Long and Winding Central Paths

Journée McTAO

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January 18th, 2016

Linear programming and its complexity

Linear programming (LP) = optimize a linear objective function under linear (affine) inequality constraints.

Definition

A **linear program** is of the form:

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax \geqslant b,\ x \in \mathbb{R}^n \end{aligned}$$

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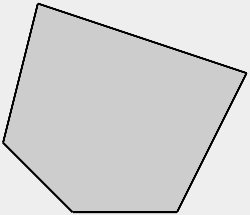

$$\begin{aligned} \text{minimize} \quad & x + 3y \\ \text{subject to} \quad & x + y \geqslant 3 \\ & 23 \geqslant x + 3y \\ & 4x \geqslant 1 + y \\ & 11 + y \geqslant 2x \\ & 2y \geqslant 2 \end{aligned}$$

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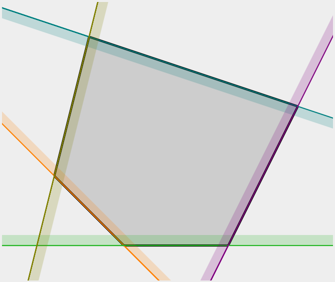

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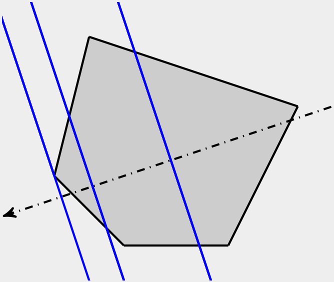

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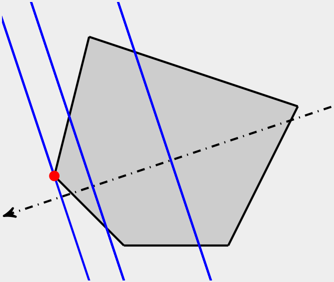

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Linear programming and its complexity (2)

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*Linear programming can be solved in **polynomial time** in the Turing Machine model.*

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= execution time bounded by a polynomial

$$P(m, n, L)$$

where:

- m = nb of inequalities
- n = dimension of the space
- L = total size of the coefficients A_{ij}, b_i, c_j in bits (sum of their $\log_2$).

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≠ **strongly polynomial** complexity

- number of arithmetic operations bounded by a polynomial in the **dimension** of the problem, i.e. $m \times n$
- the size of operands of arithmetic operations is bounded by a polynomial in L

Linear programming and its complexity (3)

9th Smale's Problem for 21st Century

Is there a strongly polynomial algorithm for linear programming?

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Existing algorithms for LP:

- simplex method (Dantzig, 1947)
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(ellipsoid method and interior point method: polynomial time)

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} polynomial time (ellipsoid method and interior point method)

Purpose of this talk

What can we say about interior point methods?

Path-following interior-point methods

Goal

Solve a convex program

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in X \end{array} \qquad (P)$$

where $X \subset \mathbb{R}^n$ is closed, convex, with non-empty interior.

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Follow the central path with $\mu \searrow 0$ up to the solution of (P) .

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Follow the central path with $\mu \searrow 0$ up to the solution of (P) **approximately**.

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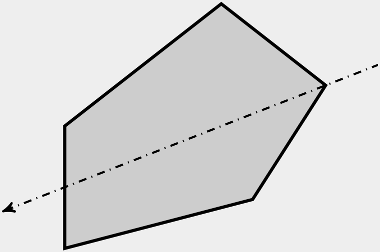

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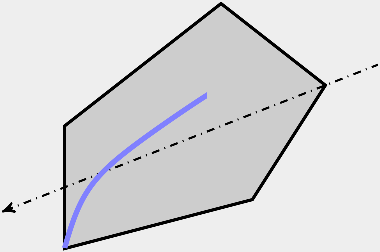

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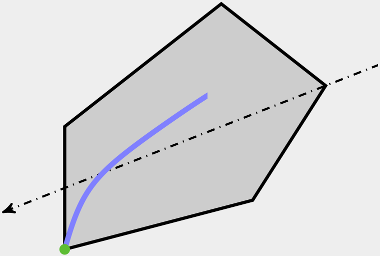

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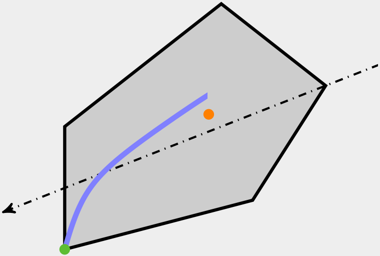

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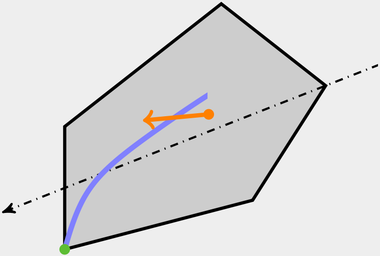

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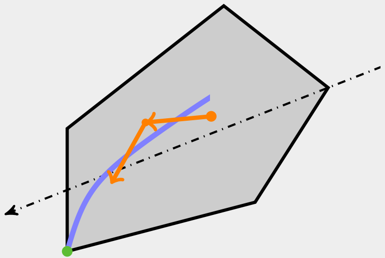

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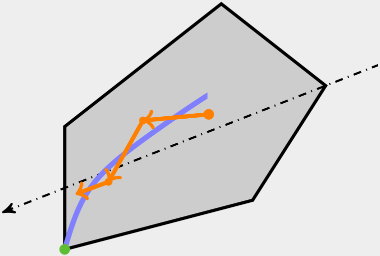

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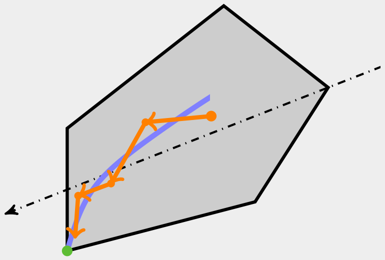

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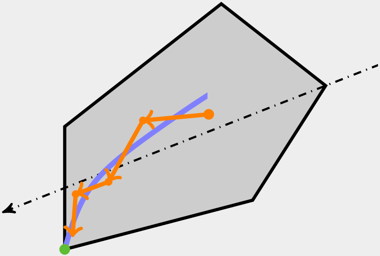

- stay in a certain "neighborhood" of the central path
- use Newton descent directions to iterate
- different choices of steps (short, long, predictor/corrector, etc)

Complexity of interior point methods

Intimately related with the geometry of the central path!

