > 0$ *such that:*

$$c(T) \sum_{n\in\mathbb{N}^*} |a_n|^2 \leqslant \int_0^T \left| \sum_{n\in\mathbb{N}^*} a_n e^{2i\pi\lambda_n t} \right|^2 \mathrm{d}t \,.$$

Consequently, if $\gamma > 0$ and $T > \frac{1}{\gamma}$, the system $\dot{x} = Ax + Bu$ is exactly controllable.

Averaged Ingham inequalities I

Let us now consider the probability space $(\Omega, \mathcal{F}, \mu)$ given by $\Omega = \{\zeta_0, \dots, \zeta_K\} \subset \mathbb{R}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and μ given by $\mu(\{\zeta_k\}) := \theta_k \in [0, 1]$.
Consider the parameter dependent system:

$$\dot{x}_\zeta = \zeta A x_\zeta + Bu\,.$$

The aim is to find $T > 0$ and $\bar{c}(T) > 0$ such that;

$$\bar{c}(T) \sum_{n\in\mathbb{N}^*} |a_n|^2 \leqslant \int_0^T \left| \sum_{k=0}^{K} \theta_k \sum_{n\in\mathbb{N}^*} a_n e^{2i\pi\lambda_n\zeta t} \right|^2 \mathrm{d}t \qquad ((a_n)_n \in X)\,.$$

Theorem

Set $\gamma > 0$ and assume $\lambda_n \in \gamma\mathbb{N}$ Then, if $T > \frac{1}{\gamma}\sum_{k=0}^{K} \frac{1}{|\zeta_k|}$, there exists a constant $\bar{c}(T) > 0$ such that:

$$\theta_0 \bar{c}(T) \sum_{n\in\mathbb{N}^*} |a_n|^2 \prod_{k=1}^{K} \sin\left(\frac{\lambda_n \pi}{\gamma}\frac{\zeta_0}{\zeta_k}\right) \leqslant \int_0^T \left| \sum_{k=0}^{K} \theta_k \sum_{n\in\mathbb{N}^*} a_n e^{2i\pi\lambda_n\zeta t} \right|^2 \mathrm{d}t \qquad ((a_n)_n \in X)\,.$$

Averaged Ingham inequalities II

Idea of the proof:

Set $f(t) = \sum_{k=0}^{K} \theta_k \sum_{n\in\mathbb{N}^*} a_n e^{2i\pi\lambda_n\zeta t}$ and notice that

$$f(t+|\zeta_K|^{-1}) - f(t) = \sum_{k=0}^{K-1} \theta_k \sum_{n\in\mathbb{N}^*} a_n \left(e^{2i\pi\lambda_n \frac{\zeta_k}{|\zeta_K|}} \right) e^{2i\pi\lambda_n\zeta t}.$$

Iterate K times and use Ingham Inequality.

Corollary

Assume in addition $\theta_0 > 0$, $\frac{\zeta_1}{\zeta_0}, \dots, \frac{\zeta_K}{\zeta_0}$ are algebraic and $\zeta_0, \dots, \zeta_K$ are $\mathbb{Q}$-linearly independent.

Then for every $T > \frac{1}{\gamma} \sum_{k=0}^{K} \frac{1}{|\zeta_k|}$, and every $\varepsilon > 0$, there exists $\bar{c}_\varepsilon(T) > 0$ such that:

$$\bar{c}_\varepsilon(T) \sum_{n\in\mathbb{N}} \frac{|a_n|^2}{|\lambda_n|^{2(1+\varepsilon)}} \leqslant \int_0^T \left| \sum_{k=0}^{K} \theta_k \sum_{n\in\mathbb{N}^*} a_n e^{2i\pi\lambda_n\zeta t} \right|^2 \mathrm{d}t\,.$$

Consequently we obtained an Ingham inequality in a weighed space.

Application to the string equation I

Consider the parameter dependent string equation:

$$\begin{aligned} \ddot{w}_\zeta(t,x) &= \zeta^2 \partial_x^2 w_\zeta(t,x) && ((t,x) \in \mathbb{R}_+^* \times (0,1))\,, \\ w_\zeta(t,0) &= u(t) && (t \in \mathbb{R}_+^*)\,, \\ w_\zeta(t,1) &= 0 && (t \in \mathbb{R}_+^*)\,, \end{aligned}$$

$$w_\zeta(0,x) = \mathrm{w}^{i,0}(x) \quad \text{and} \quad \dot{w}_\zeta(0,x) = \mathrm{w}^{i,1}(x) \qquad (x \in (0,1))\,.$$

