A non-dispersive discontinuous galerkin method for the simulation of acoustic scattering in complex media

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The time harmonic scalar wave equation

An heterogeneous Helmholtz equation

$$\begin{cases} \operatorname{div}\Big(\lambda(\mathbf{x})\nabla u(\mathbf{x})\Big) + \mu(\mathbf{x})u(\mathbf{x}) = 0 & \text{in } \Omega \\ u(\mathbf{x}) = g_D(\mathbf{x}) & \text{on } \partial\Omega_D \\ \lambda(\mathbf{x})\dfrac{\partial u}{\partial n}(\mathbf{x}) = g_N(\mathbf{x}) & \text{on } \partial\Omega_N \\ \lambda(\mathbf{x})\dfrac{\partial u}{\partial n}(\mathbf{x}) + Z(\mathbf{x})u(\mathbf{x}) = g_F(\mathbf{x}) & \text{on } \partial\Omega_F. \end{cases}$$

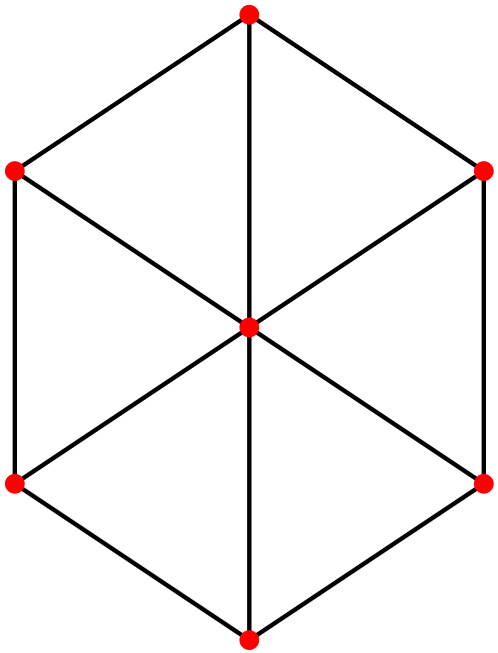
- the wave number k is included in λ and μ
- Other boundary condition can be as well considered
- The physical parameter function λ and μ are piecewise constant.
- The domain Ω is two or three dimensional with boundary $\partial\Omega$

The Classical IPDG method

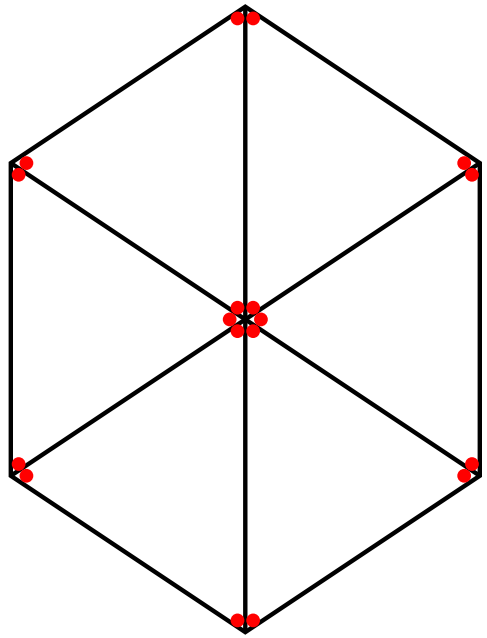
The Interior Penalty Discontinuous Galerkin method

- It has been ivented by Arnold in 1982
- It has been intensively studied during the last decade
- The main Advantages in frequency domain
  - High oscillation of the coefficients can be considered
  - Every tetrahedron is connected only to four neighbors (important for direct methods like LU)
  - Less dispersive than Continuous Finite Element
- Drawbacks
  - Hard to code
  - The unknowns at the interface between two elements are doubled (this has been corrected by the Hybridizable Discontinuous Galerkin method)

The IPDG method



$\mathbb{P}_1$ Continuous Galerkin



$\mathbb{P}_1$ Discontinuous Galerkin

The IPDG method

On each triangle K, we have the variational equation

$$\mathrm{a}_K(u, v) \;=\; 0$$

with the bilinear forms

$$\begin{cases} \mathrm{a}_K(u,v) &= \displaystyle\int_K \lambda_K \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x} \;-\; \int_{\partial K} p_K(\mathbf{x}) v(\mathbf{x}) ds_{\mathbf{x}}, \\ &- \displaystyle\int_K \mu_K u(\mathbf{x})\; v(\mathbf{x})\; d\mathbf{x} \end{cases}$$

and

$$p_K(\mathbf{x}) = \lambda_K \nabla u(\mathbf{x}) \cdot \mathbf{n}_K$$

Summing over all the tetrahedra of the mesh, we have

$$\mathrm{a}(u, v) \;=\; 0 \tag{1}$$

with the bilinear forms

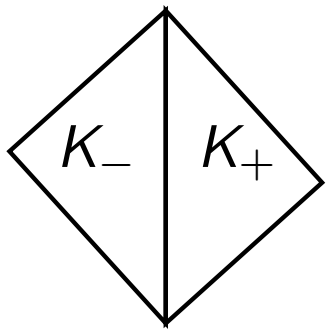
$$\begin{cases} \mathrm{a}(u,v) &= \displaystyle\sum_K \int_K \lambda_K \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x} \\ &- \displaystyle\sum_K \int_{\partial K} p_K(\mathbf{x}) v(\mathbf{x}) ds_{\mathbf{x}}, \\ &- \displaystyle\sum_K \int_K \mu_K u(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} \end{cases} \tag{2}$$

with K the tetrahedra.

The IPDG (Neumann boundary condition)

The interface term in the bilinear form is evaluated by taking into account the Neumann boundary condition and in remarking that every face is considered twice

$$\sum_K \int_{\partial K} p_K(\mathbf{x}) v(\mathbf{x}) ds_{\mathbf{x}} = \sum_T \int_T [p(\mathbf{x}) v(\mathbf{x})] ds_{\mathbf{x}} \tag{3}$$



with

$$[p(\mathbf{x}) v(\mathbf{x})] = p_+ v_+(\mathbf{x}) + p_- v_-(\mathbf{x}) \tag{4}$$

and K the tetrahedra and T the interior faces.

Due to the transmission conditions

$$p_+(\mathbf{x}) \;=\; -\,p_-(\mathbf{x}) \;=\; \{p\} \tag{5}$$

It follows that (v is discontinuous across the interface)

$$[p(\mathbf{x})v(\mathbf{x})] \;=\; \{p\}\,[v] \tag{6}$$

with

$$\begin{cases} [v] &= v_+ - v_-; \\ \{p\} &= \dfrac{p_+(\mathbf{x}) - p_-(\mathbf{x})}{2}. \end{cases} \tag{7}$$

The IPDG method

The bilinear form a takes the form

$$\begin{cases} \mathrm{a}(u,v) &= \displaystyle\sum_K \int_K \lambda_K \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) - \mu_K u(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} \\ &- \displaystyle\sum_T \int_T \{p\}[v] ds_{\mathbf{x}} \end{cases} \tag{8}$$

- The initial bilinear form is unsymmetric and not coercive
- The bilinear form is symmetrized
- An interior penalization is added in order to ensure the positivity. The coefficient α_T is a parameter which should be chosen large enough.

