Homogenization with defects

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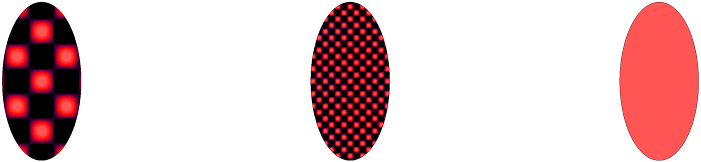
Typical homogenization problem

$$\begin{cases} -\operatorname{div}\left(a_\varepsilon \nabla u_\varepsilon\right) = f \ \text{in} & \Omega, \\ u_\varepsilon = 0 \ \text{on} & \partial\Omega. \end{cases}$$

Ω fixed smooth domain, a_ε conductivity, $0 < \alpha \leq a_\varepsilon(x) \leq \beta < +\infty$.
Matrix a_ε has ε-scale oscillations : ex. $a_\varepsilon(x) = a_{(per)}\left(\frac{x}{\varepsilon}\right)$.



Limit $\varepsilon \to 0$: $-\operatorname{div}(a^* \nabla u^*) = f$?

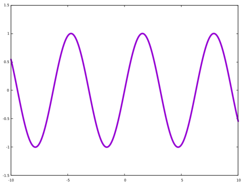
Preliminary : averages

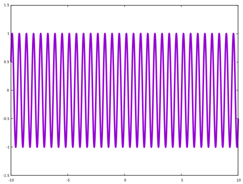
If a is periodic, then $a\left(\frac{x}{\varepsilon}\right) \underset{\varepsilon \to 0}{\longrightarrow} \langle a \rangle = \fint_Q a$ weakly.

That is, for any domain Ω, $\int_\Omega a\left(\frac{x}{\varepsilon}\right) dx \underset{\varepsilon \to 0}{\longrightarrow} |\Omega| \langle a \rangle$.

Proof in 1D: $\varepsilon = \frac{1}{N}$.

$$\int_0^1 a\left(\frac{x}{\varepsilon}\right) dx = \int_0^1 a(Nx) dx = \frac{1}{N} \int_0^N a(x) dx = \langle a \rangle.$$





Dimension 1

$$-\frac{d}{dx}\left(a\left(\frac{x}{\varepsilon}\right)\frac{du_\varepsilon}{dx}\right)=f \quad \xrightarrow[\varepsilon\to 0]{} \quad -\frac{d}{dx}\left(a^*\frac{du^*}{dx}\right)=f \quad ?$$

First guess : $a^* = \langle a \rangle$? Hum...

$$-a\left(\frac{x}{\varepsilon}\right)\frac{du_\varepsilon}{dx}=F(x):=\int_0^x f(t)dt.$$

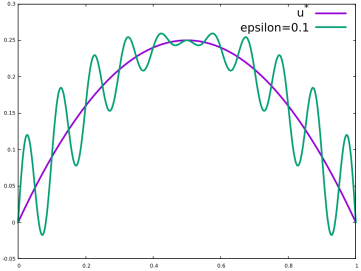
$$\frac{du_\varepsilon}{dx}=-a\left(\frac{x}{\varepsilon}\right)^{-1}F(x)\xrightarrow[\varepsilon\to 0]{}-\left\langle a^{-1}\right\rangle F(x).$$

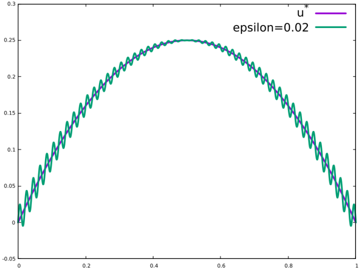
$$\frac{du^*}{dx}=-\left\langle a^{-1}\right\rangle F(x) \Rightarrow -\frac{d}{dx}\left(\left\langle a^{-1}\right\rangle^{-1}\frac{du^*}{dx}\right)=f.$$

$$a^*=\left\langle a^{-1}\right\rangle^{-1}$$

$$u_\varepsilon(x) = -\int a\left(\frac{t}{\varepsilon}\right)^{-1} F(t)dt, \quad u^*(x) = -\int \left\langle a^{-1} \right\rangle F(t)dt.$$

$$\Rightarrow \quad u_\varepsilon(x) - u^*(x) = \int \left(\left\langle a^{-1} \right\rangle - a\left(\frac{t}{\varepsilon}\right)^{-1} \right) F(t)dt = O(\varepsilon).$$





Large scale u^* + modulation of amplitude ε, at scale ε.