The adjoint problem of averaged observability is:

$$\begin{aligned} \ddot{z}_\zeta(t,x) &= \zeta^2 \partial_x^2 z_\zeta(t,x) && ((t,x) \in \mathbb{R}_+^* \times (0,1))\,, \\ 0 = z_\zeta(t,0) &= z_\zeta(t,1) && (t \in \mathbb{R}_+^*)\,, \end{aligned}$$

$$z_\zeta(0,x) = \mathrm{z}^{i,0}(x) \quad \text{and} \quad \dot{z}_\zeta(0,x) = \mathrm{z}^{i,1}(x) \qquad (x \in (0,1))\,.$$

and the averaged observability map is:

$$-\sum_{k=0}^{K} \partial_x (A_0^{-1} \dot{z}_\zeta(t,\cdot))(0)\, \zeta_k \theta_k\,.$$

Application to the string equation II

Expanding $z_\zeta(t,x)$ on the Fourier basis $\sin(\pi n\zeta x)$, i.e. $z_\zeta(t,x)=\sum_{n\in\mathbb{N}^*} a_n(t)\sin(\pi n\zeta x)$ leads to an averaged observability map of the type:

$$\sum_{k=0}^{K}\theta_k\sum_{n\in\mathbb{Z}^*} a_n e^{2i\pi\lambda_n\zeta_k t}\,,$$

with $\lambda_n=\frac{1}{2}n$.
Applying the previous corollary, we obtain:

Application to the string equation III

Proposition

Let $\varepsilon > 0$ and assume $\zeta_0, \dots, \zeta_K$ $\mathbb{Q}$-linearly independent and $\frac{\zeta_1}{\zeta_0}, \dots, \frac{\zeta_K}{\zeta_0}$ algebraic.

Then, if $(\mathrm{w}^{i,0}_{\zeta_0}, \mathrm{w}^{i,1}_{\zeta_0}), \dots, (\mathrm{w}^{i,0}_{\zeta_K}, \mathrm{w}^{i,1}_{\zeta_K}), (\mathrm{w}^{f,0}, \mathrm{w}^{f,1}) \in X_{1+\varepsilon} \times X_\varepsilon$, for every $T > 2 \sum_{k=0}^{K} \frac{1}{|\zeta_k|}$, there exists $u \in L^2([0,T])$ such that the solution $w_\zeta(t,x) = w_\zeta(t,x;u)$ satisfies:

$$\sum_{k=0}^{K} \theta_k w_{\zeta_k}(T,x) = \mathrm{w}^{f,0}(x) \quad \text{and} \quad \sum_{k=0}^{K} \theta_k \dot{w}_{\zeta_k}(T,x) = \mathrm{w}^{f,1}(x) \qquad (x \in (0,1)).$$

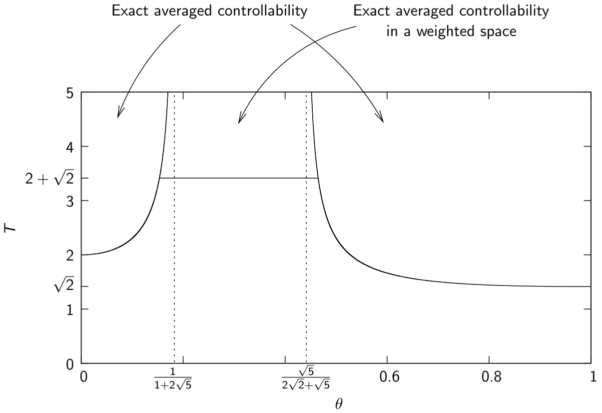
With $X_\alpha = \left\{ \varphi : x \in (0,1) \mapsto \sum_{n \in \mathbb{N}^*} a_n \sin(\pi n x), \ \sum_{n \in \mathbb{N}^*} n^{2\alpha} |a_n|^2 < \infty \right\}$.

From Dáger-Zuazua (2006),
if in addition $\frac{\zeta_l}{\zeta_k}$ algebraic for every $k \neq l$, then there exists $u \in L^2([0,T])$ such that the solution $w_\zeta(t,x) = w_\zeta(t,x;u)$ satisfies:

$$w_{\zeta_k}(T,x) = \mathrm{w}^{f,0}(x) \quad \text{and} \quad \dot{w}_{\zeta_k}(T,x) = \mathrm{w}^{f,1}(x) \qquad (x \in (0,1), \ k \in \{0, \dots, K\}).$$

Application to the string equation IV

We can also apply the perturbation argument.
For instance, for $\zeta_0 = 1$ and $\zeta_1 = \sqrt{2}$ and measure $\mu = (1-\theta)\delta_{\zeta_0} + \theta\delta_{\zeta_1}$, we obtain the set of parameters where averaged controllability holds.