According to Bayer and Lagarias (1989), the central path is

> *[…] a fundamental mathematical object underlying Karmarkar's algorithm and that the good convergence properties of Karmarkar's algorithm arise from good geometric properties […]*

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Given a linear program defined by m inequalities in dimension n,

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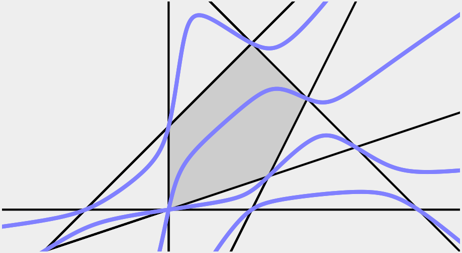

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Continuous analogue of Hirsch conjecture (Deza, Terlaky, and Zinchenko, 2009)

The total curvature of the central path is bounded by $O(m)$.

This talk

Theorem

We can construct a linear program with $3r + 4$ inequalities in dimension $2r + 2$ where the central path has a total curvature in $\Omega(2^r)$.

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\text{subject to} \quad & u_0 \leqslant t\,,\ v_0 \leqslant t^2 \\
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Our approach

Study the limit of the central path of $\mathrm{LP}(t)$ when $t \to +\infty$ through the "tropical central path".

This talk

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1. Preliminaries on tropical geometry

2. Tropicalizing the central path

3. Central paths with large curvature

Outline of the talk

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2. Tropicalizing the central path

3. Central paths with large curvature

Tropical algebra and tropical polyhedra

Tropical algebra refers to the semiring $\mathbb{R}_{\max} := \mathbb{R} \cup \{-\infty\}$ where:

- the addition $x \oplus y$ is $\max(x, y)$
- the multiplication $x \odot y$ is $x + y$

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Tropical operations extend to matrices and vectors:

$$A \oplus B = (A_{ij} \oplus B_{ij})_{ij} \qquad A \odot B = \Big(\bigoplus_k A_{ik} \odot B_{kj}\Big)_{ij}$$

Tropical algebra and tropical polyhedra (2)

A **tropical polyhedron** is the set of solutions $x \in \mathbb{R}_{\max}^n$ of a system of the form:

$$A^+ \odot x \oplus b^+ \geqslant A^- \odot x \oplus b^-$$

with $A^+, A^- \in \mathbb{R}_{\max}^{m \times n}$ and $b^+, b^- \in \mathbb{R}_{\max}^m$.

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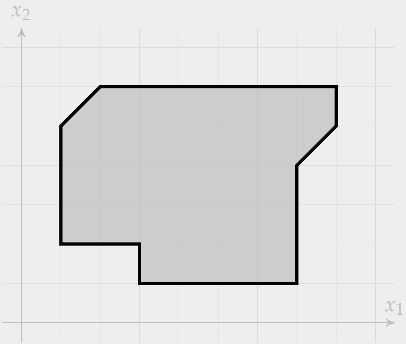

$$\begin{pmatrix} 0 & 1 \\ -\infty & 0 \\ -\infty & 0 \\ -\infty & -\infty \\ 4 & -\infty \end{pmatrix} \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} -\infty \\ -\infty \\ 4 \\ 8 \\ -\infty \end{pmatrix}$$

$$\geqslant \begin{pmatrix} -\infty & -\infty \\ -10 & -\infty \\ -3 & -\infty \\ 0 & 2 \\ -\infty & 0 \end{pmatrix} \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} 3 \\ 1 \\ -\infty \\ -\infty \\ 5 \end{pmatrix}$$

Tropical algebra and tropical polyhedra (2)

A **tropical polyhedron** is the set of solutions $x \in \mathbb{R}^n_{\max}$ of a system of the form:

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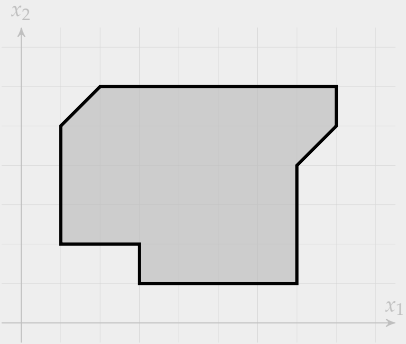

$$\begin{aligned} \max(x_1, 1 + x_2) &\geqslant 3 \\ x_2 &\geqslant \max(-10 + x_1, 1) \\ \max(x_2, 4) &\geqslant -3 + x_1 \\ 8 &\geqslant \max(x_1, 2 + x_2) \\ 4 + x_1 &\geqslant \max(x_2, 5) \end{aligned}$$

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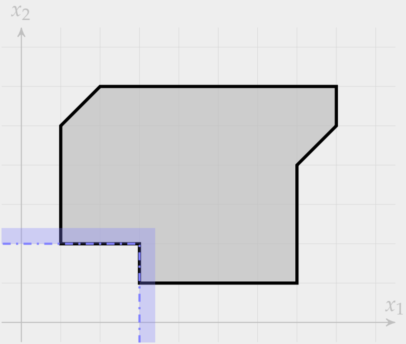

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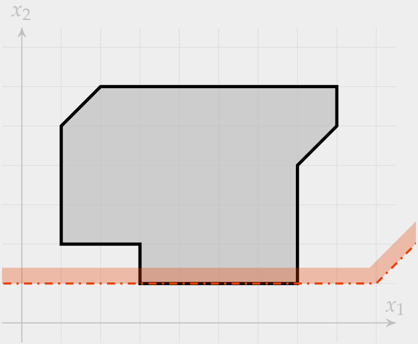

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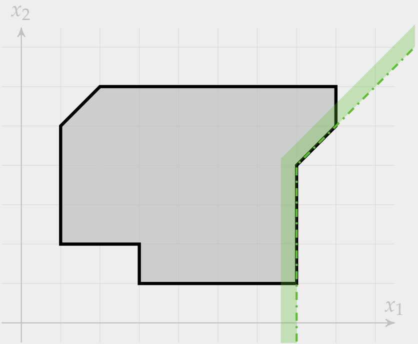

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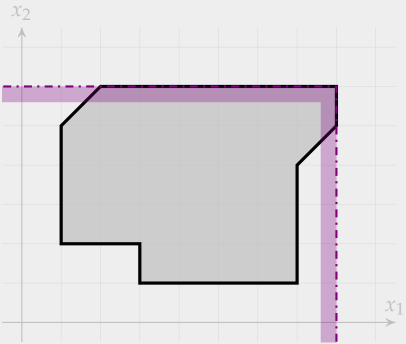