The IPDG method

The bilinear form a takes the form

$$\begin{cases} a(u,v) &= \sum_K \int_K \lambda_K \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) - \mu_K u(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} \\ &- \sum_T \int_T \{p\}[v] ds_{\mathbf{x}} \\ &- \sum_T \int_T \{q\}[u] ds_{\mathbf{x}} \quad \text{with } q = \lambda \nabla v \cdot \mathbf{n} \end{cases} \tag{8}$$

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The IPDG method

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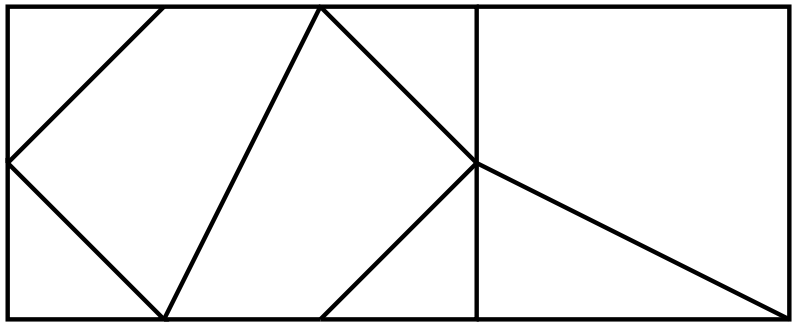
- The initial bilinear form is unsymmetric and not coercive
- The bilinear form is symmetrized
- An interior penalization is added in order to ensure the positivity. The coefficient α_T is a parameter which should be chosen large enough.

The IPDG-BEM method

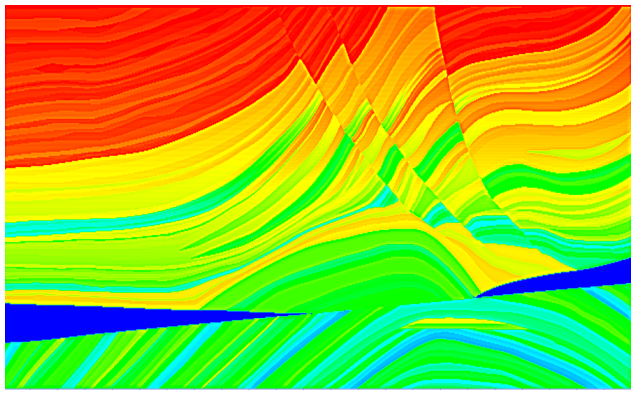
- **First point of view:** A discontinuous Galerkin method whose shape functions are quasi solution of the Helmholtz equation (constructed thanks to a Boundary element method)
  - This type of method is called Trefftz method
  - Classically, we use either plane wave basis inside each element or Bessel function
- **Second point of view:** A domain decomposition method for boundary element method with weak transmission condition

The mesh

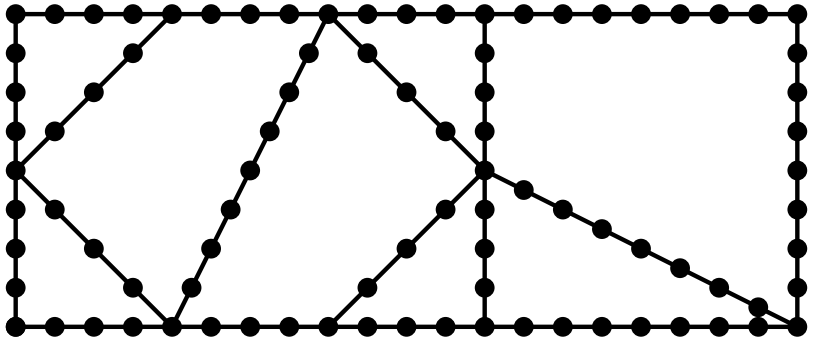
The degree of freedom of the Galerkin space



A smart finite element method



The mesh

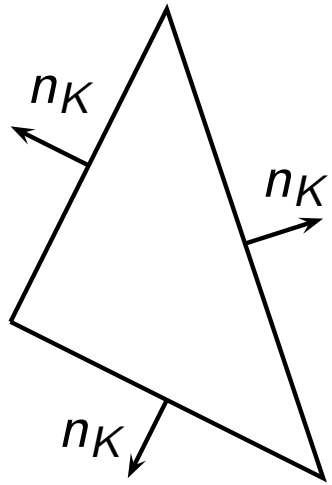


A smart finite element method

Trial and test functions

We are looking for an approximate solution define in every element

$$\begin{cases} \operatorname{div}\Big(\lambda_K \nabla u_K(\mathbf{x})\Big) \; + \; \mu_K u_K(\mathbf{x}) \; = \; 0 \text{ in } T \\ u_K(\mathbf{x}) \text{ given on } \partial T. \end{cases}$$



The Neumann trace

$$p_K \; = \; \lambda_K \, \nabla u_K \cdot \mathbf{n}_K \text{ on } \partial T$$

is the unique solution of the problem of first kind

$$\frac{V_K p_K}{\lambda_K} = \frac{M_K u_K}{2} - N_K u_K$$

with V_K and N_K the single layer and double layer operators.

The boundary element method

- u_K is approximated by a $\mathbb{P}_r$-continuous function
- p_K is approximated by a $\mathbb{P}_{r'}$-discontinuous function

We have

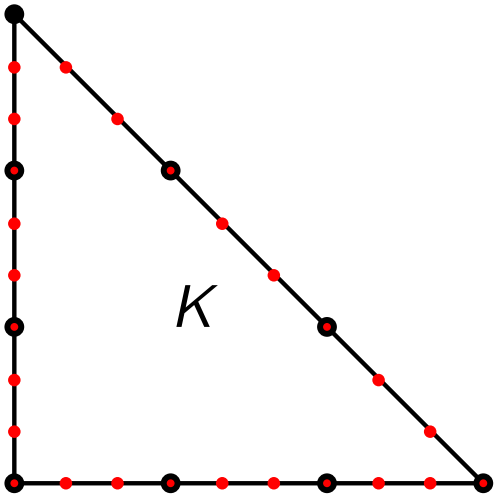
$$\begin{cases} (M_K u_K, q_k)_{\partial K} &= \displaystyle\int_{\partial K} u_K(\mathbf{x})\ q_K(\mathbf{x}) ds_{\mathbf{x}}, \\ (V_K p_K, q_k)_{\partial K} &= \displaystyle\int_{\partial K}\int_{\partial K} p_K(\mathbf{x})\ G_K(\mathbf{x}-\mathbf{y})\ q_K(\mathbf{y}) ds_{\mathbf{x}} ds_{\mathbf{y}}, \\ (N_K u_K, q_k)_{\partial K} &= \displaystyle\int_{\partial K}\int_{\partial K} p_K(\mathbf{x})\ \frac{\partial G_K}{\partial \mathbf{n_y}}(\mathbf{x}-\mathbf{y}) q_K(\mathbf{y}) ds_{\mathbf{x}} ds_{\mathbf{y}} \end{cases}$$

with

$$G(\mathbf{x}) = \frac{\exp(ik_K\|\mathbf{x}\|)}{4\pi\|\mathbf{x}\|} \text{ with } k_K = \sqrt{\frac{\mu_K}{\lambda_K}}$$

The boundary element method

- u_K is approximated by a $\mathbb{P}_r$-continuous function
- p_K is approximated by a $\mathbb{P}_{r'}$-discontinuous function



for u_K and p_K.

