Material derivatives of boundary integral operators in electromagnetism and application to inverse scattering problems

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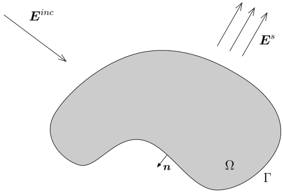
Wave days in South-West, March 9-11, 2016, Pau

Motivation

→ To use shape optimization tools for solving shape reconstruction problems

Direct problem :



$$\begin{cases} \textbf{curl}\,\textbf{curl}\,\boldsymbol{E}^s - \kappa^2 \boldsymbol{E}^s & = 0 \text{ in } \mathbb{R}^3 \backslash \overline{\Omega} \\ \boldsymbol{n} \times (\boldsymbol{E}^s + \boldsymbol{E}^{inc}) & = 0 \text{ on } \Gamma \\ \lim\limits_{|x| \to \infty} (\textbf{curl}\,\boldsymbol{E}^s \times x - i\kappa |x| \boldsymbol{E}^s) & = 0 \\ \boldsymbol{E}^s(x) = \frac{e^{i\kappa|x|}}{|x|} \boldsymbol{E}^\infty(\frac{x}{|x|}) + O\left(\frac{1}{|x|}\right) & \text{when } |x| \to \infty \end{cases}$$

Given the boundary Γ, we compute $\boldsymbol{F}(\Gamma) = \boldsymbol{E}^\infty$.

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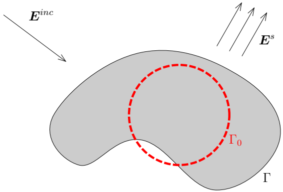
Inverse problem :



Given noisy farfield data $\boldsymbol{E}_\delta^\infty$ (and a good initial guess Γ_0), solve

$$\boldsymbol{F}(\Gamma) = \boldsymbol{E}_\delta^\infty,$$

using regularized iterative algorithm.

→ To avoid solving the direct scattering problem numerous times at each iteration step.

→ *Joint work with O. Ivanyshyn Yaman (Assistant Professor, IZTECH, Turkey).*

Outline

D. COLTON & R. KRESS, *Inverse acoustic and electromagnetic scattering theory*

- The time-harmonic Maxwell equation : $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E}^s - \kappa^2 \boldsymbol{E}^s = 0$
- $\kappa > 0$ and $G(\kappa, x-y) = \frac{e^{i\kappa|x-y|}}{4\pi|x-y|}$. The electromagnetic potential operators are :

$$\mathscr{S}_\kappa\,\varphi(x) = \frac{1}{\kappa}\int_\Gamma \mathbf{curl}\,\mathbf{curl}^x \left\{ G(\kappa, x-y)\varphi(y)) \right\} d\sigma(y)$$

$$\mathscr{D}_\kappa\,\psi(x) = \int_\Gamma \mathbf{curl}^x \left\{ G(\kappa, x-y)\psi(y) \right\} d\sigma(y)$$

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- Integral representation of the solution :

$$\boldsymbol{E}^s = \mathscr{D}_\kappa \psi + i\eta \mathscr{S}_\kappa \boldsymbol{\Lambda}\psi$$

where $\eta \in \mathbb{R}$, $\eta \neq 0$ and $\int_\Gamma \boldsymbol{\Lambda}\psi \cdot (\boldsymbol{n} \times \overline{\psi})\, d\sigma \geqslant c||\psi||^2$ (to ensure uniqueness)

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- Solving the Direct Problem $\Leftrightarrow$ solving a Boundary Integral Equation

$$\boldsymbol{n} \times (\boldsymbol{E}^s + \boldsymbol{E}^{inc}) = 0 \quad \Rightarrow \quad \frac{1}{2}\psi(x) + D_\kappa\psi(x) + i\eta S_\kappa \boldsymbol{\Lambda}\psi(x) = -\boldsymbol{n}(x) \times \boldsymbol{E}^{inc}(x)\,,\ \ x \in \Gamma$$

$$S_\kappa\varphi(x) = \boldsymbol{n}(x) \times \lim_{s\to 0}(\mathscr{S}_\kappa\varphi)(x + s\boldsymbol{n}(x)) \ \text{ and } \ \frac{1}{2}\psi(x) + D_\kappa\psi(x) = \boldsymbol{n}(x) \times \lim_{s\to 0}(\mathscr{D}_\kappa\psi)(x + s\boldsymbol{n}(x))$$

Energy spaces

J.-C. NÉDÉLEC, *Acoustic and electromagnetic equations ...*

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- The trace $\boldsymbol{n} \times \boldsymbol{E}^s \in \boldsymbol{H}^{-\frac{1}{2}}_{\mathrm{div}}(\Gamma) := \left\{\psi \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma), \boldsymbol{n} \cdot \psi = 0 \text{ and } \mathrm{div}_\Gamma\, \psi \in H^{-\frac{1}{2}}(\Gamma)\right\}$

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- The $\boldsymbol{L}^2_t$ dual space : $\boldsymbol{H}^{-\frac{1}{2}}_{\text{curl}}(\Gamma) := \left\{\psi \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma), \boldsymbol{n} \cdot \psi = 0 \text{ and } \operatorname{curl}_\Gamma \psi \in H^{-\frac{1}{2}}(\Gamma)\right\}$