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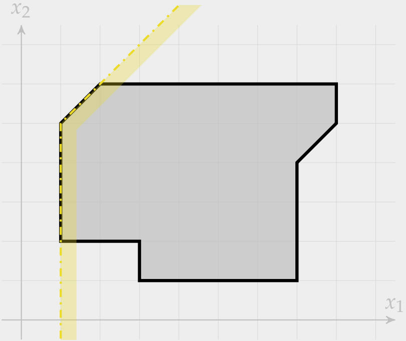

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Tropical polyhedra vs convex polyhedra

Alternative definition

Tropical polyhedra = limits of deformations of classical polyhedra through the map

$$\log_t : x \mapsto \frac{\log x}{\log t}$$

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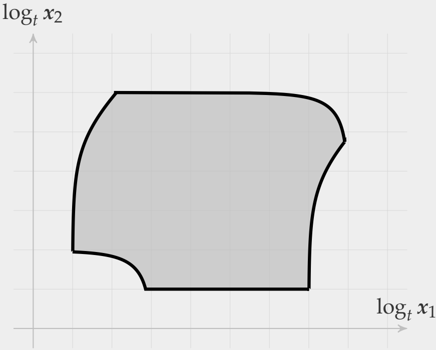

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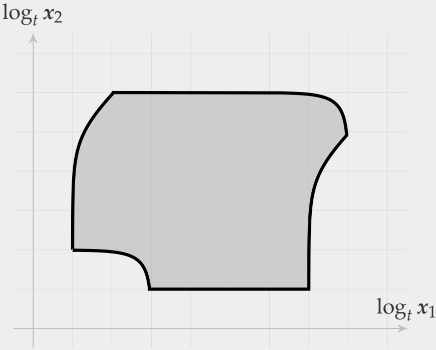
$\xrightarrow{\log_t(\cdot)}$ $t = 10$



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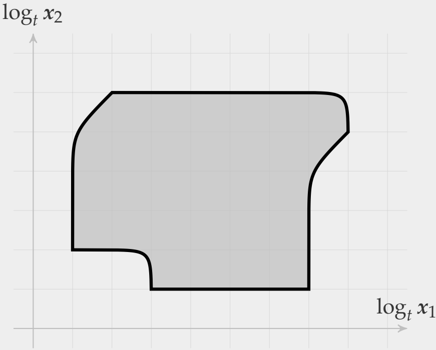

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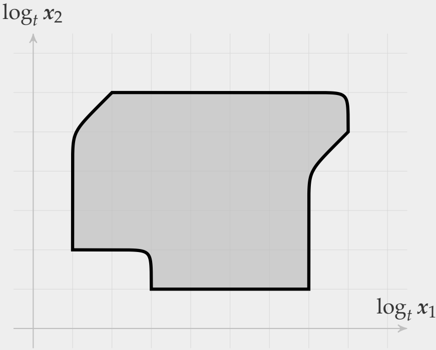

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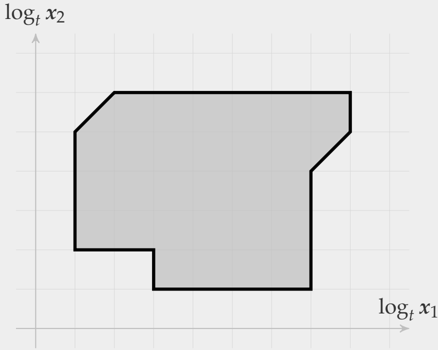

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$$\max(\log_t \boldsymbol{x}, \log_t \boldsymbol{y}) \leqslant \log_t(\boldsymbol{x}+\boldsymbol{y}) \leqslant \max(\log_t \boldsymbol{x}, \log_t \boldsymbol{y}) + \log_t 2$$

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Our goal: tropicalizing the central path

Study the central path of a parametric family of LPs:

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}(t)^\top x \\ \text{subject to} \quad & \boldsymbol{A}(t)x \leqslant \boldsymbol{b}(t)\,, \ x \geqslant 0 \end{aligned}$$

and its deformation by the map $\log_t(\cdot)$, when t goes to $+\infty$.

A possible setting for tropicalization

The entries of $A(t)$, $b(t)$ and $c(t)$ belong to the **Hardy field** $\mathbb{K}$.

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The good properties of $\mathbb{K}$

- $\mathbb{K}$ is real-closed. In particular, it is ordered:

$$f \leqslant g \qquad \text{if} \quad f(t) \leqslant g(t) \quad \text{for all } t \gg 1$$

- elements of $\mathbb{K}$ have "polynomial asymptotics":

$$f(t) \sim pt^{\alpha} \qquad \text{when} \quad t \to +\infty \qquad (p, \alpha \in \mathbb{R}\,,\ p \neq 0)$$

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Makes sense to consider a LP over $\mathbb{K}$, which encodes a family of LPs over $\mathbb{R}$:

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & Ax \leqslant b\,,\ x \in (\mathbb{K}_{\geqslant 0})^n \end{array} \qquad\qquad \begin{array}{ll} \text{minimize} & c(t)^\top x \\ \text{subject to} & A(t)x \leqslant b(t)\,,\ x \in (\mathbb{R}_{\geqslant 0})^n \end{array}$$

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$$\mathsf{val}(\boldsymbol{x}) := \lim_{t \to +\infty} \log_t |\boldsymbol{x}(t)|$$

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The valuation maps the "classical" laws to the tropical ones: $\forall x, y \in \mathbb{K}_{\geqslant 0}$,

$$\mathsf{val}(x + y) = \max(\mathsf{val}(x), \mathsf{val}(y))$$

$$\mathsf{val}(x \cdot y) = \mathsf{val}(x) + \mathsf{val}(y)$$

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Theorem

Let $\mathcal{P} \subset (\mathbb{K}_{\geqslant 0})^n$ be a convex polyhedron. Then