Two-scale expansion

Ansatz: $u_\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \dots$

$u_k(x,y)$ are periodic wrt y.

$x =$ slow (macro) variable, $y = \frac{x}{\varepsilon} =$ fast (micro) variable.

Idea: as $\varepsilon \to 0$, x and y become independent variables.

Differentiation rule: $\nabla_x \left(u_k\left(x, \frac{x}{\varepsilon}\right)\right) = \nabla_x u_k\left(x, \frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon}\nabla_y u_k\left(x, \frac{x}{\varepsilon}\right).$

Insert this into $-\mathrm{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = f.$

Hierarchy of equations

$$-\operatorname{div}\left(a\left(x,\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right)=f$$

$$u_\varepsilon(x)=u_0\left(x,\frac{x}{\varepsilon}\right)+\varepsilon u_1\left(x,\frac{x}{\varepsilon}\right)+\varepsilon^2 u_2\left(x,\frac{x}{\varepsilon}\right)+\dots$$

$$\nabla_x\left(u_k\left(x,\frac{x}{\varepsilon}\right)\right)=\nabla_x u_k\left(x,\frac{x}{\varepsilon}\right)+\frac{1}{\varepsilon}\nabla_y u_k\left(x,\frac{x}{\varepsilon}\right).$$

$$\varepsilon^{-2} : -\operatorname{div}_y\left(a(y)\nabla_y u_0\right)=0.$$

$$\varepsilon^{-1} : -\operatorname{div}_y\left(a(y)\nabla_y u_1\right)=\operatorname{div}_y\left(a(y)\nabla_x u_0\right)+\operatorname{div}_x\left(a(y)\nabla_y u_0\right).$$

$$\varepsilon^{0} : -\operatorname{div}_y\left(a(y)\left(\nabla_y u_2\right)\right)=\operatorname{div}_x\left(a(y)\left(\nabla_x u_0+\nabla_y u_1\right)\right)+\operatorname{div}_y\left(a(y)\nabla_x u_1\right)+f.$$

$$\dots$$

$$\varepsilon^{i} : -\operatorname{div}_y\left(a(y)\nabla_y u_{i+2}\right)=F(u_i,u_{i+1}).$$

Periodic cell problem:

$$\begin{cases} -\operatorname{div}_y(a(y)\nabla_y v) = g, \\ v \text{ periodic.} \end{cases}$$

Lemma

This problem has a solution if and only if

$$\int_Q g = 0.$$

And this solution is unique up to the addition of a constant.

Tool: Fredholm's alternative (or Lax-Milgram).

Corrector equation

$$\varepsilon^{-2}: \quad -\mathrm{div}_y\left(a(y)\nabla_y u_0\right) = 0 + \text{pbc} \Rightarrow \nabla_y u_0 = 0\text{:}\ \ u_0(x,y) = u_0(x).$$

$$\varepsilon^{-1}: \quad -\mathrm{div}_y\left(a(y)\nabla_y u_1\right) = \mathrm{div}_y\left(a(y)\nabla_x u_0\right) + \cancel{\mathrm{div}_x\left(a(y)\nabla_y u_0\right)}.$$

Introduce the corrector problem ($e_1, \ldots, e_d$ canonical basis of $\mathbb{R}^d$)

$$\begin{cases} -\mathrm{div}_y\left(a(y)\nabla_y w_i\right) = \mathrm{div}_y\left(a(y)e_i\right), \\ \qquad w_i \text{ periodic.} \end{cases}$$

$$u_1(x,y) = \sum_{i=1}^{d} \frac{\partial u_0}{\partial x_i}(x) w_i(y).$$

Homogenized equation

$$\varepsilon^0 : \; -\mathrm{div}_x\left(a(y)\left(\nabla_x u_0 + \nabla_y u_1\right)\right) + \mathrm{div}_y\left(a(y)\left(\nabla_x u_1 + \nabla_y u_2\right)\right) = f(x).$$