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- The $\boldsymbol{L}^2_t$ dual space : $\boldsymbol{H}^{-\frac{1}{2}}_{\text{curl}}(\Gamma) := \left\{\psi \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma), \boldsymbol{n} \cdot \psi = 0 \text{ and } \operatorname{curl}_\Gamma \psi \in H^{-\frac{1}{2}}(\Gamma)\right\}$
- The single layer boundary integral operator :

$$S_\kappa \varphi(x) = \kappa\, \boldsymbol{n}(x) \times \int_\Gamma G(\kappa, x-y)\varphi(y)d\sigma(y) - \frac{1}{\kappa}\overrightarrow{\textbf{curl}}_\Gamma \int_\Gamma G(\kappa, x-y)\operatorname{div}_\Gamma \varphi(y)d\sigma(y)$$

has a hypersingular kernel but is bounded on $\boldsymbol{H}^{-\frac{1}{2}}_{\text{div}}(\Gamma)$

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- $\boldsymbol{L}^2_{\rm t}(\Gamma) := \left\{\boldsymbol{\psi} \in \boldsymbol{L}^2(\Gamma), \boldsymbol{n}\cdot\boldsymbol{\psi} = 0\right\}$.
- The trace $\boldsymbol{n}\times\boldsymbol{E}^s \in \boldsymbol{H}^{-\frac{1}{2}}_{\rm div}(\Gamma) := \left\{\boldsymbol{\psi} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma), \boldsymbol{n}\cdot\boldsymbol{\psi} = 0 \text{ and } \mathrm{div}_\Gamma\,\boldsymbol{\psi} \in H^{-\frac{1}{2}}(\Gamma)\right\}$
- The $\boldsymbol{L}^2_{\rm t}$ dual space : $\boldsymbol{H}^{-\frac{1}{2}}_{\rm curl}(\Gamma) := \left\{\boldsymbol{\psi} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma), \boldsymbol{n}\cdot\boldsymbol{\psi} = 0 \text{ and } \mathrm{curl}_\Gamma\,\boldsymbol{\psi} \in H^{-\frac{1}{2}}(\Gamma)\right\}$
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  has a hypersingular kernel but is bounded on $\boldsymbol{H}^{-\frac{1}{2}}_{\rm div}(\Gamma)$
- The double layer boundary integral operator :

$$D_\kappa\boldsymbol{\psi}(x) = \int_\Gamma \boldsymbol{n}(x)\times\mathbf{curl}^x\{G(\kappa, x-y)\boldsymbol{\psi}(y)\}d\sigma(y)$$

  has a strongly singular kernel but is compact on $\boldsymbol{H}^{-\frac{1}{2}}_{\rm div}(\Gamma)\cap\boldsymbol{L}^2_{\rm t}(\Gamma)$

- Different choices of $\boldsymbol{\Lambda}$

J. R. MAUTZ & R. F. HARRINGTON, 1979 $\boldsymbol{\Lambda}\psi = \boldsymbol{n} \times \psi$

D. COLTON & R. KRESS, *Inverse acoustic ...* $\boldsymbol{\Lambda}\psi = \boldsymbol{n} \times V_0^2 \psi$ $**$

O. STEINBACH & M. WINDISCH, 2009 $\boldsymbol{\Lambda}\psi = \boldsymbol{n} \times V_0 \psi + \overrightarrow{\mathbf{curl}}_\Gamma V_0 \,\mathrm{div}_\Gamma\, \psi$ $**$

D. LEVADOUX & F. MILLOT & S. PERNET, 2010 $\boldsymbol{\Lambda} = (\mathrm{I} + \frac{\Delta_\Gamma}{\kappa^2})^{-\frac{1}{2}} \left(\mathrm{I} - \frac{1}{\kappa^2} \overrightarrow{\mathbf{curl}}_\Gamma \,\mathrm{curl}_\Gamma \right)$

$**$ *V_0 is the single layer boundary integral operator related to the Laplace equation*

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D. Colton & R. Kress, *Inverse acoustic ...* $\boldsymbol{\Lambda}\psi = \boldsymbol{n} \times V_0^2 \psi$ **

O. Steinbach & M. Windisch, 2009 $\boldsymbol{\Lambda}\psi = \boldsymbol{n} \times V_0 \psi + \overrightarrow{\textbf{curl}}_\Gamma V_0 \,\mathrm{div}_\Gamma\, \psi$ **

D. Levadoux & F. Millot & S. Pernet, 2010 $\boldsymbol{\Lambda} = (\mathrm{I} + \frac{\Delta_\Gamma}{\kappa^2})^{-\frac{1}{2}} \left(\mathrm{I} - \frac{1}{\kappa^2} \overrightarrow{\textbf{curl}}_\Gamma \,\mathrm{curl}_\Gamma \right)$

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- Factorization of the boundary to far-field operator

$$\mathscr{S}_\kappa \varphi(x) \underset{|x|\to\infty}{\rightsquigarrow} \mathscr{S}_\kappa^\infty \varphi(\hat{x}) \quad \text{and} \quad \mathscr{D}_\kappa \psi(x) \underset{|x|\to\infty}{\rightsquigarrow} \mathscr{D}_\kappa^\infty \psi(\hat{x}) \qquad \hat{x} = \frac{x}{|x|}$$

$$\boldsymbol{F}(\Gamma) = \boldsymbol{E}^\infty = \left[\mathscr{D}_\kappa^\infty + i\eta \mathscr{S}_\kappa^\infty \boldsymbol{\Lambda}\right] \left[\frac{1}{2}\mathrm{I} + D_\kappa + i\eta S_\kappa \boldsymbol{\Lambda}\right]^{-1} (-\boldsymbol{n} \times \boldsymbol{E}^{inc})_{||\Gamma}$$

$\boldsymbol{E}^\infty$ is an analytical function defined on the unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3$.