- geometric nodes for p_K
- geometric nodes for u_K

We could not use the same mesh

Idea: the Neumann trace should be computed precisely

Remark: p_K needs not to be discontinuous everywhere, only at the geometric singularity.

The symmetric variational formulation

The bilinear form a takes the form

$$\int_\Omega \lambda \nabla u \cdot \nabla v - \mu u v d\mathbf{x} = \int_\Gamma \frac{[p][v] + \{p\}\{v\}}{2} ds_\mathbf{x} + \int_{\partial\Omega} pv ds_\mathbf{x}$$

In the same way, we have

$$\int_\Omega \lambda \nabla u \cdot \nabla v - \mu u v d\mathbf{x} = \int_\Gamma \frac{[u][q] + \{u\}\{q\}}{2} ds_\mathbf{x} + \int_{\partial\Omega} uq ds_\mathbf{x}$$

Subtracting the two last equations:

$$\begin{cases} 0 &= \displaystyle\int_\Gamma \frac{[u][q] + \{u\}\{q\}}{2} ds_\mathbf{x} + \int_{\partial\Omega} uq ds_\mathbf{x} \\ &- \displaystyle\int_\Gamma \frac{[p][v] + \{p\}\{v\}}{2} ds_\mathbf{x} - \int_{\partial\Omega} pv ds_\mathbf{x} \end{cases}$$

No more volume terms; only boundary terms

The symmetric variational formulation

We remark that for the exact solution $[u] = 0$ and $\{p\} = 0$ on Γ and $u = g_D$ on $\partial\Omega_D$ and $\partial_n u = g_N$ on $\partial\Omega_D$ and $\partial_n u + Zu = g_F$ on $\partial\Omega_F$.

$$
\left\{
\begin{array}{rcl}
a(u,p;v,q) & = & \displaystyle\int_\Gamma \frac{\{u\}\{q\} + \{p\}\{v\} - [p][v] - [u][q]}{2} ds_{\mathbf{x}} \\
& - & \displaystyle\int_{\partial\Omega_D} pv + uq ds_{\mathbf{x}} \\
& + & \displaystyle\int_{\partial\Omega_N} pv + uq ds_{\mathbf{x}} \\
& + & \displaystyle\int_{\partial\Omega_F} pv + uq + 2Zuv ds_{\mathbf{x}}
\end{array}
\right.
$$

$$
\ell_1(v,q) = -2\int_{\partial\Omega_D} g_D q ds_{\mathbf{x}} + 2\int_{\partial\Omega_N} g_N v ds_{\mathbf{x}} + 2\int_{\partial\Omega_F} g_F v ds_{\mathbf{x}}
$$

The symmetric variational formulation

Adding the penalization terms ($[u] = 0$ on Γ and $u = g$ on $\partial\Omega$):

$$\underbrace{\int_\Gamma \alpha[u][v] + \int_{\partial\Omega_D} \alpha uv}_{b(u,p;v,q)} = \underbrace{\int_{\partial\Omega_D} \alpha g_D v}_{\ell_2(v)}$$

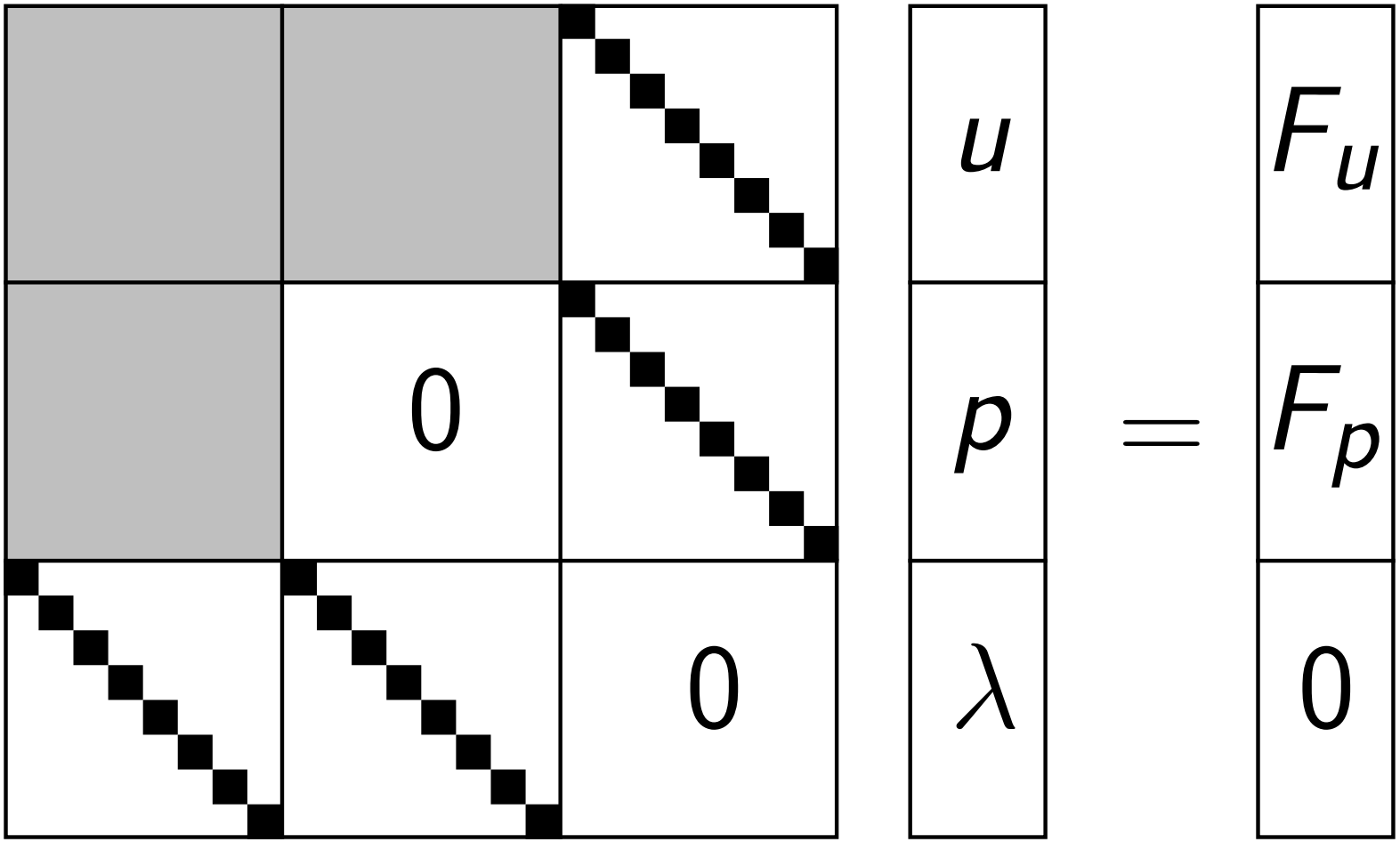
This leads to the formulation

$$a(u,p;v,q) + b(u,p;v,q) = \ell_1(q) + \ell_2(v).$$

Why the symmetry is important ?

