Boundary stabilization of a cross-diffusion system in a time-dependent domain using backstepping

Jean Cauvin-Vila,[@,*,†] Virginie Ehrlacher,[*,†] Amaury Hayat[*]

[@]jean.cauvin-vila@enpc.fr, [*]CERMICS, Ecole des Ponts, [†]Team MATHERIALS, Inria Paris

Abstract

We study a cross-diffusion model for a Physical Vapor Deposition process [A.Bakhta, V.Ehrlacher, '18]. The problem is set in a one-dimensional moving domain. We are able to stabilize the *linearized system* around uniform steady states in arbitrary time. To our knowledge, this is one of the first control results for cross-diffusion systems. For the analysis, we adapt the backstepping technique [J-M.Coron, H-M.Nguyen, '17] to a case where the domain extends with time. This work should be seen as a fundamental brick on the way to the local stabilization of the nonlinear system.

Statement of the problem

In the model, a solid layer is composed of $n+1$ different chemical species (volumic fraction $u_i \geq 0$, $\sum_{i=0}^n u_i = 1$) and occupies a domain of the form $(0, e(t))$, where $e(t) > 0$ denotes the thickness of the layer and is determined by the fluxes ϕ_i of atoms that are absorbed at the surface of the layer: $e(t) = e_0 + \int_0^t \sum_{i=0}^n \phi_i(s)ds$.

The cross-diffusion equation in the bulk, together with the flux boundary conditions, form the system set on $(0, e(t))$:

$$\begin{cases} \partial_t u - \partial_x(\mathbf{A}(u)\partial_x u) = 0, & x \in (0, e(t)),\ t > 0, \\ (\mathbf{A}(u)\partial_x u)(t, 0) = 0, & t > 0, \\ (\mathbf{A}(u)\partial_x u)(t, e(t)) + e'(t)u(t, e(t)) = \varphi(t), & t > 0, \\ u(0, x) = u^0(x), & \end{cases} \tag{1}$$

where $u \in \mathbb{R}^n$ is the volumic fraction vector, $\varphi \in \mathbb{R}^n_+$ is the incoming flux at the boundary and $\mathbf{A}(u) \in \mathbb{R}^{n\times n}$ is the diffusion matrix that is formally derived from a stochastic lattice hopping model and satisfies: $(K_{ij} > 0)$

$$(\mathbf{A}(u)\partial_x u)_i = \sum_{1\leq j\neq i\leq n} (K_{ij} - K_{i0})(u_j\partial_x u_i - u_i\partial_x u_j) + K_{i0}\partial_x u_i.$$

Bakhta and Ehrlacher proved existence of weak solutions and long-time asymptotics for (1) in the case of *constant external fluxes* $\overline{\varphi}$: the solution converges in a rescaled L^1 sense to the uniform steady state $\bar{u}$ with speed at least $\frac{1}{\sqrt{t}}$. Numerical tests suggested a convergence rate of $\frac{1}{t}$.

Objective : derive a feedback control $\varphi(t, u(t))$ to stabilize (1) in finite time around the target state $(\bar{u}, \bar{e})$.

After linearization of (1) around $\bar{u}$ with respect to the fluxes $\delta\varphi = \varphi - \overline{\varphi}$ and the volumic fractions $\delta u := u - \bar{u}$, and after showing, thanks to the entropic structure of the system, that $A(\bar{u})$ can be diagonalized with positive eigenvalues, it remains to analyze the scalar problem: $(\sigma > 0,\ \bar{v} > 0)$

$$\begin{cases} \partial_t z - \sigma\partial^2_{xx} z = 0, & x \in (0, \bar{e}(t)),\ t > 0, \\ \sigma\partial_x z(t, \bar{e}(t)) + \bar{v}z(t, \bar{e}(t)) = \delta\psi(t), & t > 0, \\ \sigma(\partial_x z)(t, 0) = 0, & t > 0, \\ z(0, x) = z^0(x), & x \in (0, \bar{e}_0). \end{cases} \tag{2}$$

where $\delta\psi(t) := \delta\varphi(t) - \delta e'(t)\bar{u}$.