A penalty method to get the "best" control I

The averaged control do not give any information on the variance's outputs. We remind that the averaged control of minimum $L^2([0,T],U)$-norm is a minimizer of:

$$\min \quad \mathcal{J}_0(u) := \frac{1}{2}\|u(t)\|^2_{L^2([0,T],U)}$$
$$\Big| \quad \int_\Omega \left(x_\zeta(T;u) - \mathrm{x}^f_\zeta\right) \mathrm{d}\mu_\zeta = 0\,.$$

Let us now add a penalty term,

$$\min \quad \mathcal{J}_\kappa(u) := \frac{1}{2}\|u(t)\|^2_{L^2([0,T],U)} + \kappa \int_\Omega \left\|x_\zeta(T;u) - \mathrm{x}^f_\zeta\right\|^2_X \mathrm{d}\mu_\zeta$$
$$\Big| \quad \int_\Omega \left(x_\zeta(T;u) - \mathrm{x}^f_\zeta\right) \mathrm{d}\mu_\zeta = 0\,. \qquad (\kappa \geqslant 0)\,,$$

Notice that when x^f_ζ is independent of ζ and $\int_\Omega \left(x_\zeta(T;u) - \mathrm{x}^f\right) \mathrm{d}\mu_\zeta = 0$, then $\int_\Omega \left\|x_\zeta(T;u) - \mathrm{x}^f\right\|^2_X \mathrm{d}\mu_\zeta$ is the variance of the outputs.

If the system has the averaged controllability property, then for every $\kappa \geqslant 0$, $\mathcal{J}_\kappa$ has one and only one minimizer $u_\kappa \in L^2([0,T],U)$.

Let us study the behavior as $\kappa \to \infty$.

A penalty method to get the "best" control II

Theorem

Set $T > 0$ and assume that the system $(\star)$ in controllable in average in time T. Let us define $(\mathrm{x}^\star_\zeta)_\zeta \in L^2(\Omega, X; \mu)$ be the minimizer of the minimization problem:

$$\min \quad \left\|\mathrm{x}_\zeta - \mathrm{x}^f_\zeta\right\|_{L^2(\Omega,X;\mu)}$$
$$\left|\quad \begin{aligned} &(\mathrm{x}_\zeta)_\zeta \in \overline{\{x_\zeta(T;u)\,,\ u \in L^2([0,T],U)\}}\,,\\ &\int_\Omega (\mathrm{x}_\zeta - \mathrm{x}^f_\zeta)\,\mathrm{d}\mu_\zeta\,. \end{aligned}\right.$$

If $\lim_{\kappa\to\infty} \left(\left\|x_\zeta(T;u_\kappa) - \mathrm{x}^f_\zeta\right\|_{L^2(\Omega,X;\mu)}\right) = 0$, *then,* $\mathrm{x}^\star_\zeta = \mathrm{x}^f_\zeta$ *and*

- *If* $\left(\|u_\kappa\|_{L^2([0,T],U)}\right)_{\kappa\geqslant 0}$ *is bounded, then up to a subsequence,* $(u_\kappa)_\kappa$ *converges to a control which steers exactly* x^i_ζ *to* $\mathrm{x}^\star_\zeta$ *and realise the minimum of:*

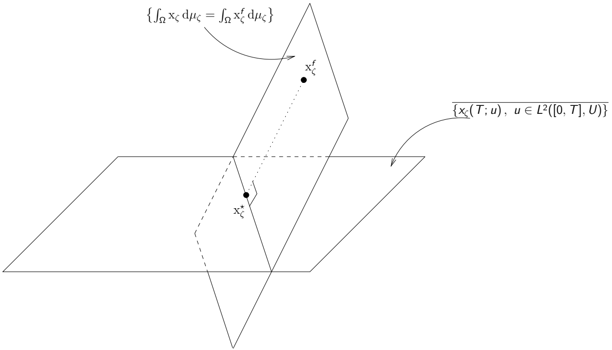
$$\min \quad \tfrac{1}{2}\|u\|^2_{L^2([0,T],U)}$$
$$\left|\quad \|x_\zeta(T;u) - \mathrm{x}^\star_\zeta\|_{L^2(\Omega,X;\mu)} = 0\,.\right.$$

- *If* $\left(\|u_\kappa\|_{L^2([0,T],U)}\right)_{\kappa\geqslant 0}$ *is unbounded, then* x^i_ζ *can be approximatively steered to* $\mathrm{x}^\star_\zeta$.