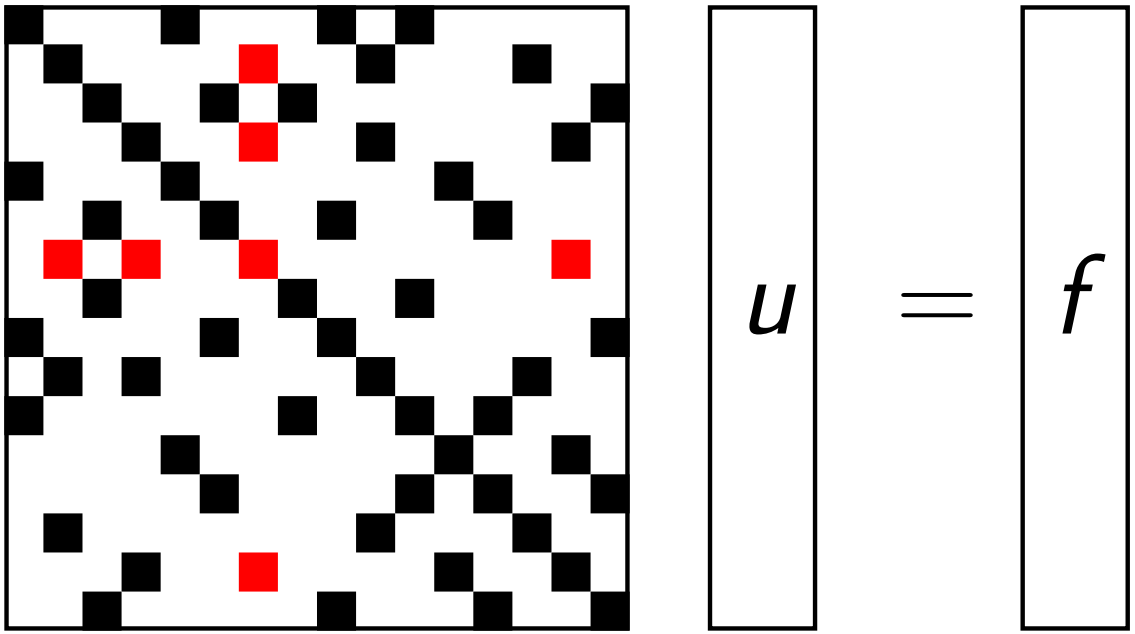
- for the linear algebra solver: it needs less memory
- for boundary element method: it has been observed that these methods are more stable.

The Lagrangian formulation



The reduced formulation

Eliminating p and λ



- Symmetric block sparse matrix
- No need to assembly the augmented matrix

The numerical dispersion or the numerical pollution

One dimensional wave equation

$$-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}(x,t)+\frac{\partial^2 u}{\partial x^2}(x,t) \;=\; f(x,t) \tag{9}$$

Time harmonic domain

$$\begin{aligned} u(x,t) &= \Re\Big(u(x)\exp(-i\omega t)\Big) \\ f(x,t) &= \Re\Big(f(x)\exp(-i\omega t)\Big) \end{aligned} \tag{10}$$

The functions u satisfies the Helmholtz equation

$$\frac{d^2 u}{dx^2}(x) \;+\; k^2\, u(x) \;=\; f(x) \tag{11}$$

with k the wave number

$$k^2 \;=\; \frac{\omega^2}{c^2} \tag{12}$$

Limiting absorption principle

Let f be compactly supported

$$f(x) = 0 \text{ for } x \notin [x_-, x_+] \tag{13}$$

This problem is not well-posed

- the associated bilinear form is not coercive

$$a(u, v) = \int_{\mathbb{R}} \frac{du}{dx}(x)\overline{\frac{dv}{dx}}'(x) - k^2 u(x)\overline{v}(x)dx \tag{14}$$

- no solution in $H^1(\mathbb{R})$. Solutions are $H^1_{\text{loc}}(\mathbb{R})$.
- not unique in $H^1_{\text{loc}}(\mathbb{R})$.

Limiting absorption principle

Introduce a perturbation of the wave equation, with $\varepsilon > 0$ a small factor

$$\frac{d^2 u_\varepsilon}{dx^2}(x) + (k + i\varepsilon)^2 u_\varepsilon(x) = f(x) \quad \forall x \in \mathbb{R}. \tag{15}$$

which can be interpreted as a damping term in time domain

$$\frac{d^2 u_\varepsilon}{dx^2}(x) - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + \varepsilon c\right)^2 u_\varepsilon(x) = f(x) \quad \forall x \in \mathbb{R}. \tag{16}$$

The function u is then defined as the limit

$$u(x) = \lim_{\varepsilon \to 0^+} u_\varepsilon(x) \tag{17}$$

The outgoing Green function is then given by

$$G(x) = \frac{\exp(ik|x|)}{2ik} \tag{18}$$

The solution is given by:

$$u(x) = G * f(x) = \int_{\mathbb{R}} G(x-y)f(y)dy \tag{19}$$

$$u(x) = u_+(x) + u_-(x) \tag{20}$$

Propagative term with direction x growing

$$u_+(x) = \left(\int_{-\infty}^{x} \frac{f(y)\exp(-iky)}{2ik} \right) dy \ \exp(ikx) \tag{21}$$

Propagative term with direction x decreasing

$$u_-(x) = \left(\int_{x}^{+\infty} \frac{f(y)\exp(iky)}{2ik} \right) dy \ \exp(-ikx) \tag{22}$$

Plane wave

For f compactly supported

$$f(x) = 0 \text{ for } x \notin [x_-, x_+] \tag{23}$$

The exact solution of the Helmholtz equation is given by

$$u(\mathbf{x}) = \int_{x_-}^{x_+} f(y) \exp ik(x-y)dx = U_+ \exp(ikx) \text{ if } x > x_+,$$

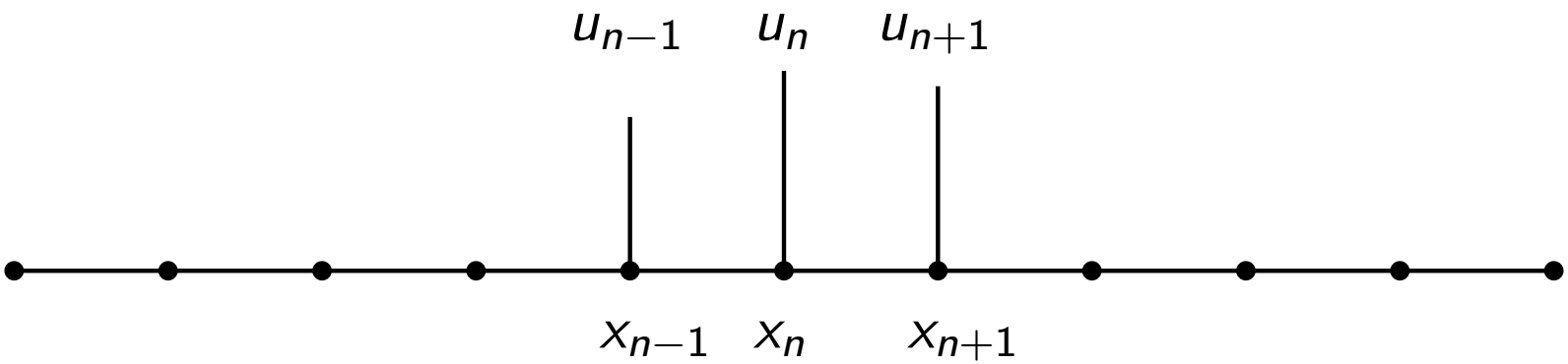
$$u(\mathbf{x}) = \int_{x_-}^{x_+} f(y) \exp -ik(x-y)dx = U_- \exp(-ikx) \text{ if } x < x_-,$$