- Choose a reference domain Ω_{ref} with a smooth orientable boundary Γ_{ref}

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- Define an open set of admissible parametrizations :

$$\mathcal{Q}_{ad} \subset \{\boldsymbol{q} \in \mathscr{C}^{1,\alpha}(\Gamma_{\mathrm{ref}}, \mathbb{R}^3) \mid \Gamma_{\boldsymbol{q}} := \boldsymbol{q}(\Gamma_{\mathrm{ref}}) \text{ is diffeomorphic to } \Gamma_{\mathrm{ref}}\}$$

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- We restrict the (IP) to the family of parametrised boundaries $\{\Gamma_{\boldsymbol{q}} \,|\, \boldsymbol{q} \in \mathcal{Q}_{ad}\}$:

$$\boxed{\text{Find } \boldsymbol{q} \text{ such that } \boldsymbol{F}(\Gamma_{\boldsymbol{q}}) = \boldsymbol{E}_{\delta}^{\infty} \text{ !}}$$

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- We analyze the Fréchet differentiability of $\boldsymbol{q} \in \mathcal{Q}_{ad} \mapsto \boldsymbol{F}(\Gamma_{\boldsymbol{q}})$:

2 ways for computing Fréchet derivatives

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  1. Characterization of the first Fréchet derivative in the direction $\boldsymbol{\xi} \in \mathscr{C}^{1,\alpha}(\Gamma_{\mathrm{ref}}, \mathbb{R}^3)$:

     R. KRESS, *Electromagnetic waves scattering : Scattering by obstacles*, (2001)

$$\left\{\begin{array}{rcl} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E}^{s}_{\boldsymbol{q},\boldsymbol{\xi}} - \kappa^2 \boldsymbol{E}^{s}_{\boldsymbol{q},\boldsymbol{\xi}} & = & 0 \text{ in } \mathbb{R}^3 \backslash \overline{\Omega}_{\boldsymbol{q}} \\ \boldsymbol{n}_{\boldsymbol{q}} \times \boldsymbol{E}^{s}_{\boldsymbol{q},\boldsymbol{\xi}} & = & \boldsymbol{f}_{\boldsymbol{q},\boldsymbol{\xi}} \text{ on } \Gamma_{\boldsymbol{q}} \\ \lim\limits_{|x|\to\infty} (\mathbf{curl}\,\boldsymbol{E}^{s}_{\boldsymbol{q},\boldsymbol{\xi}} \times x - i\kappa|x|\boldsymbol{E}^{s}_{\boldsymbol{q},\boldsymbol{\xi}}) & = & 0 \\ \partial_{\boldsymbol{q}} \boldsymbol{F}[\Gamma_{\boldsymbol{q}}]\boldsymbol{\xi} = \boldsymbol{E}^{\infty}_{\boldsymbol{q},\boldsymbol{\xi}} & & \end{array}\right.$$

$$\boldsymbol{f}_{\boldsymbol{q},\boldsymbol{\xi}} = -\left(\boldsymbol{\xi} \circ \boldsymbol{q}^{-1} \cdot \boldsymbol{n}_{\boldsymbol{q}}\right)(\boldsymbol{n}_{\boldsymbol{q}} \times \mathbf{curl}(\boldsymbol{E}^{s}_{\boldsymbol{q}} + \boldsymbol{E}^{inc})) \times \boldsymbol{n}_{\boldsymbol{q}} - \frac{1}{\kappa^2} \mathbf{curl}_{\Gamma_{\boldsymbol{q}}} \left((\boldsymbol{\xi} \circ \boldsymbol{q}^{-1} \cdot \boldsymbol{n}_{\boldsymbol{q}})\, \mathrm{div}_{\Gamma_{\boldsymbol{q}}} (\boldsymbol{n}_{\boldsymbol{q}} \times \mathbf{curl}(\boldsymbol{E}^{s}_{\boldsymbol{q}} + \boldsymbol{E}^{inc}))\right)$$

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$$\boldsymbol{f}_{\boldsymbol{q},\boldsymbol{\xi}} = -\left(\boldsymbol{\xi} \circ \boldsymbol{q}^{-1} \cdot \boldsymbol{n}_{\boldsymbol{q}}\right) (\boldsymbol{n}_{\boldsymbol{q}} \times \mathbf{curl}(\boldsymbol{E}^s_{\boldsymbol{q}} + \boldsymbol{E}^{inc})) \times \boldsymbol{n}_{\boldsymbol{q}} - \frac{1}{\kappa^2} \mathbf{curl}_{\Gamma_{\boldsymbol{q}}} \left((\boldsymbol{\xi} \circ \boldsymbol{q}^{-1} \cdot \boldsymbol{n}_{\boldsymbol{q}}) \operatorname{div}_{\Gamma_{\boldsymbol{q}}} (\boldsymbol{n}_{\boldsymbol{q}} \times \mathbf{curl}(\boldsymbol{E}^s_{\boldsymbol{q}} + \boldsymbol{E}^{inc}))\right)$$