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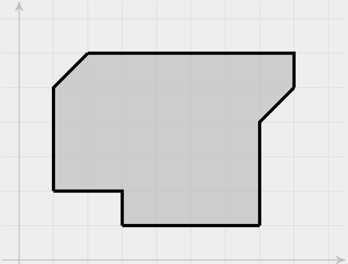
$$\begin{aligned} t^3 &\leqslant \boldsymbol{x}_1 + t\boldsymbol{x}_2 \\ t^{-10}\boldsymbol{x}_1 + t &\leqslant \boldsymbol{x}_2 \\ t^{-3}\boldsymbol{x}_1 &\leqslant \boldsymbol{x}_2 + t^4 \\ \boldsymbol{x}_1 + t^2\boldsymbol{x}_2 &\leqslant t^8 \\ \boldsymbol{x}_2 + t^5 &\leqslant t^4\boldsymbol{x}_1 \end{aligned} \xrightarrow{\ \mathsf{val}(\cdot)\ }$$



Outline of the talk

1. Preliminaries on tropical geometry
2. Tropicalizing the central path
3. Central paths with large curvature

The central path over the Hardy field

Given $\boldsymbol{A} \in \mathbb{K}^{m \times n}$, $\boldsymbol{b} \in \mathbb{K}^m$ and $\boldsymbol{c} \in \mathbb{K}^n$, consider the following LP:

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \leqslant \boldsymbol{b}\,,\ \boldsymbol{x} \geqslant 0 \\ & \boldsymbol{x} \in \mathbb{K}^n \end{aligned} \qquad \mathbf{LP}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$$

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Proposition

For all $\mu \in \mathbb{K}_{>0}$, the log-barrier problem over the Hardy field

$$\begin{array}{ll} \textit{minimize} & c^\top x - \mu\Big(\sum_{j=1}^n \log(x_j) + \sum_{i=1}^m \log(w_i)\Big) \\ \textit{subject to} & Ax + w = b\,,\ x > 0\,,\ w > 0\,, \end{array}$$

has a unique solution (x^μ, w^μ).

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The expansion of our o-minimal structure with the function log is also o-minimal (van den Dries et al., 1994).

$\Longrightarrow$ the resulting Hardy field still has nice model theoretic properties.

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The proposition is valid over the reals, so it is still valid over the Hardy field.

The tropical central path

Two points of view:

- over the Hardy field, the central path of $\mathbf{LP}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$

$$\boldsymbol{\mu} \mapsto \boldsymbol{\mathcal{C}}(\boldsymbol{\mu})$$

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For all $\boldsymbol{\mu} \in \mathbb{K}_{>0}$, we have

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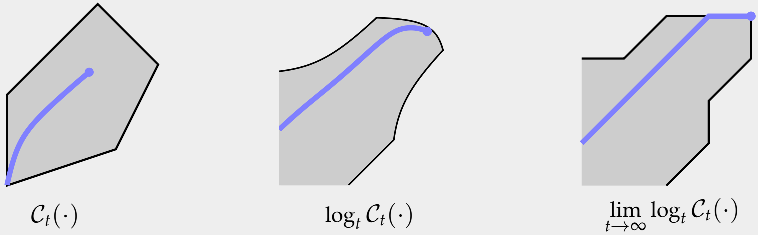

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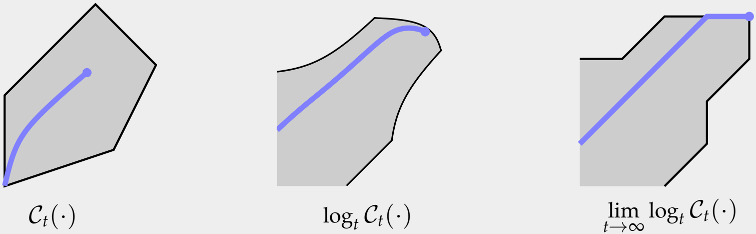

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The **tropical central path** is defined as the map

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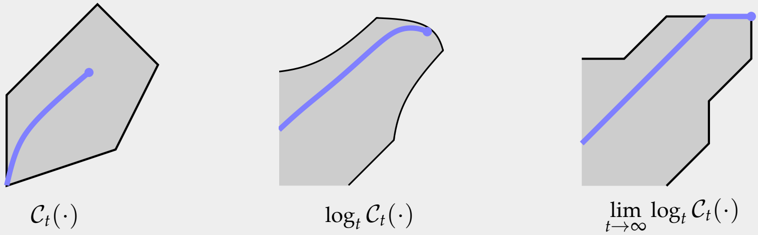

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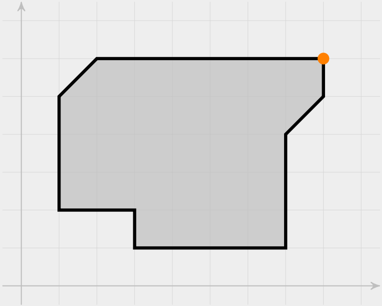
Geometric characterization of the tropical central path

Relies on the notion of **barycenter** of a tropical polyhedron $\mathcal{P}$
$=$ greatest point of the set $\mathcal{P}$ for the coordinate-wise order $\leqslant$



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Remark

The tropical central path does not depend on the representation of $\boldsymbol{\mathcal{P}}$.

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$$(\mathcal{P})\quad \begin{aligned} \text{minimize}\quad & \boldsymbol{x}_1 + t^3\boldsymbol{x}_2 \\ & \boldsymbol{x}_1 + \boldsymbol{x}_2 \leqslant 2 \\ & t\boldsymbol{x}_1 \leqslant 1 + t^2\boldsymbol{x}_2 \\ & t\boldsymbol{x}_2 \leqslant 1 + t^3\boldsymbol{x}_1 \\ & \boldsymbol{x}_1 \leqslant t^2\boldsymbol{x}_2 \\ & \boldsymbol{x}_1, \boldsymbol{x}_2 \geqslant 0 \end{aligned}$$

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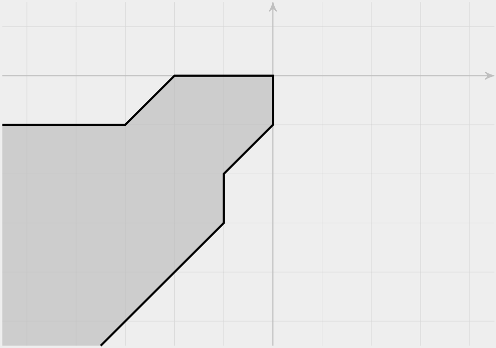

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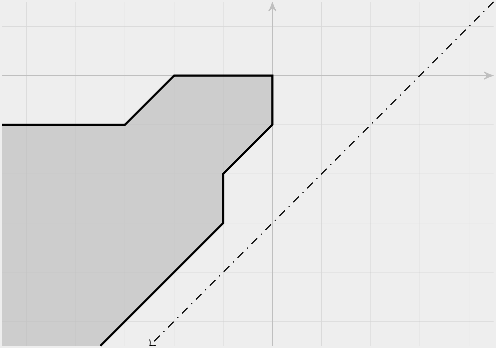