To understand the direction of propagation, these expressions should be multiplyed by $\exp(-i\omega t)$

$$\begin{array}{ll} \text{direction x growing} & \exp(ikx - i\omega t) \\ \text{direction x decreasing} & \exp(-ikx - i\omega t) \end{array} \tag{24}$$

At infinity, the solution is a plane with speed $c = \pm\omega/k$

Second order finite differences on infinite grid



The sequence u_n aims in approximating the fonction u at

$$x_n = n\ h \tag{25}$$

For the Helmholtz equation, it reads

$$\frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} \ + \ k^2\ u_n = f_n \tag{26}$$

In $\ell^2(\mathbb{Z})$, this equation is not well-posed

$$\ell^2(\mathbb{Z}) \ = \ \left\{ (u_n \in \mathbb{C})_{n\in\mathbb{Z}} \ : \ \sum_{n=-\infty}^{+\infty} |u_n|^2 < +\infty \right\} \tag{27}$$

A discrete absorption principle

Adding a damping term which will tend to 0 with factor $\varepsilon > 0$, we have

$$\frac{u_{\varepsilon,n+1} - 2u_{\varepsilon,n} + u_{\varepsilon,n-1}}{h^2} - (k + i\varepsilon)^2 u_{\varepsilon,n} = f_n \quad \forall n \in \mathbb{Z}. \tag{28}$$

This problem becomes well posed in $\ell^2(\mathbb{Z})$ since one can apply the Lax-Milgram theorem.

The outgoing solution is then defined by:

$$u_n = \lim_{\varepsilon \to 0^+} u_{\varepsilon,n} \tag{29}$$

The discrete Green function

The source term is of the form

$$f_0 = 1 \quad \text{and} \quad f_n = 0 \text{ for } n \neq 0 \tag{30}$$

For $n > 0$, we have the birecurrent sequence

$$u_{n+1} - (2 - k^2 h^2) u_n + u_{n-1} = 0. \tag{31}$$

Characteristic equation

$$r^2 - (2 - k^2 h^2) r + 1 = 0 \tag{32}$$

For h small, $\Delta = (2 - k^2 h^2)^2 - 4 < 0$. The two roots associated to this equation are given by

$$r_h^+ = \exp(i\alpha_h) \text{ and } r_h^- = \exp(-i\alpha_h) \tag{33}$$

The discrete Green function

We have two solutions

$$\exp(i\alpha_h n) \text{ and } \exp(-i\alpha_h n). \tag{34}$$

To better understand this expression we introduce $k_h = \alpha_h / h$

$$\exp(ik_h x_n) \text{ and } \exp(-ik_h x_n). \tag{35}$$

and we multiply both terms by $\exp(-i\omega t)$

$$\exp(ik_h x_n - i\omega t) \text{ and } \exp(-ik_h x_n - i\omega t) \tag{36}$$

- The first wave is a numerical wave propagating in the direction x-growing with speed $c_h = k_h/\omega$
- The second wave is a numerical wave propagating in the direction x-decreasing with speed $c_h = k_h/\omega$

The discrete Green function

To summarize we have for $n \geq 0$

$$u_n = \beta \exp(ik_h x_n). \tag{37}$$

In the same way, we have for $n \leq 0$

$$u_n = \gamma \exp(-ik_h x_n). \tag{38}$$

We determine β and γ thanks to the realtion

$$\begin{cases} u_{0+} = u_{0-} \\ u_{-1} - (2 - k^2 h^2) u_0 + u_1 = h^2 \end{cases} \tag{39}$$

This leads to

$$u_n = \frac{\exp(ik_h |x_n|)}{2 \exp(ik_h h) - 2 + k^2 h^2} \tag{40}$$

The discrete Green function

The discrete Green function is exactly the function which has been computed. Let us denote it by G_n

$$G_n = \frac{h^2 \exp(ik_h|x_n|)}{2 \exp(ik_h h) - 2 + k^2 h^2} \tag{41}$$

Let us now consider the solution of the problem with a general source term (defined by the absorption principle)

$$\frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} - k^2 u_n = f_n \quad \forall n \in \mathbb{Z}. \tag{42}$$

Acting by linearity, we have

$$u_n = \sum_{p \in \mathbb{Z}} G_{n-p} f_p \quad \text{(discrete convolution operator)} \tag{43}$$

The notion of discrete Green function

Let us now consider a source term with compact support included in $[x_-, x_+]$. We have $f_n = 0$ if $x_n \notin [x_-, x_+]$.
The expression

$$u_n = \sum_{p \in \mathbb{Z}} G_{n-p} f_p \quad \text{(discrete convolution operator)} \tag{44}$$

can be simplified for n such that $x_n > x_+$

$$u_n = A_h \exp(ik_h x_n) \tag{45}$$

and such that $x < x_-$

$$u_n = B_h \exp(-ik_h x_n) \tag{46}$$

Two sources of errors:

- error in amplitude A_h and B_h do not have a correct value
- error in phase k_h does not have a correct value (this is the pollution effect)

Conclusion for the second order finite differences

$$\delta k_h = k_h - k \;=\; \underset{h\to 0}{O}\left(\frac{k^3 h^2}{24}\right) \tag{47}$$

- The numerical wave number is larger than the wave number.
- Phase shift for every wave length

$$\delta\varphi_h \;=\; \frac{k^3 h^2 \lambda}{24} \;\simeq\; \frac{2\pi}{24}\, k^2 h^2 \tag{48}$$