  2. We can differentiate the integral representation of $\boldsymbol{F}(\Gamma_{\boldsymbol{q}})$

     R. Potthast, *Domain derivatives in electromagnetic scattering*, (1996)

A. KIRSCH, *The domain derivative and two applications in inverse scattering theory*, (1993)

T. HOHAGE, *Iterative Methods in Inverse Obstacle Scattering ... ,* PhD thesis, (1999)

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→ First order linearization yields : $\boldsymbol{F}(\Gamma_{\boldsymbol{q}}^\delta) + \partial_{\boldsymbol{q}}\boldsymbol{F}[\Gamma_{\boldsymbol{q}}^\delta]\boldsymbol{\xi} \approx \boldsymbol{E}_\delta^\infty$,

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1. Choose an initial guess $\Gamma_0 := \boldsymbol{q}_0(\Gamma_{\text{ref}})$

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→ First order linearization yields : $\boldsymbol{F}(\Gamma_{\boldsymbol{q}}^\delta) + \partial_{\boldsymbol{q}} \boldsymbol{F}[\Gamma_{\boldsymbol{q}}^\delta]\boldsymbol{\xi} \approx \boldsymbol{E}_\delta^\infty$,

1. Choose an initial guess $\Gamma_0 := \boldsymbol{q}_0(\Gamma_{\mathrm{ref}})$
2. For the current reconstruction $\Gamma_N^\delta := \boldsymbol{q}_N^\delta(\Gamma_{\mathrm{ref}})$, evaluate $\boldsymbol{F}(\Gamma_N^\delta)$
   If the stopping rule is not satisfied at the Nth regularized Newton step :

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   If the stopping rule is not satisfied at the Nth regularized Newton step :
3. Compute the next iterate $\boldsymbol{q}_{N+1}^\delta$ by minimizing the least square (IRGNM)

$$\boldsymbol{q}_{N+1}^\delta := \operatorname{argmin}_{\boldsymbol{q}} \left[\|\boldsymbol{F}(\Gamma_N^\delta) + \partial_{\boldsymbol{q}}\boldsymbol{F}[\Gamma_N^\delta](\boldsymbol{q} - \boldsymbol{q}_N^\delta) - \boldsymbol{E}_\delta^\infty\|^2 + \alpha_N \|\boldsymbol{q} - \boldsymbol{q}_0\|^2 \right].$$

   using conjugate gradient method

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   using conjugate gradient method

→ Main disadvantage : numerous solution of the direct problem to compute $\partial_{\boldsymbol{q}}\boldsymbol{F}[\Gamma_N^\delta]\boldsymbol{\xi}$

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0 \gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

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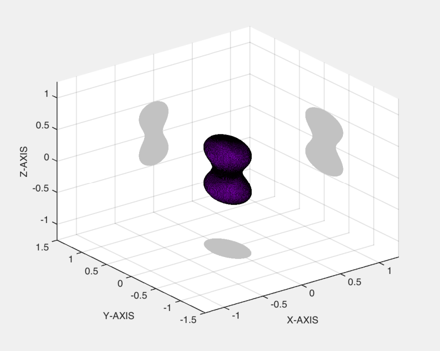

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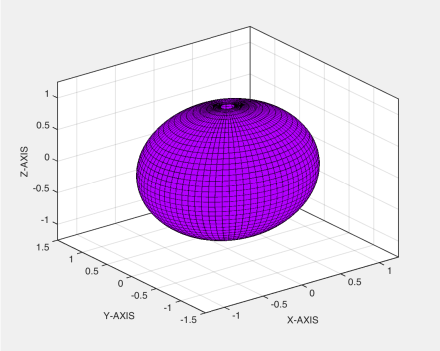

INITIAL GUESS

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
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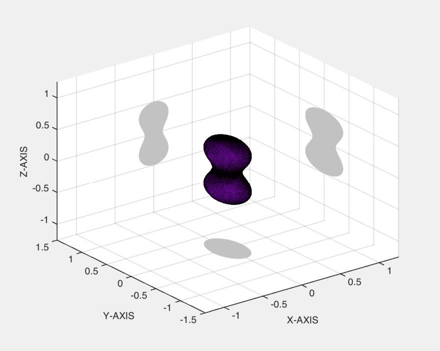

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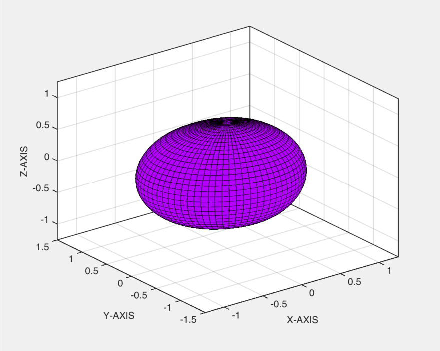

ITÉRATION $N = 1$

Algorithm 1 : numerical experiments

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1,0,0)$, $\boldsymbol{p}_1 = (0,0,1)$ and $\boldsymbol{d}_2 = (0,1,0)$, $\boldsymbol{p}_1 = (1,0,0)$
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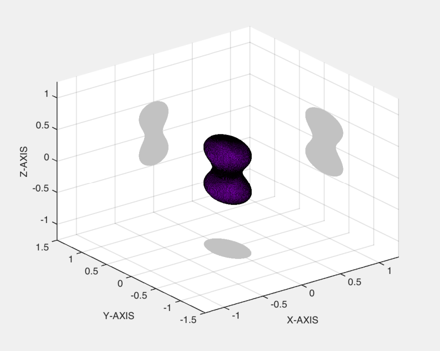