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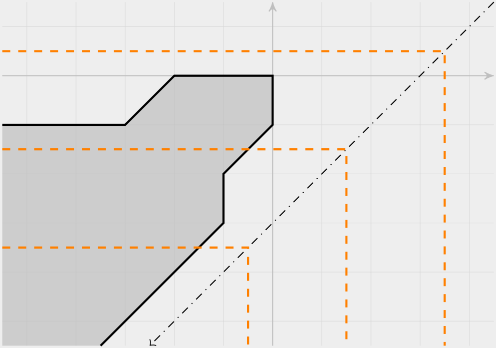

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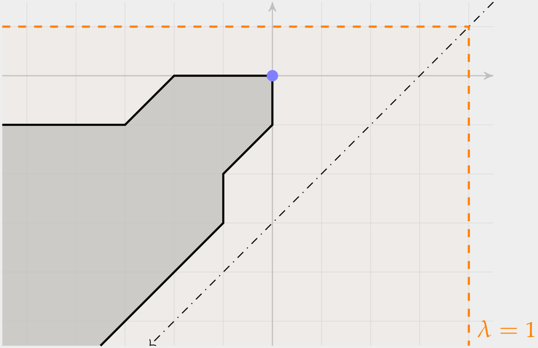

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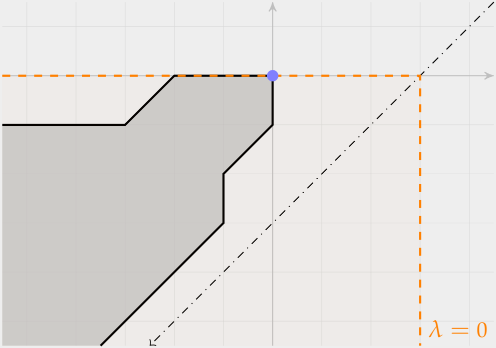

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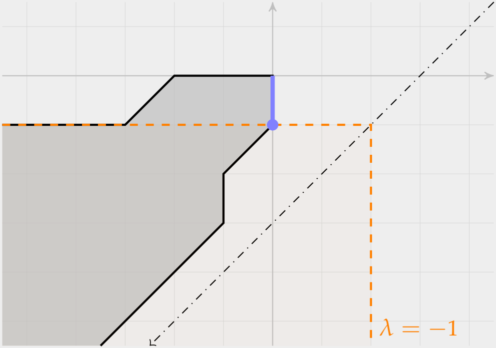

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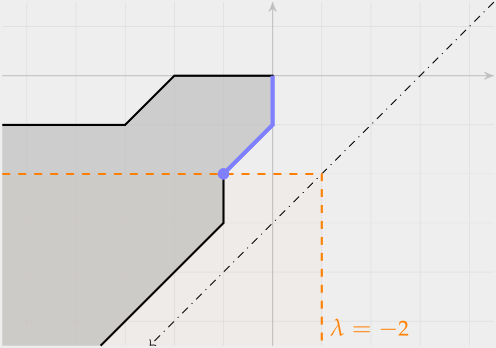

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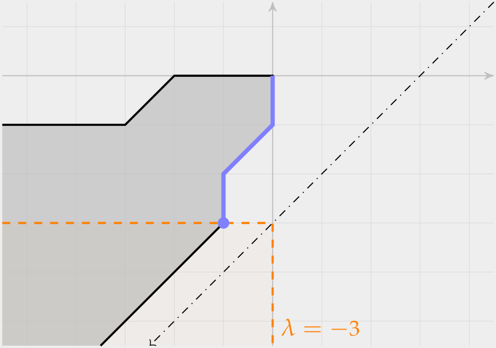

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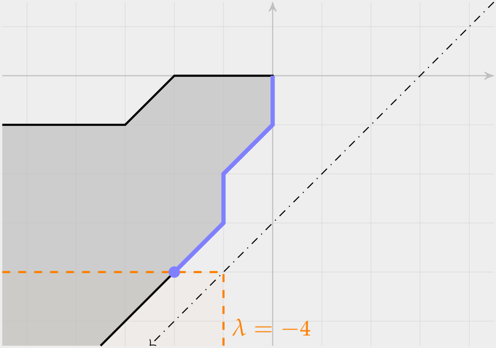

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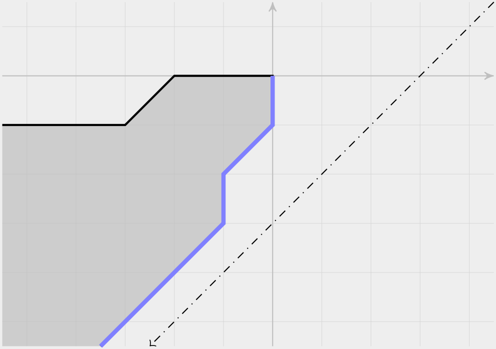

Sketch of the proof

Let us fix $\lambda \in \mathbb{R}$, and let $\boldsymbol{\mu} := t^{\lambda}$.

Consider the penalized function

$$\Phi(\boldsymbol{x}, \boldsymbol{w}) = \frac{\boldsymbol{c}^\top \boldsymbol{x}}{\boldsymbol{\mu}} - \left(\sum_{j=1}^{n} \log(\boldsymbol{x}_j) + \sum_{i=1}^{m} \log(\boldsymbol{w}_i)\right)$$

defined over the (relative) interior of $\boldsymbol{\mathcal{P}}$.

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- if $\mathsf{val}(\boldsymbol{c}) \odot \mathsf{val}(\boldsymbol{x}) > \lambda$,

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- if $\mathsf{val}(c) \odot \mathsf{val}(x) \leqslant \lambda$, then

$$[\Phi(\boldsymbol{x}, \boldsymbol{w})](t) = -\Big(\sum_{j=1}^n \mathsf{val}(x_j) + \sum_{i=1}^m \mathsf{val}(w_i)\Big)\log t + O(1)$$

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Let us fix $\lambda \in \mathbb{R}$, and let $\boldsymbol{\mu} := t^{\lambda}$.