Integrate over Q: periodicity $\int_Q \mathrm{div}_y(g(y))dy = 0.$

$$-\mathrm{div}_x\left(\int_Q a(y)\left(\nabla_x u_0(x) + \nabla_y u_1(x,y)\right)\right) = f(x).$$

$$\begin{cases} -\mathrm{div}_y\left(a(y)\left(e_i + \nabla_y w_i\right)\right) = 0, \\ \qquad\qquad w_i \text{ periodic.} \end{cases}$$

$$-\mathrm{div}_x\left(a^* \nabla_x u_0(x)\right) = f(x),$$

$$\nabla_y u_1(x,y) = \sum_{i=1}^{d} \frac{\partial u_0}{\partial x_i}(x) \nabla_y w_i(y).$$

$$a^*_{ij} = \int_Q \left[a(y)\left(\delta_{ij} + \nabla w_i(y) \cdot e_j\right)\right] dy.$$

Homogenized equation with a formula for a^*.

Back to 1D

$$\begin{cases} -\operatorname{div}_y \left(a(y)\left(e_i + \nabla_y w_i\right)\right) = 0, \\ \qquad\qquad w_i \text{ periodic.} \end{cases}$$

$$a^*_{ij} = \int_Q \left[a(y)\left(\delta_{ij} + \nabla w_i(y)\cdot e_j\right)\right] dy.$$

In 1D: $e_i = 1,\qquad -\frac{d}{dy}\left(a(y)\left(1+\frac{dw}{dy}\right)\right) = 0,\qquad w$ periodic.

$$1 + \frac{dw}{dy} = \frac{C}{a(y)}.$$

Integrate on $[0,1]$: $1 + \underbrace{w(1) - w(0)}_{=0} = C\left\langle a^{-1}\right\rangle.\quad 1 + \frac{dw}{dy} = \left\langle a^{-1}\right\rangle^{-1} a(y)^{-1}.$

$$a^* = \int_0^1 a(y)\left(1 + \frac{dw}{dy}(y)\right) dy = \left\langle a a^{-1}\left\langle a^{-1}\right\rangle^{-1}\right\rangle = \left\langle a^{-1}\right\rangle^{-1}.$$

Strong convergence

Theorem (Allaire, JKO, BLP, ...)

$$\left\| u_\varepsilon(x) - u_0(x) - \varepsilon \sum_{i=1}^{d} \frac{\partial u_0}{\partial x_i}(x) w_i\left(\frac{x}{\varepsilon}\right) \right\|_{H^1(\Omega)} \underset{\varepsilon \to 0}{\longrightarrow} 0,$$

that is,

$$\| u_\varepsilon - u_0 \|_{L^2(\Omega)} \underset{\varepsilon \to 0}{\longrightarrow} 0,$$

$$\left\| \nabla u_\varepsilon(x) - \nabla u_0(x) - \sum_{j=1}^{d} \frac{\partial u_0}{\partial x_i}(x) \nabla w_i\left(\frac{x}{\varepsilon}\right) \right\|_{L^2(\Omega)} \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

- Correctors allow for strong H^1 convergence.
- Convergence rate ε on interior domains.
- Convergence rate $\sqrt{\varepsilon}$ due to boundary layers.

Theorem (Murat-Tartar, 1978)

There exists a^*, elliptic and bounded, and $u^* \in H^1_0(\Omega)$, such that, up to extraction,

$$u_\varepsilon \rightharpoonup u^* \quad H^1(\Omega), \quad a_\varepsilon \nabla u_\varepsilon \rightharpoonup a^* \nabla u^* \quad L^2,$$

and

$$-\mathrm{div}(a^* \nabla u^*) = f \ \ in\ \Omega.$$

- Much more general than periodic: a_ε elliptic bounded only.
- Possibility of including correctors.
- Limit up to extraction of a subsequence.
- No "usable" formula for a^*.