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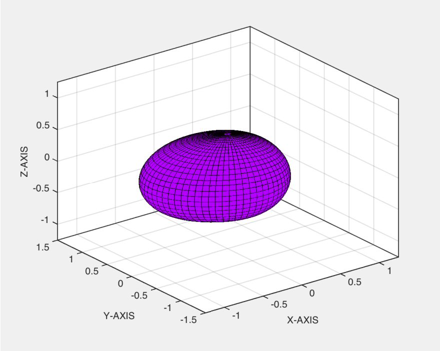

ITÉRATION $N = 2$

Algorithm 1 : numerical experiments

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0 \gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

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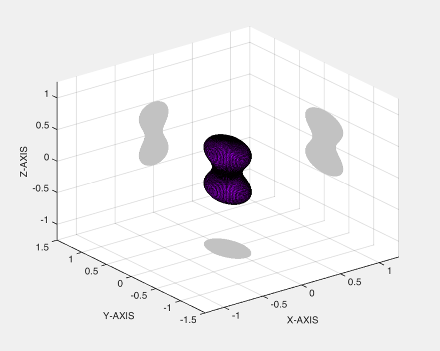

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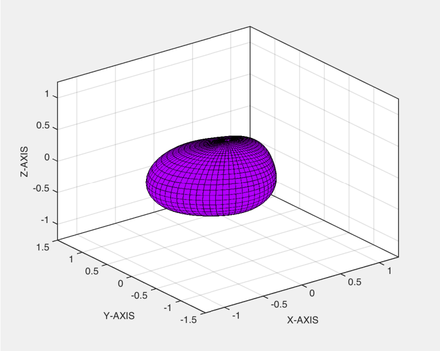
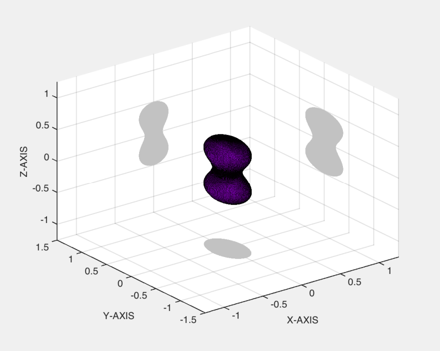


ITÉRATION $N = 3$

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2\right)^{\frac{1}{2}} \leqslant \tau\delta$$



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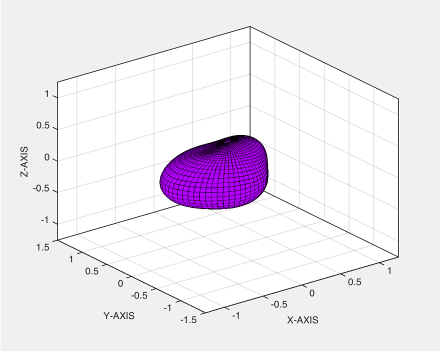
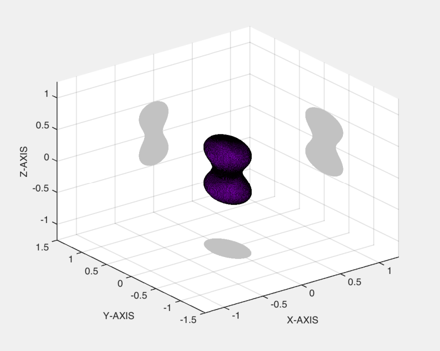


ITÉRATION $N = 4$

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1,0,0)$, $\boldsymbol{p}_1 = (0,0,1)$ and $\boldsymbol{d}_2 = (0,1,0)$, $\boldsymbol{p}_1 = (1,0,0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2\right)^{\frac{1}{2}} \leqslant \tau\delta$$



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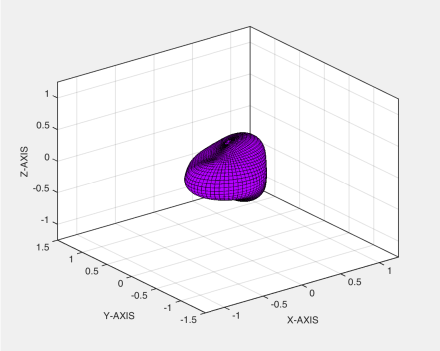

ITÉRATION $N = 5$

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1,0,0)$, $\boldsymbol{p}_1 = (0,0,1)$ and $\boldsymbol{d}_2 = (0,1,0)$, $\boldsymbol{p}_1 = (1,0,0)$
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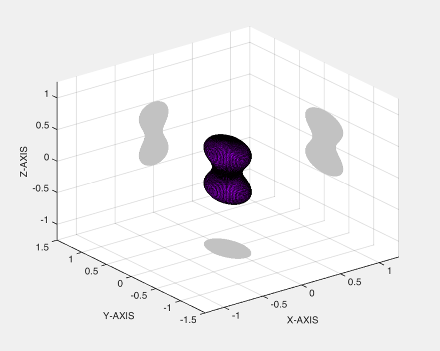

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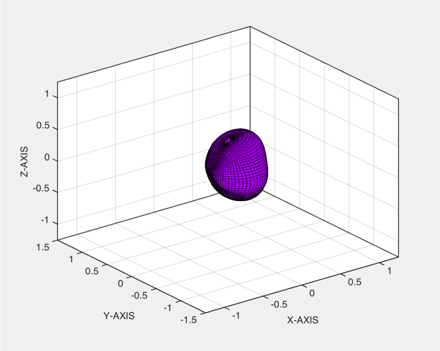

