Boundary stabilization of a cross-diffusion system in a time-dependent domain using backstepping

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Abstract

We study a one-dimensional cross-diffusion model for a Physical Vapor Deposition process [A.Bakhta, V.Ehrlacher, '18]. Using the backstepping technique [J-M.Coron, H-M.Nguyen, '17], we are able to show finite time stabilization to the uniform steady state, in arbitrary time, for the *linearized system*. The novelty of this research lies in the analysis of the time dependence in the system. Ongoing work is concerned with the treatment of the nonlinear terms.

Statement of the problem

In the model, a solid layer is composed of $n+1$ different chemical species and occupies a domain of the form $(0, e(t))$, where $e(t) > 0$ denotes the thickness of the layer and is determined by the fluxes ϕ_i of atoms that are absorbed at the surface of the layer: $e(t) = e_0 + \int_0^t \sum_{i=0}^n \phi_i(s) ds$.

The cross-diffusion equation in the bulk, together with the flux boundary conditions, form the system set on $(0, e(t))$:

$$\begin{cases} \partial_t u - \partial_x(\mathbf{A}(u)\partial_x u) = 0, & x \in (0, e(t)),\ t > 0, \\ (\mathbf{A}(u)\partial_x u)(t, 0) = 0, & t > 0, \\ (\mathbf{A}(u)\partial_x u)(t, e(t)) + e'(t)u(t, e(t)) = \phi(t), & t > 0, \\ u(0, x) = u^0(x), \end{cases} \tag{1}$$

where $u \in \mathbb{R}^n$ is the volumic fraction vector, $\phi \in \mathbb{R}^n_+$ is the incoming flux at the boundary and $\mathbf{A}(u) \in \mathbb{R}^{n\times n}$ is the diffusion matrix that is formally derived from a stochastic lattice hopping model and satisfies:

$$(\mathbf{A}(u)\nabla u)_i = \sum_{0\le j\ne i\le n} K_{ij}(u_j \nabla u_i - u_i \nabla u_j).$$

Bakhta and Ehrlacher proved existence of weak solutions and long-time asymptotics for (1) in the case of *constant external fluxes*: the solution converges in a rescaled sense to the uniform steady state $\bar{f}$ with speed at least $\frac{1}{\sqrt{t}}$. Numerical tests suggested a convergence rate of $\frac{1}{t}$.

Objective : control the incoming fluxes $\phi(t, u(t))$ to get finite time stabilization of the uniform steady state .

Entropy variable, linearization and uncoupling

We introduce the scaled variable $v(t, y) = u(t, e(t)y)$ in order to work on a fix domain. It introduces a first order term in the equation. After a change of variable $w := Dh(v)$ related to the Boltzmann *entropy* h of the system, and after linearization around the steady state at first order in the fluxes $\delta\phi$ and the solution $z := \delta w = w - \bar{w}$, the system (1) is formally locally equivalent to:

$$\begin{cases} \partial_t z - \frac{1}{\bar{e}(t)^2}\mathbf{C}^{-1}\mathbf{M}(\bar{w})\partial^2_{yy} z - \frac{V}{\bar{e}(t)} y \partial_y z = 0, & y \in (0,1),\ t > 0, \\ \frac{1}{\bar{e}(t)}\mathbf{C}^{-1}\mathbf{M}(\bar{w})\partial_y z(t,1) + V z(t,1) = \phi_{eff}(t), & t > 0, \\ \frac{1}{\bar{e}(t)}(\partial_y z)(t,0) = 0, & t > 0, \end{cases} \tag{2}$$

where $V > 0$, $\mathbf{M}$ and $\mathbf{C}$ are symmetric positive definite matrices. Therefore $\mathbf{C}^{-1}\mathbf{M}$ can be diagonalized, the equations can be uncoupled and we are left to consider the *scalar* version of (2).

Backstepping and stabilization

The idea is to map the original equation to a *stable target equation* and to get the result using the reverse transformation. The transformation and its inverse are of the form

$$g(t, y) := z(t, y) - \int_0^y k(t, y, s) z(t, s) ds,$$
$$z(t, y) = g(t, y) + \int_0^y l(t, y, s) g(t, s) ds.$$

It is natural here to consider as a target the same equation with a $-\lambda g$ damping term on the RHS that provides $\lambda-$exponential stability in L^2 norm. Then the kernels k and l must solve a pde in a triangle $\{0 \le s \le y \le 1\}$:

$$\begin{cases} k_t - \frac{1}{\bar{e}(t)^2}(k_{yy} - k_{ss}) - \frac{V}{\bar{e}(t)}((sk)_s + yk_y) + \lambda k = 0, \\ \frac{1}{\bar{e}(t)^2} k_s(t, y, 0) = 0, \\ \frac{1}{\bar{e}(t)^2} k(t, y, y) = -\frac{\lambda}{2} y, \end{cases}$$

We recover L^2 stability for any exponential rate for (2) by making sure that $l(t)$ is not growing too fast, since it holds:

$$||z(t)||_{L^2} \le C(1 + ||l(t)||_{L^2})e^{-\lambda t}||z(0)||_{L^2}.$$

Hence controlling $||l(t)||_{L^2}$ is the main matter. Adapting the method from Coron and Nguyen, we are even able to show finite time stabilization by applying this technique for well-chosen sequences of time intervals and increasing associated damping rates λ_n:

Theorem. *Let $T > 0$. There exists a functional $K : [0, T) \to H^1(0, 1)^*$ such that, for every $z_0 \in H^1(0, 1)$, if $z \in C^0([0, T], L^2)$ is the solution of (2) with $\phi_{eff}(t)$ defined by:*

$$\phi_{eff}(t) := K(t)z(t, \cdot),$$

then:

$$z(t, \cdot) \to 0 \text{ in } L^2(0, 1) \text{ as } t \to T^-.$$

$$K(t)z = \frac{1}{\bar{e}(t)} k_n(t, 1, 1) z(t, 1) + \int_0^1 \left(\frac{1}{\bar{e}(t)}\partial_y k_n(t, 1, s) + V k_n(t, 1, s)\right) z(s) ds,$$

for $t_n \le t < t_{n+1}$, for some $(t_n)_{\mathbb{N}}$ and some kernels $(k_n)_{\mathbb{N}}$ defined previously by backstepping.