Literature

- Bensoussan-Lions-Papanicolaou, 1978. periodic case
- Jikov-Kozlov-Oleinik, 1994. periodic + stochastic + general theory
- Cioranescu-Donato, 1999. periodic case
- Allaire, 2002. periodic + general theory
- Tartar, 2009. general theory
- Shen, 2018. periodic, fine estimates
- Armstrong-Kuusi-Mourrat, 2019. stochastic

Setting of the problem

$$-\operatorname{div}\left[a\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon\right] = f$$

a: elliptic and bounded.

$$a(x) = a_{per}(x) + \widetilde{a}(x), \qquad \lim_{|x|\to+\infty} \widetilde{a}(x) = 0.$$

Standard homogenization: $u^\varepsilon \to u^*$, $\quad \nabla u^\varepsilon \rightharpoonup \nabla u^*$,

$$-\operatorname{div}\left[a^* \nabla u^*\right] = f$$

Formula for a^*? Strong convergence in H^1? Corrector.

Corrector equation

$$-\operatorname{div}\left(a\left(e_i + \nabla w_i\right)\right) = 0,$$

$$a(x) = a_{per}(x) + \widetilde{a}(x), \qquad \lim_{|x|\to+\infty} \widetilde{a}(x) = 0.$$

One expects

$$w_i = w_i^{per} + \widetilde{w}_i, \qquad \text{"}\nabla \widetilde{w}_i(x) \underset{|x|\to+\infty}{\longrightarrow} 0\text{"}.$$

$$a^*_{ij} = \left\langle a\left(\delta_{ij} + \nabla w_i \cdot e_j\right)\right\rangle$$

The case $\tilde{a} \in L^2$

Hypotheses:

- $a_{per} \in C^{0,\alpha}$ is periodic, elliptic and bounded.
- $\tilde{a} \in L^2 \cap L^\infty$, $a_{per} + \tilde{a}$ elliptic.

Theorem (XB/Le Bris/Lions, 2012, Milan J. Math.)

The corrector problem $-\operatorname{div}\left[(a_{per} + \tilde{a})(\nabla w_i + e_i)\right] = 0$ *has a solution such that* $\nabla w_i \in \mathcal{H}$. *Such a solution is unique up to the addition of a constant. Further,*

$$\nabla w_i = \nabla w_i^{per} + \nabla \tilde{w}_i, \quad \nabla \tilde{w}_i \in L^2.$$

$$\mathcal{H} = \left\{ f_{per} + g, \quad f_{per} \text{ periodic}, \ g \in L^2 \right\}$$

$\widetilde{a} \in L^2$: ideas of proof

Existence: look for w_i as $w_i = w_i^{per} + \widetilde{w}_i$.

$$-\operatorname{div}\left[\left(a_{per} + \widetilde{a}\right)\nabla \widetilde{w}_i\right] = \operatorname{div}\underbrace{\left[\widetilde{a}\left(\nabla w_i^{per} + e_i\right)\right]}_{:=g \in L^2}$$

Solve this problem by regularization:

$$-\operatorname{div}\left[\left(a_{per} + \widetilde{a}\right)\nabla \widetilde{w}_{i,\gamma}\right] + \gamma \widetilde{w}_{i,\gamma} = \operatorname{div}(g)$$

Solution in H^1. Bounds:

$$c\left\|\nabla \widetilde{w}_{i,\gamma}\right\|_{L^2}^2 + \gamma\left\|w_{i,\gamma}\right\|_{L^2}^2 \leq \|g\|_{L^2}\left\|\nabla \widetilde{w}_{i,\gamma}\right\|_{L^2}.$$

Hence $\left\|\nabla \widetilde{w}_{i,\gamma}\right\|_{L^2} \leq C$, and $\sqrt{\gamma}\left\|\widetilde{w}_{i,\gamma}\right\|_{L^2} \leq C$.
Pass to the limit $\gamma \to 0$.