ITÉRATION $N = 6$

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
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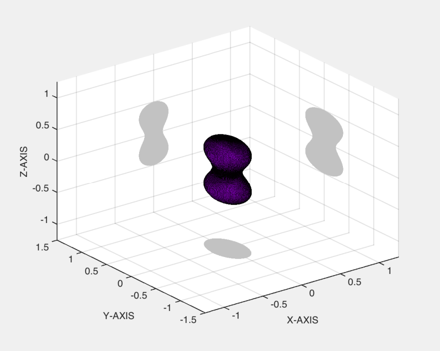
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2 \right)^{\frac{1}{2}} \leqslant \tau\delta$$



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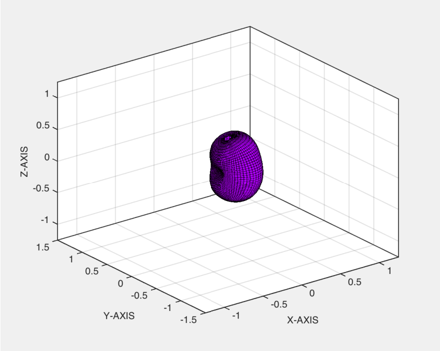
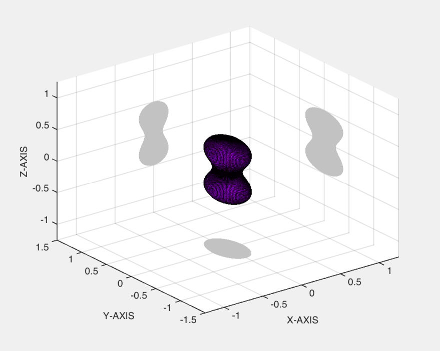


ITÉRATION $N = 7$

Algorithm 1 : numerical experiments

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1,0,0)$, $\boldsymbol{p}_1 = (0,0,1)$ and $\boldsymbol{d}_2 = (0,1,0)$, $\boldsymbol{p}_1 = (1,0,0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2\right)^{\frac{1}{2}} \leqslant \tau\delta$$



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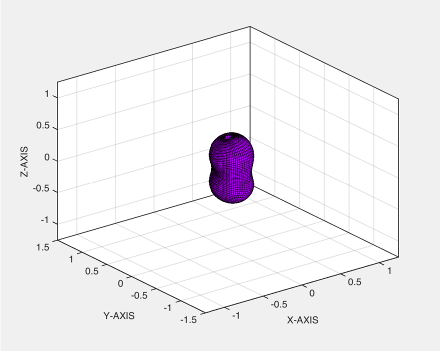
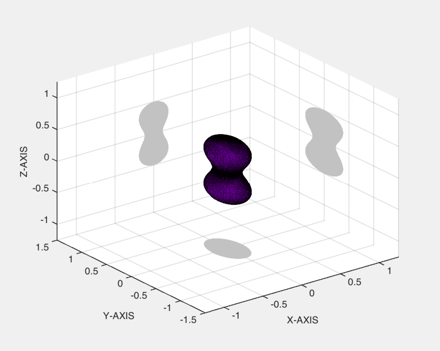


ITÉRATION $N = 8$

Algorithm 1 : numerical experiments

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2\right)^{\frac{1}{2}} \leqslant \tau\delta$$



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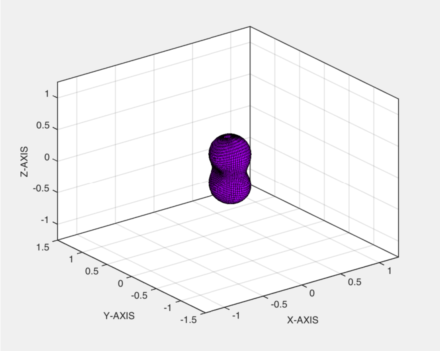
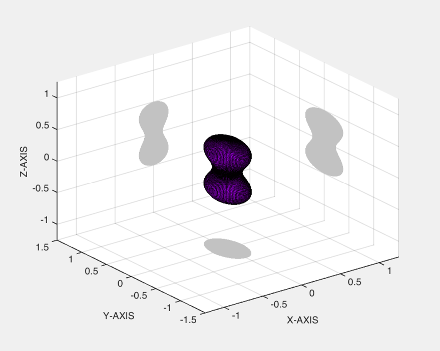


ITÉRATION $N = 9$

Algorithm 1 : numerical experiments

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2\right)^{\frac{1}{2}} \leqslant \tau\delta$$



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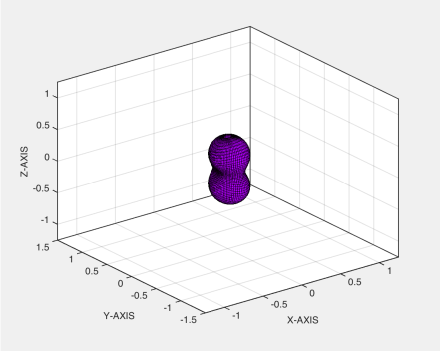
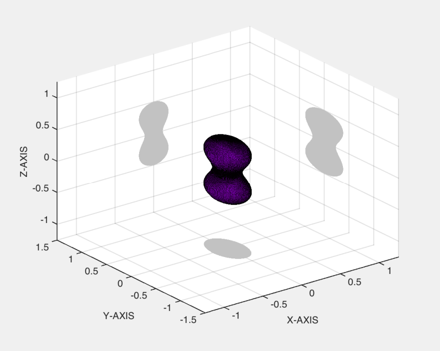