A penalty method to get the "best" control III

$\mathrm{x}_\zeta^\star$ is the orthogonal projection of x_ζ^f in $L^2(\Omega, X; \mu)$ on

$$\overline{\{x_\zeta(T;u)\,,\ u \in L^2([0,T],U)\}} \cap \left\{ (\mathrm{x}_\zeta)_\zeta\,,\ \int_\Omega (\mathrm{x}_\zeta - \mathrm{x}_\zeta^f)\, \mathrm{d}\mu_\zeta \right\}.$$



A penalty method to get the "best" control IV

		$\left(\Vert x_\zeta(T;u_\kappa) - \mathrm{x}^f_\zeta\Vert_{L^2(\Omega,X;\mu)}\right)_\kappa$	
		converge to 0	do not converge to 0
$\left(\Vert u_\kappa\Vert_{L^2}\right)_\kappa$	bounded	simultaneous exact controllability	simultaneous exact controllability to $\mathrm{x}^\star_\zeta$
	unbounded	simultaneous approximate controllability	simultaneous approximate controllability to $\mathrm{x}^\star_\zeta$

Notice that if $\mathrm{Card}\,\Omega < \infty$ and $\dim X < \infty$, the sequence $(\Vert u_\kappa\Vert_{L^2([0,T],U)})_\kappa$ is bounded.

When the system is simultaneously controllable I

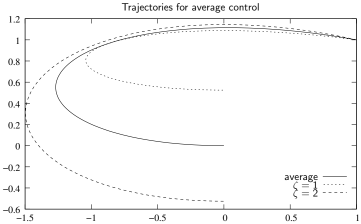
Numerical example

Let us consider the probability space $\Omega = \{1, 2\}$ and the probability measure μ is given by $\mu(\{1\}) = \mu(\{2\}) = \frac{1}{2}$. The parameter dependent system under consideration is:

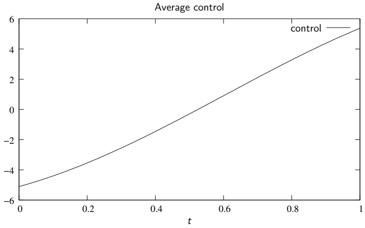
$$\dot{x}_\zeta = \zeta A x_\zeta + Bu \qquad x_\zeta(0) = \mathrm{x}^i,$$

with: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathrm{x}^i = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

In addition, we fix the final target to $\mathrm{x}^f = (0\ 0)^\top$ and the final time $T = 1$.



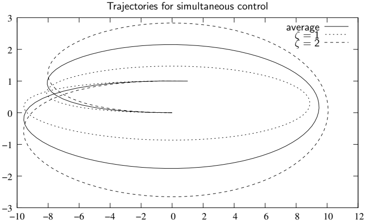
Controlled trajectories in the phase plan using the averaged control. The variance at final time is 2.75e−01.



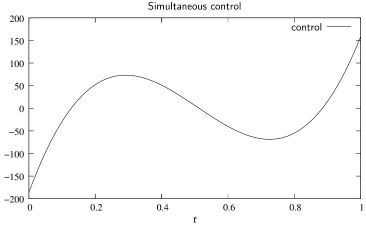
Averaged control, the norm of the control is 3.19.

When the system is simultaneously controllable II

Numerical example



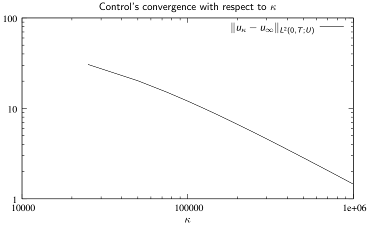
Controlled trajectories in the phase plan using the simultaneous control. The variance at final time is `4.74e-20`.



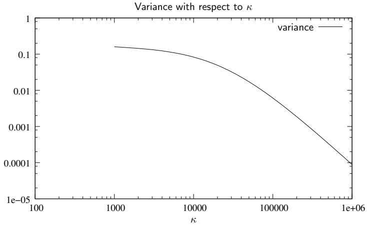
Simultaneous control, the norm of the control is `6.34e+01`.

When the system is simultaneously controllable III

Numerical example



Plot of the L^2-distance between the exact simultaneous control and the optimal control of the minimisation problem indexed with κ. This distance behaves as $C\kappa^{-0.98}$.