Backstepping approach

The idea is to map the original equation (2) to a *stable target equation* and to get the result using the reverse transformation. We consider a kernel transformation of the form

$$g(t, x) := z(t, x) - \int_0^x k(t, x, y)z(t, y)dy, \tag{3}$$

and show that it is invertible as:

$$z(t, x) = g(t, x) + \int_0^x l(t, x, y)g(t, y)dy. \tag{4}$$

It is natural here to consider as a target the same problem with an additional damping term λg and homogeneous boundary conditions, which provides $\lambda-$exponential stability in L^2 norm for g. The assignment of homogeneous boundary conditions for g imposes the form of the feedback:

$$\delta\psi_\lambda(t) := \sigma k_\lambda(t; \bar{e}(t), \bar{e}(t))z_\lambda(t, \bar{e}(t)) + \int_0^{\bar{e}(t)} \left[\sigma\partial_x k_\lambda(t; \bar{e}(t), y) + \bar{v}k_\lambda(t; \bar{e}(t), y)\right] z_\lambda(t, y)dy. \tag{5}$$

We show that then the kernels k and l must solve similar problems (up to the sign of λ in the PDE): $(D_t := \{(x, y) \in (\mathbb{R}_+)^2,\quad 0 < y \leq x < \bar{e}(t)\}.)$

$$\begin{cases} \partial^2_{xx} l_\lambda(t; x, y) - \partial^2_{yy} l_\lambda(t; x, y) = -\frac{\lambda}{\sigma} l_\lambda(t; x, y) & (x, y) \in D_t, \\ \partial_y l_\lambda(t; x, 0) = 0 & x \in (0, \bar{e}(t)), \\ \frac{d}{dx} l_\lambda(t; x, x) = -\frac{\lambda}{2\sigma} & x \in (0, \bar{e}(t)), \\ l_\lambda(t; 0, 0) = 0. & \end{cases} \tag{6}$$

According to (4) and since the target problem is stable, it holds:

$$\begin{aligned} \|z_\lambda(t)\|_{L^2(0,\bar{e}(t))} &\leq \left(1 + \|l_\lambda(t)\|_{L^2(D_t)}\right) \|g_\lambda(t)\|_{L^2(0,\bar{e}(t))} \leq \left(1 + \|l_\lambda(t)\|_{L^2(D_t)}\right) e^{-\lambda t}\|g^0_\lambda\|_{L^2(0,\bar{e}_0)} \\ &\leq C\left(1 + \|l_\lambda(t)\|_{L^2(D_t)}\right) e^{-\lambda t}\|z^0\|_{L^2(0,\bar{e}_0))}. \end{aligned}$$

Hence it suffices to show well-posedness for (6) and to obtain estimates on $||l(t)||_{L^2}$ to get stability of z_λ in L^2.

Main result

Adapting the method from Coron and Nguyen, we are able to show finite time stabilization by applying this technique for well-chosen sequences of time intervals $[t_{n-1}, t_n)$ and increasing associated damping rates λ_n:

Theorem. *Problem (2) is stabilizable in L^2, in any finite time $T > 0$. More precisely, for any $T > 0$, there exist some sequences of times $(t_n) \to T^-$ and $\lambda_n \to \infty$ such that, if the feedback is defined as $\delta\psi_{\lambda_n}$ in (5) in time intervals $[t_{n-1}, t_n)$, then there exists a unique weak solution z_T in L^2 to (2) and it holds:*

$$\|z_T(t, \cdot)\|_{L^2((0,\bar{e}(t))} \to 0 \text{ as } t \to T^-.$$

This result is generalized to abstract volume-filling cross-diffusion systems with a favorable entropic structure.

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