Consider the penalized function

$$\Phi(x, w) = \frac{c^\top x}{\mu} - \left(\sum_{j=1}^{n} \log(x_j) + \sum_{i=1}^{m} \log(w_i)\right)$$

$$\frac{c^\top x}{\mu} \sim \gamma t^{(\mathsf{val}(c) \odot \mathsf{val}(x)) - \lambda}$$

$$\sum_{j=1}^{n} \log(x_j) + \sum_{i=1}^{m} \log(w_i) = \left(\textstyle\sum_{j=1}^{n} \mathsf{val}(x_j) + \sum_{i=1}^{m} \mathsf{val}(w_i)\right) \log t + O(1)$$

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$+$ the point $\mathsf{val}(\boldsymbol{x}^{\boldsymbol{\mu}}, \boldsymbol{w}^{\boldsymbol{\mu}})$ maximizes the function

$$(x, w) \mapsto \sum_{j=1}^{n} x_j + \sum_{i=1}^{m} w_i$$

over the tropical polyhedron $\mathcal{P} \cap \{c^{\top} \odot x \leqslant \lambda\}$.

Outline of the talk

1. Preliminaries on tropical geometry

2. Tropicalizing the central path

3. Central paths with large curvature

Our counterexample to the continuous Hirsch conjecture

$$
\begin{aligned}
\text{minimize} \quad & v_0 \\
\text{subject to} \quad & u_0 \leqslant t \\
& v_0 \leqslant t^2 \\
& \left.\begin{aligned}
u_i &\leqslant t u_{i-1} \\
u_i &\leqslant t v_{i-1} \\
v_i &\leqslant t^{1-\frac{1}{2^i}} (u_{i-1} + v_{i-1})
\end{aligned}\right\} \text{for } 1 \leqslant i \leqslant r \\
& u_r \geqslant 0 \, , \; v_r \geqslant 0
\end{aligned}
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The point $\mathcal{C}^{\text{trop}}(\lambda)$ is the greatest point of

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\begin{aligned}
& v_0 \leqslant \lambda \\
& u_0 \leqslant 1\,, \quad v_0 \leqslant 2 \\
& \left.\begin{aligned}
u_i &\leqslant 1 + u_{i-1} \\
u_i &\leqslant 1 + v_{i-1} \\
v_i &\leqslant \left(1-\frac{1}{2^i}\right) + \max(u_{i-1}, v_{i-1})
\end{aligned}\right\} \text{for } 1 \leqslant i \leqslant r
\end{aligned}
$$

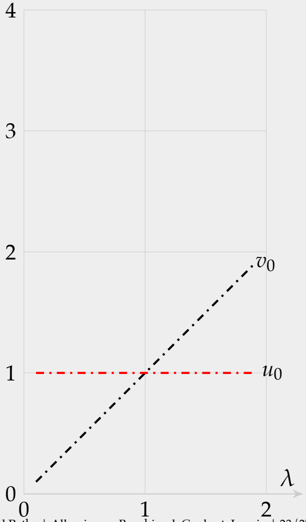
Our counterexample to the continuous Hirsch conjecture

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{v}_0 \\ \text{subject to} \quad & \boldsymbol{u}_0 \leqslant t \\ & \boldsymbol{v}_0 \leqslant t^2 \\ & \left. \begin{aligned} \boldsymbol{u}_i &\leqslant t\boldsymbol{u}_{i-1} \\ \boldsymbol{u}_i &\leqslant t\boldsymbol{v}_{i-1} \\ \boldsymbol{v}_i &\leqslant t^{1-\frac{1}{2^i}}(\boldsymbol{u}_{i-1} + \boldsymbol{v}_{i-1}) \end{aligned} \right\} \text{for } 1 \leqslant i \leqslant r \\ & \boldsymbol{u}_r \geqslant 0\,,\ \boldsymbol{v}_r \geqslant 0 \end{aligned}$$

Tropical central path

The point $\mathcal{C}^{\text{trop}}(\lambda)$ is given by

$$\begin{aligned} u_0 &= 1 \\ v_0 &= \min(\lambda, 2) \\ & \left. \begin{aligned} u_i &= 1 + \min(u_{i-1}, v_{i-1}) \\ v_i &= \left(1 - \frac{1}{2^i}\right) + \max(u_{i-1}, v_{i-1}) \end{aligned} \right\} \text{for } 1 \leqslant i \leqslant r \end{aligned}$$



Our counterexample to the continuous Hirsch conjecture

$$
\begin{aligned}
\text{minimize} \quad & \boldsymbol{v}_0 \\
\text{subject to} \quad & \boldsymbol{u}_0 \leqslant t \\
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& \left.\begin{aligned}
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\boldsymbol{u}_i &\leqslant t\boldsymbol{v}_{i-1} \\
\boldsymbol{v}_i &\leqslant t^{1-\frac{1}{2^i}}(\boldsymbol{u}_{i-1} + \boldsymbol{v}_{i-1})
\end{aligned}\right\} \text{ for } 1 \leqslant i \leqslant r \\
& \boldsymbol{u}_r \geqslant 0\,,\ \boldsymbol{v}_r \geqslant 0
\end{aligned}
$$

Tropical central path

The point $\mathcal{C}^{\mathrm{trop}}(\lambda)$ is given by

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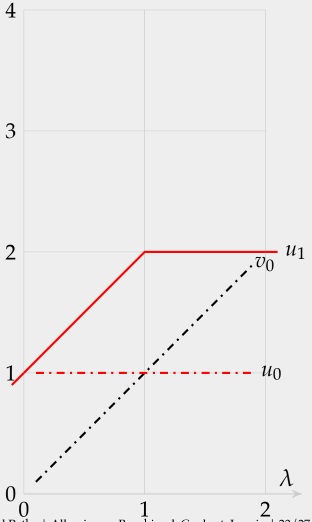
u_0 &= 1 \\
v_0 &= \min(\lambda, 2) \\
&\left.\begin{aligned}
u_i &= 1 + \min(u_{i-1}, v_{i-1}) \\
v_i &= \left(1 - \frac{1}{2^i}\right) + \max(u_{i-1}, v_{i-1})
\end{aligned}\right\} \text{ for } 1 \leqslant i \leqslant r
\end{aligned}
$$