Plot of the variance at final state $\left(\int \|x_\zeta(T) - \mathrm{x}^f\|_X^2 \mathrm{d}\mu_\zeta\right)$ with respect to κ. The variance behaves as $C\kappa^{-1.95}$.

When the system is not simultaneously controllable I

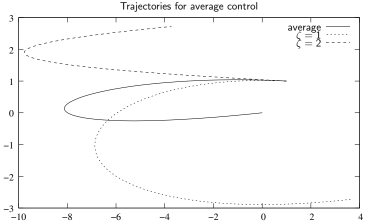
Numerical example

Let us consider the probability space $\Omega = \{1, 2\}$ and the probability density μ given by $\mu(\{1\}) = \mu(\{2\}) = \frac{1}{2}$. The parameter dependent system under consideration is:

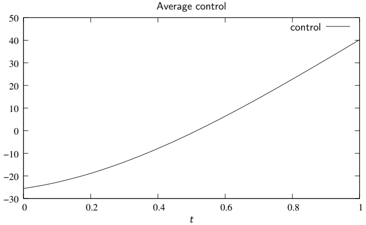
$$\dot{x}_\zeta = A_\zeta x_\zeta + Bu \qquad x_\zeta(0) = \mathrm{x}^i ,$$

with: $A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathrm{x}^i = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

The final time T is set to 1 and the target x^f is $(0\ 0)^\top$.



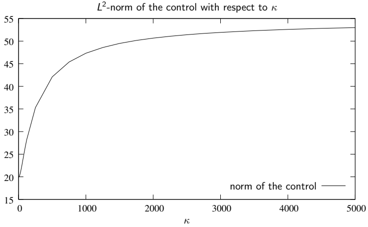
Controlled trajectories in the phase plan using the averaged control. The variance at final time is 2.13e+01.



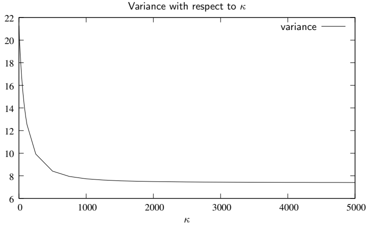
Averaged control, the L^2-norm of the control is 1.99e+01.

When the system is not simultaneously controllable II

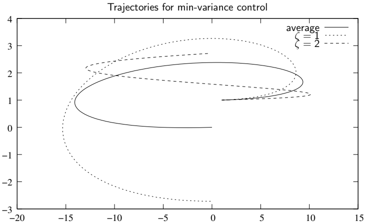
Numerical example



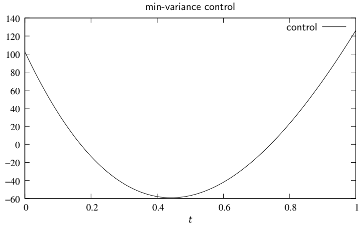
Plot of the norm of the control with respect to κ.



Plot of the variance at final state with respect to κ.



Controlled trajectories in the phase plan using the optimal control for $\kappa = 5.10^3$.



Optimal control for $\kappa = 5.10^3$ its L^2-norm is 5.30e+01.

Conclusion and open questions

- Few averaged controllability results exists in PDE context.

- Averaged Ingham inequality in general means:

$$\exists c, C > 0\,,\, c\sum_n |a_n|^2 \leqslant \int_0^T \left| \int_\Omega \sum_n a_n e^{2i\pi\lambda_n\zeta t}\, \mathrm{d}\mu_\zeta \right|^2 \mathrm{d}t \leqslant C\sum_n |a_n|^2\,.$$

In particular, for $\Omega = \mathbb{R}$, we end up with:

$$c\sum_n |a_n|^2 \leqslant \int_0^T |a_n\hat{\mu}(-\lambda_n t)|^2\, \mathrm{d}t \leqslant C\sum_n |a_n|^2\,.$$

That is to say, $\{\hat{\mu}(-\lambda_n\,\cdot)\}_n$ is a Riesz basis.

- Still for averaged Ingham inequalities, are they true for $\mathrm{d}\mu_\zeta = \frac{1}{2\varepsilon}\mathbf{1}_{(1-\varepsilon,1+\varepsilon)}(\zeta)\,\mathrm{d}\zeta$, with $\varepsilon > 0$ small enough?
- How to compute numerically average or simultaneous controls when $\operatorname{card}\Omega = \infty$?

Thank you for your attention.