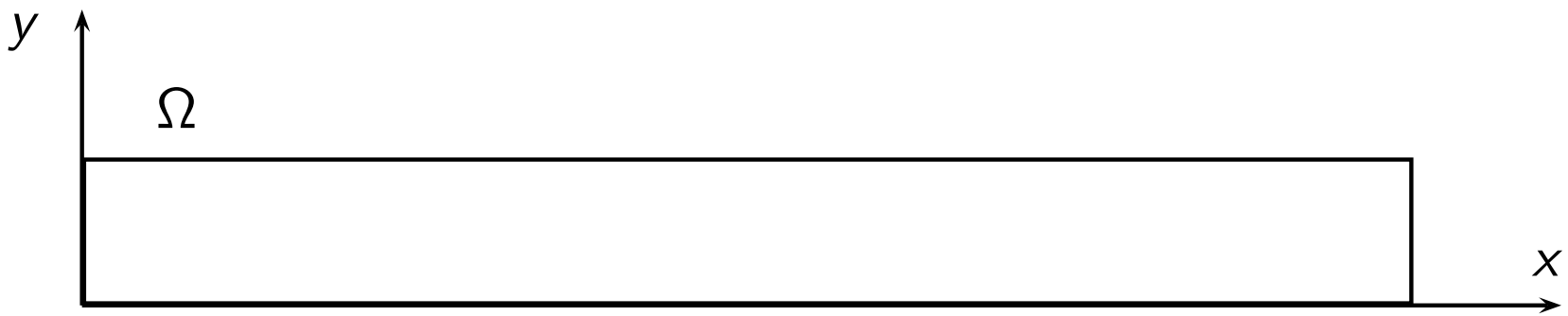
- 100% error at distance L_h defined by

$$\delta k_h\, L_h \;=\; 2\pi \quad\Longrightarrow\quad L_h \;\simeq\; 24\,\frac{2\pi}{k^3 h^2} \tag{49}$$

- 100% error at distance N_h wavelength

$$N_h \;=\; \frac{kL_h}{2\pi} \;\simeq\; \frac{24}{k^2 h^2}. \tag{50}$$

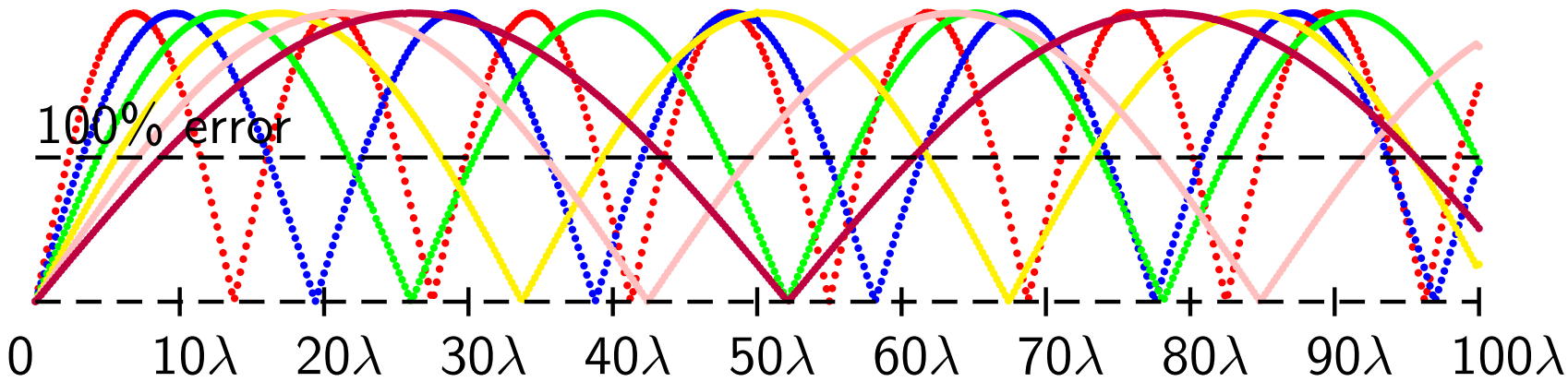
A numerical simulation



$$
\begin{cases}
\Delta u(\mathbf{x}) + k^2\, u(\mathbf{x}) = 0 & \text{in } \Omega \\
u(\mathbf{x}) = 1 & \text{at } x = 0, \\
\dfrac{\partial u}{\partial n}(\mathbf{x}) = 0 & \text{at } \mathbf{x} = 100 \text{ or } 200\lambda \\
\frac{\partial u}{\partial n}(\mathbf{x}) + iku(\mathbf{x}) = 0 & \text{at } y = 0 \text{ and } \lambda
\end{cases}
\tag{51}
$$

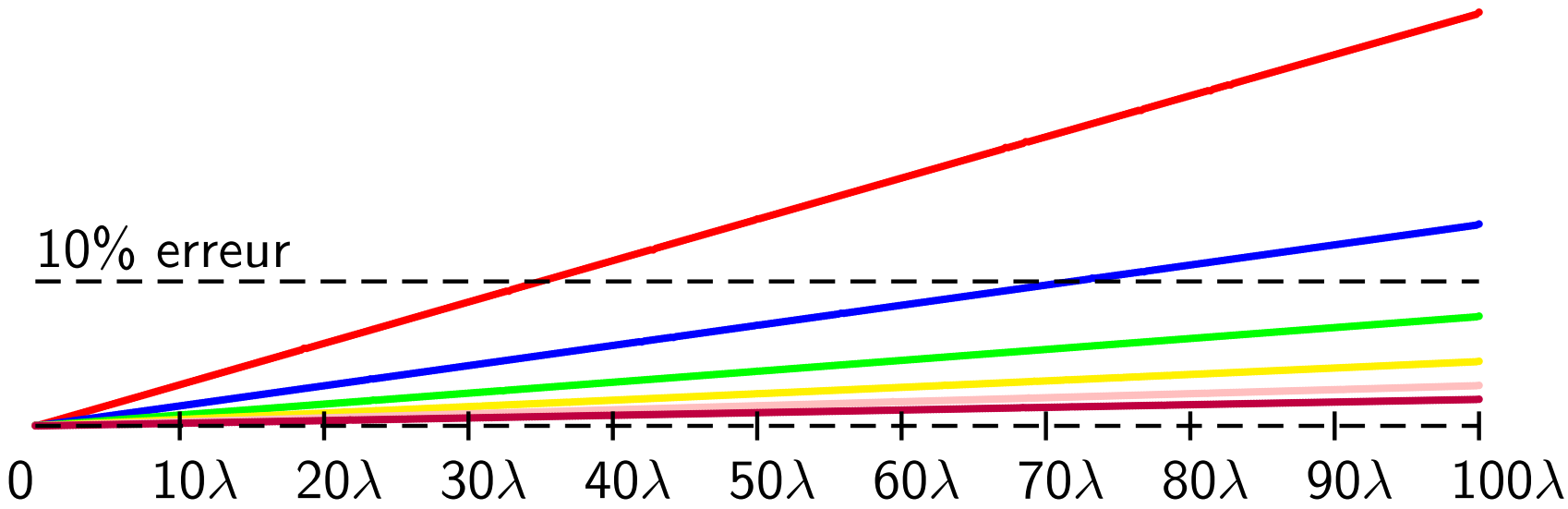
Error at every wavelength in $\mathbb{P}_1$ for 10, 12, 14, 16, 18 segments per λ

- 20 segments per λ, 2400 dof per λ^2
- 18 segments per λ, 1944 dof per λ^2
- 16 segments per λ, 1536 dof per λ^2
- 14 segments per λ, 1176 dof per λ^2
- 12 segments per λ, 864 dof per λ^2
- 10 segments per λ, 600 dof per λ^2