$\widetilde{a} \in L^2$: Ideas of proof

Uniqueness: v_i difference of two solutions: $\nabla v_i = g_{per} + \widetilde{g}$, $\widetilde{g} \in L^2$.

- $\nabla v_i = \nabla v_i^{per} + \nabla \widetilde{v}_i$, with $\nabla \widetilde{v}_i \in L^2$.
- If $\nabla \widetilde{v}_i = 0$: v_i^{per} solution to $-\operatorname{div}\left[(a_{per} + \widetilde{a}) \nabla v_i^{per} \right] = 0$. Integrate over $\mathscr{A}_R$: $\displaystyle\int_{\mathscr{A}_R} |\nabla v_i^{per}|^2 \leq \text{boundary terms} = o(R^d)$. Hence, $\nabla v_i^{per} = 0$.
- If $\nabla v_i^{per} = 0$: $\widetilde{v}_i$ solution to $-\operatorname{div}[(a_{per} + \widetilde{a}) \nabla \widetilde{v}_i] = 0$. Integrate over $\mathscr{A}_R$: $\displaystyle\int_{\mathscr{A}_R} |\nabla \widetilde{v}_i|^2 \leq \text{boundary terms} = o(1)$. Hence, $\nabla \widetilde{v}_i = 0$.

$\mathscr{A}_R = \left\{ x \in \mathbb{R}^d, \quad R < |x| < 2R \right\}$

Using the corrector

$$a^*_{ij} = \langle a(\delta_{ij} + \nabla w_i \cdot e_j) \rangle = \lim_{\varepsilon \to 0} a\left(\frac{x}{\varepsilon}\right)\left(\delta_{ij} + \nabla w_i\left(\frac{x}{\varepsilon}\right) \cdot e_j\right).$$

But : $\nabla \widetilde{w}_i\left(\frac{x}{\varepsilon}\right) \underset{\varepsilon \to 0}{\longrightarrow} 0$. Finer approximation of the solution:

$$u^\varepsilon - u^* \longrightarrow 0 \qquad \text{in} \quad L^2$$

$$\nabla u^\varepsilon - \nabla u^* \rightharpoonup 0 \qquad \text{in} \quad L^2.$$

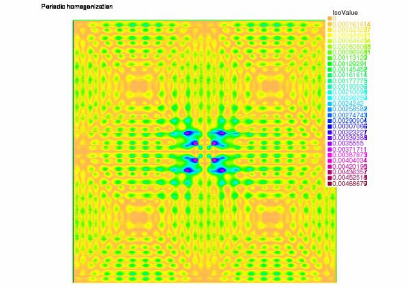
$$u^\varepsilon(x) - u^*(x) - \varepsilon \sum_{i=1}^d w_i\left(\frac{x}{\varepsilon}\right) \partial_{x_i} u^*(x) \longrightarrow 0 \qquad \text{in} \quad H^1.$$

$$u^\varepsilon(x) - u^*(x) - \varepsilon \sum_{i=1}^d w_i\left(\frac{x}{\varepsilon}\right) \partial_{x_i} u^*(x) \longrightarrow 0 \qquad \text{in} \quad W^{1,\infty}.$$

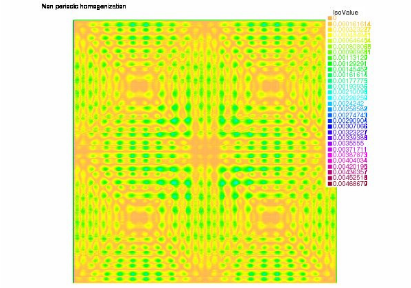
↑

Here you need the exact corrector $(a_{per} + \widetilde{a})$

Error comparison



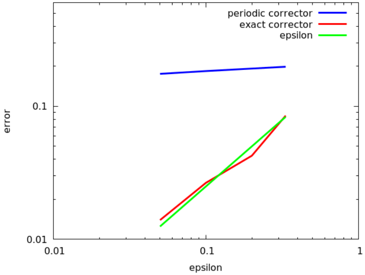
Relative error with w_i^{per}



Relative error with $w_i^{per} + \widetilde{w}_i$

Error comparison

$1/\varepsilon$	$\delta^{\varepsilon}_{per}$	δ^{ε}
3	0.198112	0.0850091
5	0.191952	0.0425306
10	0.183784	0.0266084
20	0.175248	0.0139564