ITÉRATION $N = 10$

Algorithm 1 : numerical experiments

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0 \gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2 \right)^{\frac{1}{2}} \leqslant \tau\delta$$



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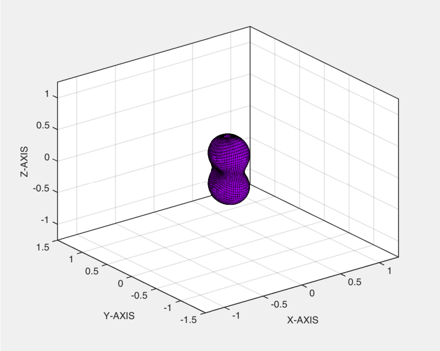
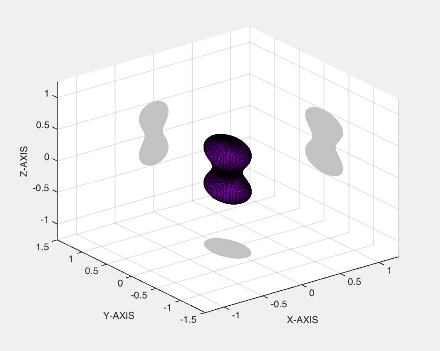


ITÉRATION $N = 11$

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2\right)^{\frac{1}{2}} \leqslant \tau\delta$$



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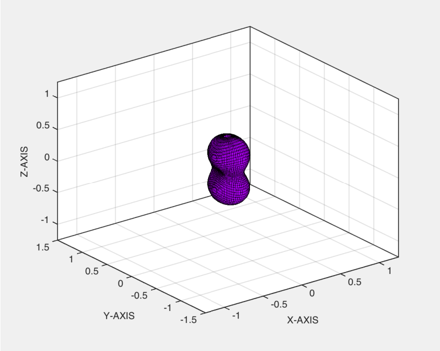
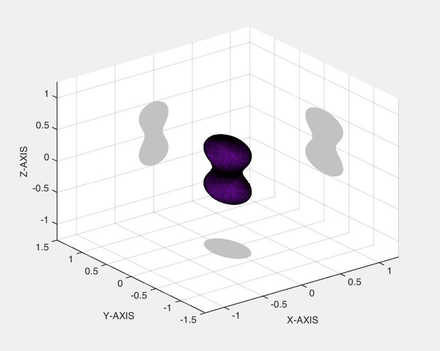


ITÉRATION $N = 12$

Algorithm 1 : numerical experiments

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1,0,0)$, $\boldsymbol{p}_1 = (0,0,1)$ and $\boldsymbol{d}_2 = (0,1,0)$, $\boldsymbol{p}_1 = (1,0,0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2\right)^{\frac{1}{2}} \leqslant \tau\delta$$



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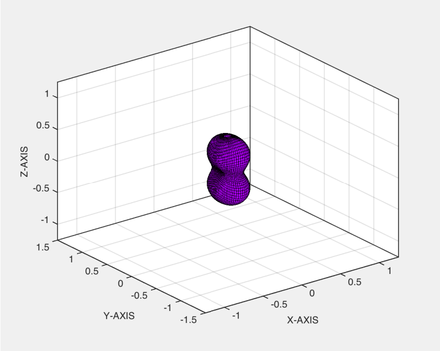
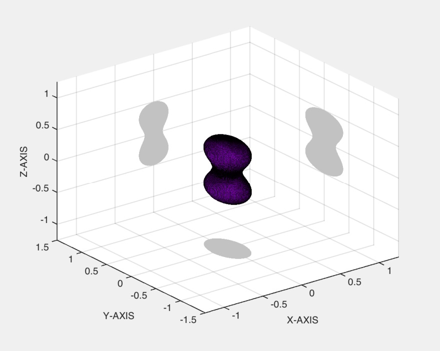


ITÉRATION $N = 13$

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2\right)^{\frac{1}{2}} \leqslant \tau\delta$$



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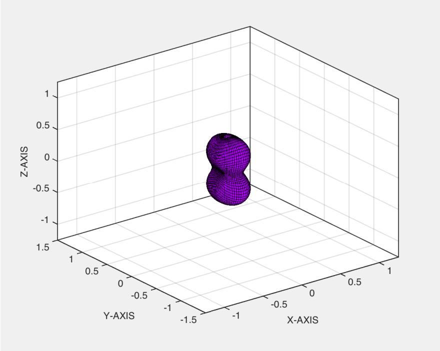
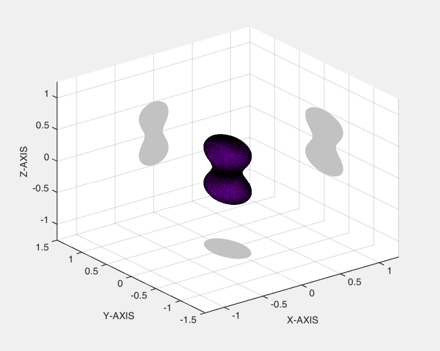


ITÉRATION $N = 14$

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2 \right)^{\frac{1}{2}} \leqslant \tau\delta$$