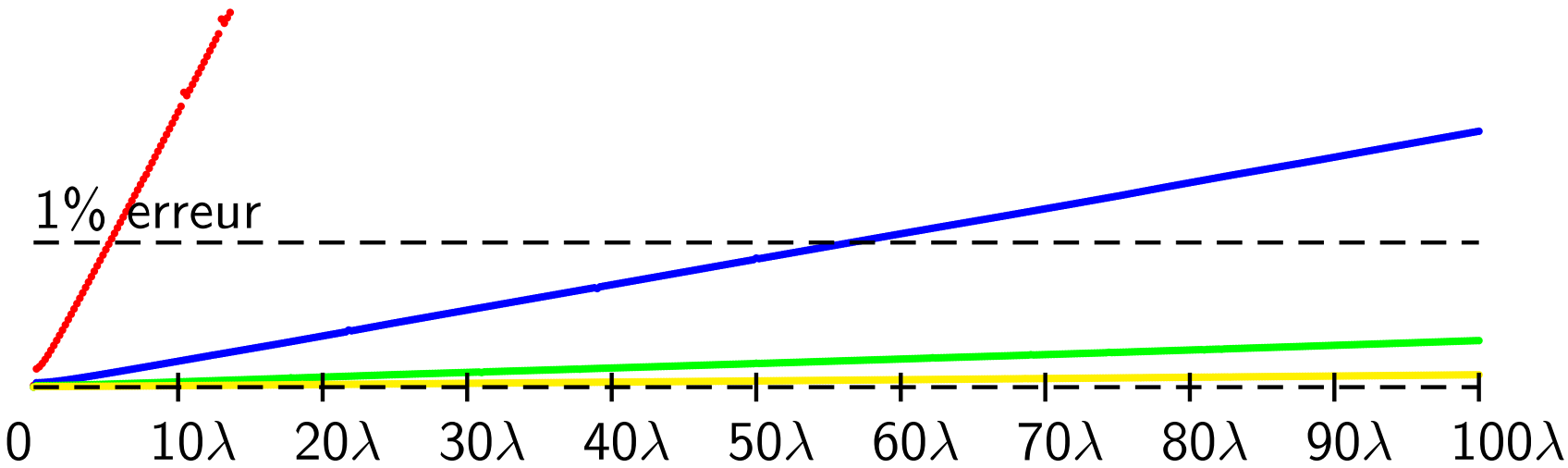
Error at every wavelength in $\mathbb{P}_2$ for 10, 12, 14, 16, 18 segments per λ

- 20 segments per λ, 4800 dof per λ^2
- 18 segments per λ, 3888 dof per λ^2
- 16 segments per λ, 3072 dof per λ^2
- 14 segments per λ, 2352 dof per λ^2
- 12 segments per λ, 1728 dof per λ^2
- 10 segments per λ, 1200 dof per λ^2



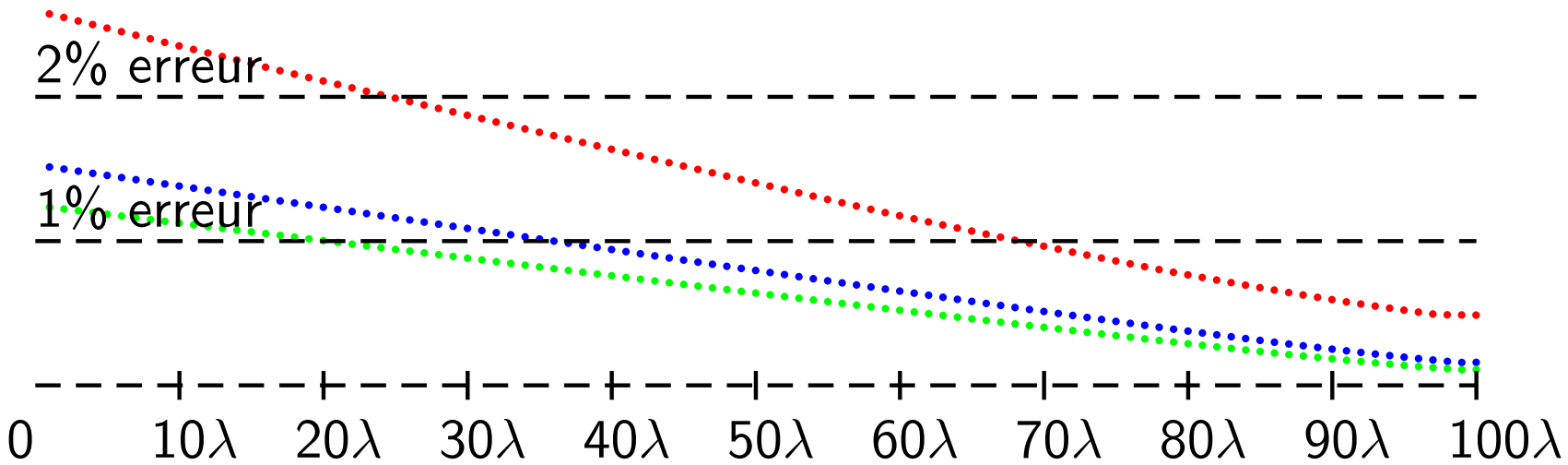
Error at every wavelength in $\mathbb{P}_3$ for 4, 6, 8, 10 segments per λ

- 10 segments per λ, 2000 dof per λ^2
- 8 segments per λ, 1280 dof per λ^2
- 6 segments per λ, 720 dof per λ^2
- 4 segments per λ, 320 dof per λ^2



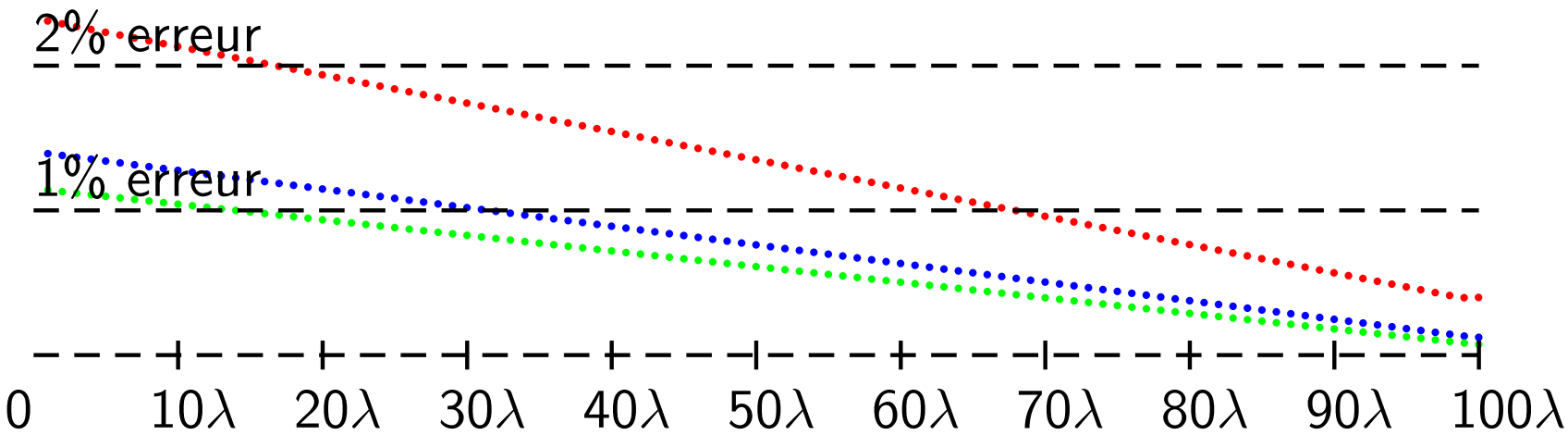
DG-BEM: modulus error for a 100 λ domain