$$\delta^{\varepsilon}_{per} = \left\| u^{\varepsilon} - u^* - \varepsilon \sum_{i=1}^{d} w_i^{per}\left(\frac{x}{\varepsilon}\right)\partial_{x_i} u^* \right\|_{W^{1,\infty}}$$

$$\delta^{\varepsilon} = \left\| u^{\varepsilon} - u^* - \varepsilon \sum_{i=1}^{d} w_i\left(\frac{x}{\varepsilon}\right)\partial_{x_i} u^*(x) \right\|_{W^{1,\infty}}$$

$$-\operatorname{div}\left[a_{per}\left(\nabla w_i^{per} + e_i\right)\right] = 0,$$

$$-\operatorname{div}\left[(a_{per} + \tilde{a})(\nabla w_i + e_i)\right] = 0.$$

The case $\widetilde{a} \in L^r$, $r < +\infty$

Hypotheses:

- $a_{per} \in C^{0,\alpha}$ is periodic, elliptic and bounded.
- $\widetilde{a} \in L^r \cap L^\infty$, $r < +\infty$, $a_{per} + \widetilde{A}$ elliptic.

Theorem (XB/Le Bris/Lions, 2015, Comm. PDE)

The corrector problem $-\operatorname{div}\left[\left(a_{per} + \widetilde{a}\right)\left(\nabla w_i + e_i\right)\right] = 0$ *has a solution such that* $\nabla w_i \in \mathscr{H}$*. Such a solution is unique up to the addition of a constant. Further,*

$$\nabla w_i = \nabla w_i^{per} + \nabla \widetilde{w}_i, \quad \nabla \widetilde{w}_i \in L^r.$$

$$\mathscr{H} = \left\{ f_{per} + g, \quad f_{per} \text{ periodic}, \ g \in L^r \right\}$$

Ideas of proof

Uniqueness: v_i difference of two solutions.

- Argument as in $r = 2$ case: $\nabla v_i = \nabla v_i^{per} + \nabla \tilde{v}_i$.
- Periodic part: as in $r = 2$ case.
- Local part $\tilde{v}_i$: $\nabla \tilde{v}_i \in L^r$

$$-\operatorname{div}\left[a_{per} \nabla \tilde{v}_i\right] = \operatorname{div} \underbrace{\left(\tilde{a} \nabla \tilde{v}_i\right)}_{\in L^{r/2}}.$$

Hence $\nabla \tilde{v}_i \in L^{r/2}$ (Avellaneda-Lin, 1987).
And so on... $\nabla \tilde{v}_i \in L^2$.

Existence: solve $-\operatorname{div}\left(\left(a_{per} + \tilde{a}\right) \nabla \tilde{w}_i\right) = \operatorname{div}\underbrace{\left(\tilde{a}\left(\nabla w_i^{per} + e_i\right)\right)}_{g \in L^r}$.

Riesz operator

Corollary (XB/Le Bris/Lions, 2018, Comm. PDE)

Assume:

- $a_{per} \in C^{0,\alpha}$ *is periodic, elliptic and bounded.*
- $\tilde{a} \in L^r \cap L^\infty$, $r < +\infty$, $a_{per} + \tilde{a}$ *elliptic.*
- $p \in \mathbb{R}^d$.

Then, for any $f \in L^q$, $1 < q < +\infty$, *the equation*

$$-\mathrm{div}(a \nabla v) = \mathrm{div}(f)$$

admits a unique solution such that $\nabla v \in L^q$. *Moreover, the operator* $f \mapsto \nabla v$ *is continuous from* L^q *to* L^q.

Concentration-compactness method. Extension of Avellaneda-Lin, 1987.