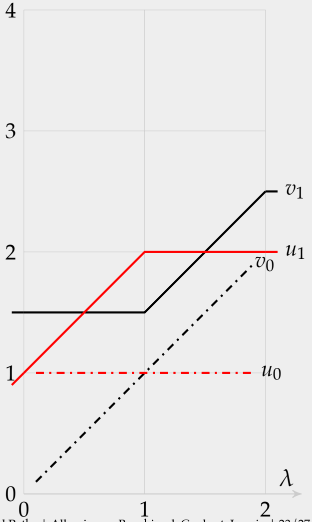
Our counterexample to the continuous Hirsch conjecture

$$
\begin{aligned}
\text{minimize} \quad & v_0 \\
\text{subject to} \quad & u_0 \leqslant t \\
& v_0 \leqslant t^2 \\
& \left.\begin{aligned} u_i &\leqslant t u_{i-1} \\ u_i &\leqslant t v_{i-1} \\ v_i &\leqslant t^{1-\frac{1}{2^i}}(u_{i-1}+v_{i-1}) \end{aligned}\right\} \text{ for } 1 \leqslant i \leqslant r \\
& u_r \geqslant 0\,,\ v_r \geqslant 0
\end{aligned}
$$

Tropical central path

The point $\mathcal{C}^{\text{trop}}(\lambda)$ is given by

$$
\begin{aligned}
u_0 &= 1 \\
v_0 &= \min(\lambda, 2) \\
&\left.\begin{aligned} u_i &= 1 + \min(u_{i-1}, v_{i-1}) \\ v_i &= \left(1 - \frac{1}{2^i}\right) + \max(u_{i-1}, v_{i-1}) \end{aligned}\right\} \text{ for } 1 \leqslant i \leqslant r
\end{aligned}
$$



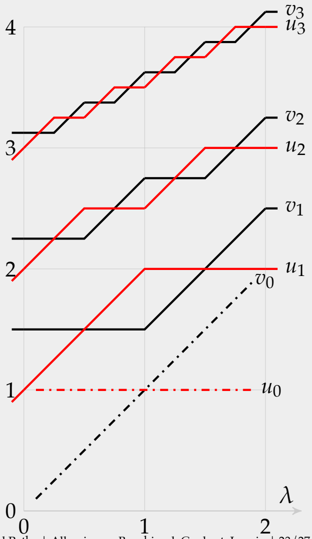
Our counterexample to the continuous Hirsch conjecture

$$\begin{aligned} \text{minimize} \quad & v_0 \\ \text{subject to} \quad & u_0 \leqslant t \\ & v_0 \leqslant t^2 \\ & \left.\begin{aligned} u_i &\leqslant t u_{i-1} \\ u_i &\leqslant t v_{i-1} \\ v_i &\leqslant t^{1-\frac{1}{2^i}} (u_{i-1} + v_{i-1}) \end{aligned}\right\} \text{ for } 1 \leqslant i \leqslant r \\ & u_r \geqslant 0 \, , \; v_r \geqslant 0 \end{aligned}$$

Tropical central path

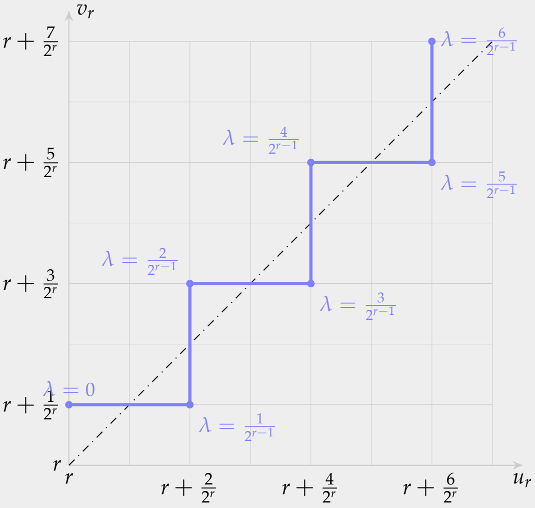
The point $\mathcal{C}^{\text{trop}}(\lambda)$ is given by

$$\begin{aligned} u_0 &= 1 \\ v_0 &= \min(\lambda, 2) \\ &\left.\begin{aligned} u_i &= 1 + \min(u_{i-1}, v_{i-1}) \\ v_i &= \left(1 - \frac{1}{2^i}\right) + \max(u_{i-1}, v_{i-1}) \end{aligned}\right\} \text{ for } 1 \leqslant i \leqslant r \end{aligned}$$



Our counterexample to the continuous Hirsch conjecture (2)

In the (u_r, v_r)-plane, the tropical central path looks like a staircase with 2^r steps:



Total curvature

Definition

The **total curvature** of a curve is defined as

Total curvature

Definition

The **total curvature** of a curve is defined as

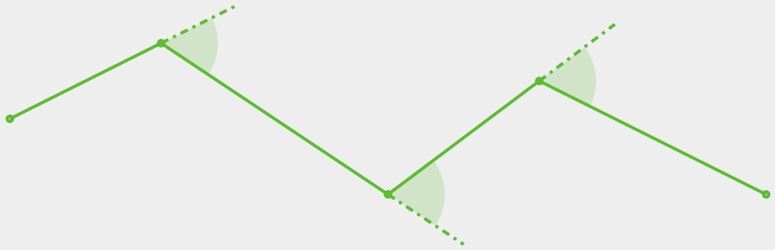
- **polygonal curve**:

Total curvature

Definition

The **total curvature** of a curve is defined as

- **polygonal curve**: sum of the angles

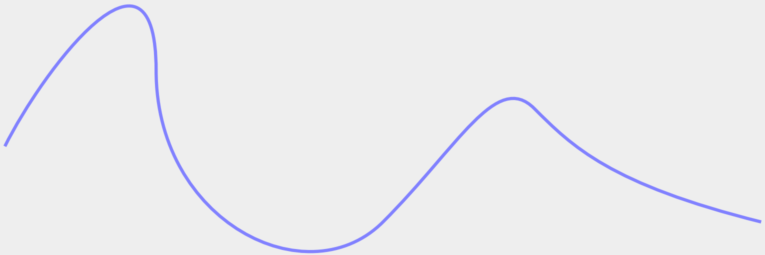


Total curvature

Definition

The **total curvature** of a curve is defined as

- **polygonal curve**: sum of the angles
- **arbitrary curve**:

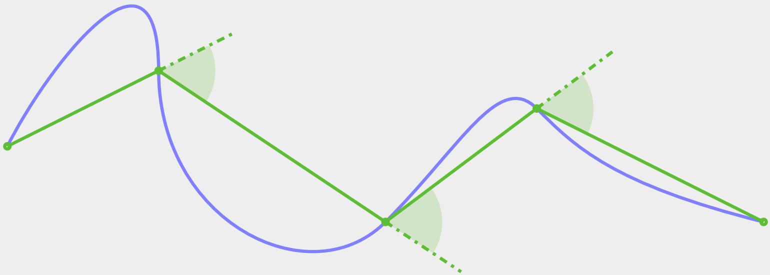


Total curvature

Definition

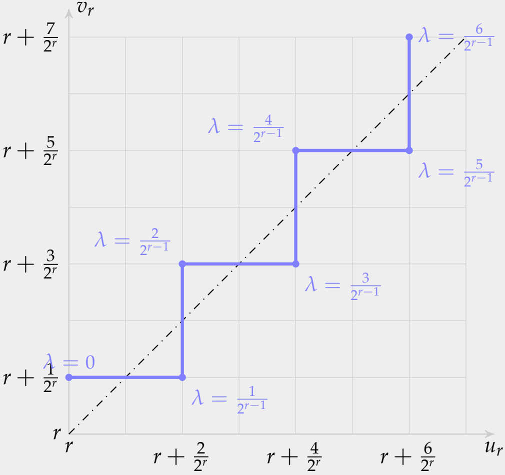
The **total curvature** of a curve is defined as

- **polygonal curve**: sum of the angles
- **arbitrary curve**: sup of total curvature of inscribed polygonal curves



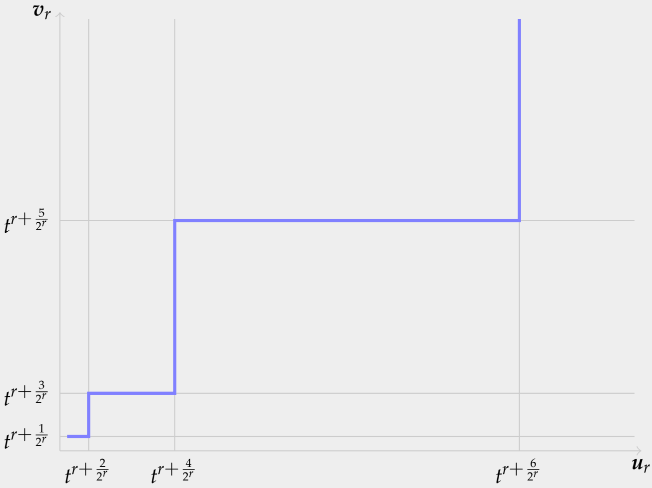
Curvature analysis

In the $(\boldsymbol{u}_r, \boldsymbol{v}_r)$-plane, the tropical central path looks like:



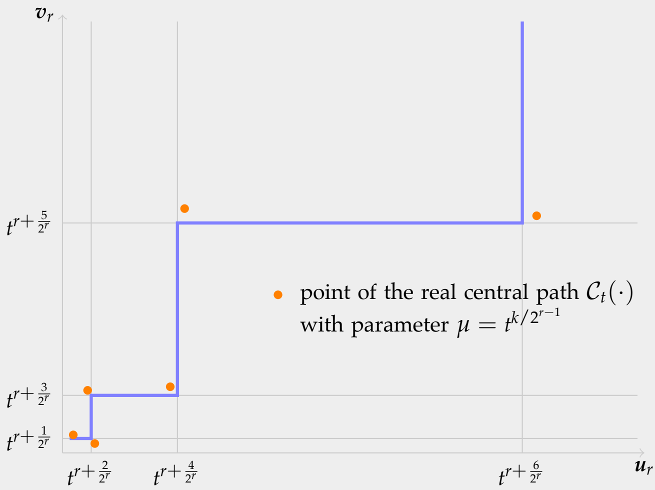
Curvature analysis

In the (u_r, v_r)-plane, the preimage under $\log_t$ of the tropical central path looks like:



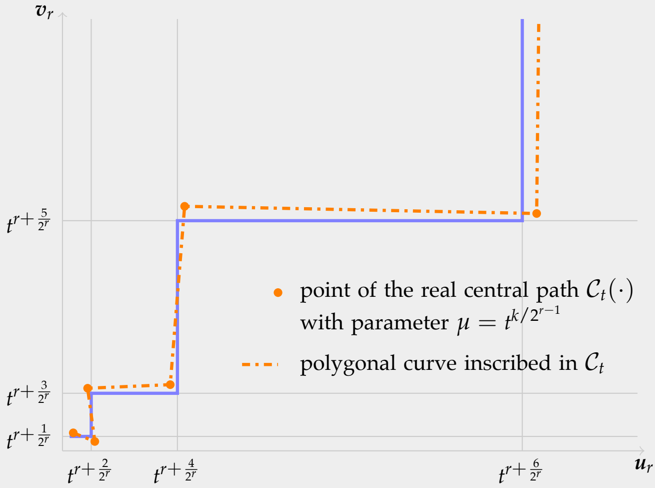
Curvature analysis

In the (u_r, v_r)-plane, the preimage under $\log_t$ of the tropical central path looks like:



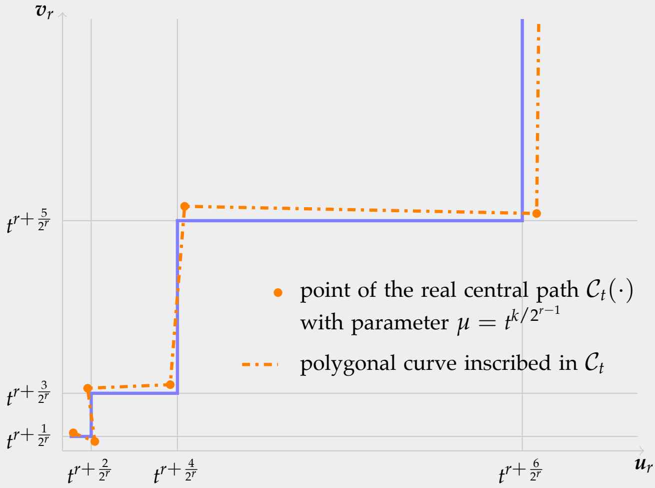
Curvature analysis

In the (u_r, v_r)-plane, the preimage under $\log_t$ of the tropical central path looks like:



Curvature analysis

In the (u_r, v_r)-plane, the preimage under $\log_t$ of the tropical central path looks like:



$\Longrightarrow$ $\liminf(\text{total curvature of } \mathcal{C}_t) \geqslant (2^r - 1)\frac{\pi}{2}$ when $t \to +\infty$.

Thank you!

Long and winding central paths, arXiv:1405.4161

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