- 90 segments per λ, 560 dof per λ^2
- 70 segments per λ, 560 dof per λ^2
- 40 segments per λ, 320 ddl per λ^2



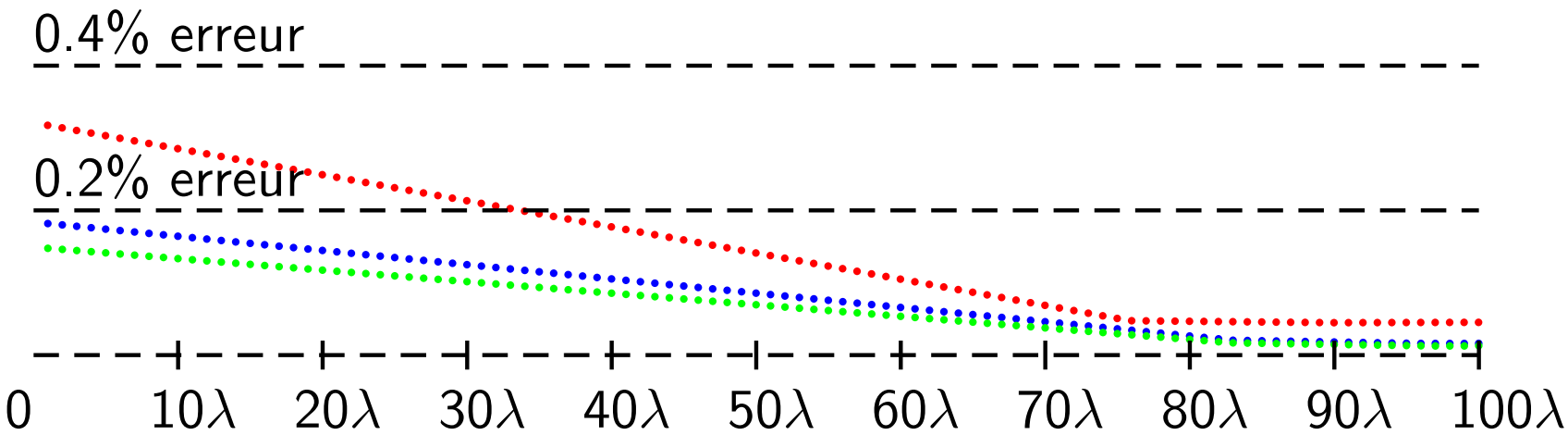
DG-BEM Phase error for 100 λ domain

- 90 segments per λ, 560 dof per λ^2
- 70 segments per λ, 560 dof per λ^2
- 40 segments per λ, 320 dof per λ^2



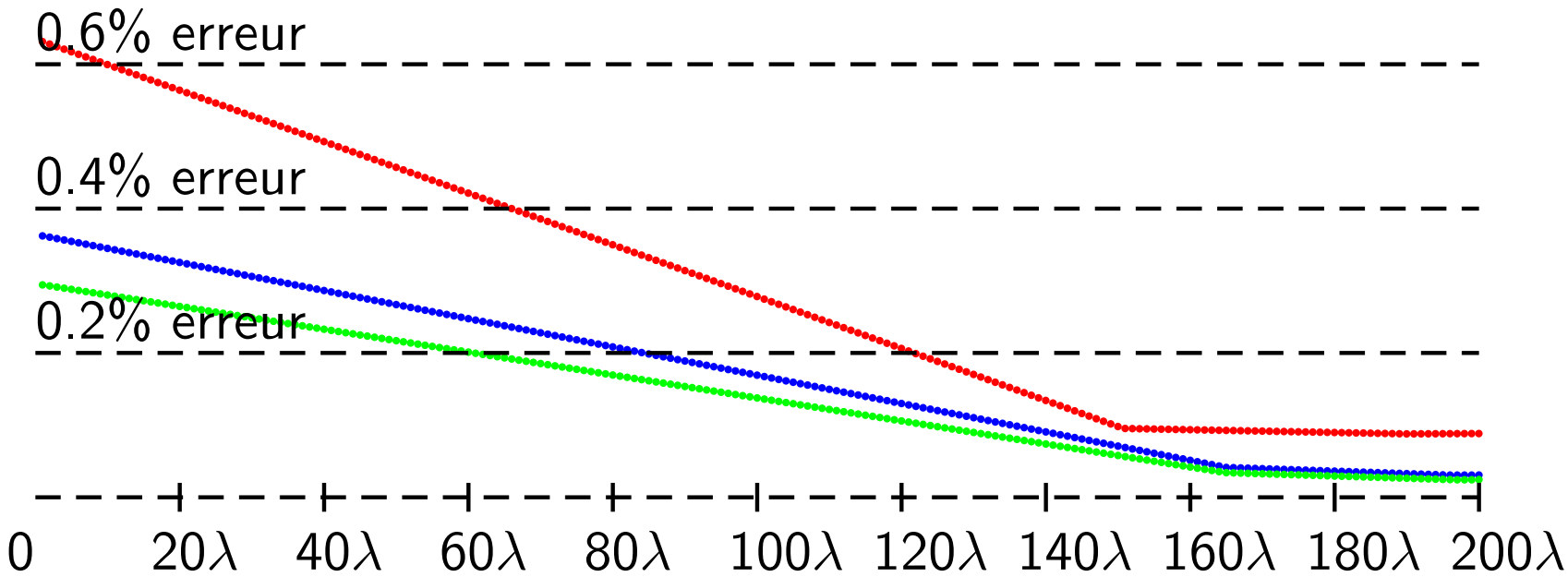
DG-BEM: Phase error for a 100 λ domain

- 90 segments per λ, 560 dof per λ^2
- 70 segments per λ, 560 dof per λ^2
- 40 segments per λ, 320 dof per λ^2



DG-BEM: Phase Error for a 200 λ domain

- 90 segments per λ, 560 dof per λ^2
- 70 segments per λ, 560 dof per λ^2
- 40 segments per λ, 320 dof per λ^2



Poly degree	nodes per λ	Method	Error at 175λ for $\mathbb{P}_2$ Error at 500λ for $\mathbb{P}_3$
$m = 2$	12	IPDG	72 %
		BEM-STDG	22 %
	16	IPDG	67 %
		BEM-STDG	5.6 %
	24	IPDG	13 %
		BEM-STDG	0.8 %
$m = 3$	12	IPDG	19 %
		BEM-STDG	1.6 %
	18	IPDG	1.7 %
		BEM-STDG	0.1 %
	24	IPDG	0.3 %
		BEM-STDG	0.02 %

Conclusion

- The IPDG-BEM is low dispersive
- The inversion of the system analytic accelerator (FMM) and algebra accelerator (ACA). This has been tested on a triangle
- The method has also been implemented in 3D

All details in the prepublication:

A Symmetric Trefftz-DG Formulation based on a Local Boundary Element Method for the Solution of the Helmholtz Equation, H Barucq and al.

https://hal.inria.fr/hal-01218784