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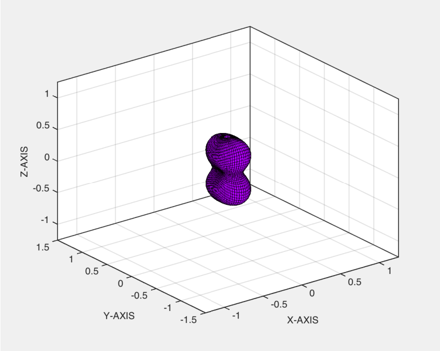
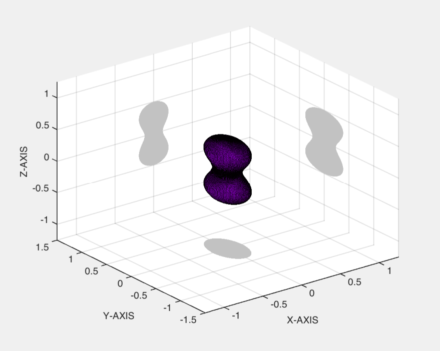


ITÉRATION $N = 15$

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1,0,0)$, $\boldsymbol{p}_1 = (0,0,1)$ and $\boldsymbol{d}_2 = (0,1,0)$, $\boldsymbol{p}_1 = (1,0,0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2\right)^{\frac{1}{2}} \leqslant \tau\delta$$



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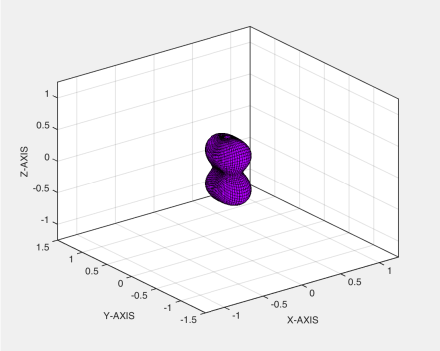

ITÉRATION $N = 16$

O. IVANYSHYN YAMAN, Nonlinear Boundary Integral Equations in Inverse Scattering, PhD thesis (2007)

$$\text{Recall} \quad \boldsymbol{F}(\Gamma_{\boldsymbol{q}}) = \mathscr{F}^{\infty}[\boldsymbol{q}]\ \mathrm{I}_{op}[\boldsymbol{q}]^{-1}\ (-\boldsymbol{n}_{\boldsymbol{q}} \times \boldsymbol{E}^{inc})_{|\Gamma_{\boldsymbol{q}}}$$

O. IVANYSHYN YAMAN, Nonlinear Boundary Integral Equations in Inverse Scattering, PhD thesis (2007)

$$\text{Recall} \quad \boldsymbol{F}(\Gamma_{\boldsymbol{q}}) = \mathscr{F}^{\infty}[\boldsymbol{q}]\, \mathrm{I}_{op}[\boldsymbol{q}]^{-1}\, (-\boldsymbol{n}_{\boldsymbol{q}} \times \boldsymbol{E}^{inc})_{|\Gamma_{\boldsymbol{q}}}$$

(IP) Given noisy farfield data $\boldsymbol{E}_{\delta}^{\infty}$, solve $\left\{ \begin{array}{lcl} \mathrm{I}_{op}[\boldsymbol{q}]\psi & = & (-\boldsymbol{n}_{\boldsymbol{q}} \times \boldsymbol{E}^{inc})_{|\Gamma_{\boldsymbol{q}}} = \boldsymbol{f}[\boldsymbol{q}] \\ \mathscr{F}^{\infty}[\boldsymbol{q}]\psi & = & \boldsymbol{E}_{\delta}^{\infty} \end{array} \right.$

O. IVANYSHYN YAMAN, Nonlinear Boundary Integral Equations in Inverse Scattering, PhD thesis (2007)

Recall $\quad \boldsymbol{F}(\Gamma_{\boldsymbol{q}}) = \mathscr{F}^{\infty}[\boldsymbol{q}]\, \mathrm{I}_{op}[\boldsymbol{q}]^{-1}\, (-\boldsymbol{n}_{\boldsymbol{q}} \times \boldsymbol{E}^{inc})_{|\Gamma_{\boldsymbol{q}}}$

(IP) Given noisy farfield data $\boldsymbol{E}^{\infty}_{\delta}$, solve $\left\{ \begin{array}{lll} \mathrm{I}_{op}[\boldsymbol{q}]\psi & = & (-\boldsymbol{n}_{\boldsymbol{q}} \times \boldsymbol{E}^{inc})_{|\Gamma_{\boldsymbol{q}}} = \boldsymbol{f}[\boldsymbol{q}] \\ \mathscr{F}^{\infty}[\boldsymbol{q}]\psi & = & \boldsymbol{E}^{\infty}_{\delta} \end{array} \right.$

→ First order linearization yields :

$$\begin{pmatrix} \mathrm{I}_{op}[\boldsymbol{q}]\psi \\ \mathscr{F}^{\infty}[\boldsymbol{q}]\psi \end{pmatrix} + \begin{pmatrix} \mathrm{I}_{op}[\boldsymbol{q}] & \partial_{\boldsymbol{q}}(\mathrm{I}_{op}[\boldsymbol{q}]\psi) \\ \mathscr{F}^{\infty}[\boldsymbol{q}] & \partial_{\boldsymbol{q}}(\mathscr{F}^{\infty}[\boldsymbol{q}]\psi) \end{pmatrix} \begin{pmatrix} \nu \\ \xi \end{pmatrix} \approx \begin{pmatrix} \boldsymbol{f}[\boldsymbol{q}] + \partial_{\boldsymbol{q}}\boldsymbol{f}[\boldsymbol{q}]\xi \\ \boldsymbol{E}^{\infty}_{\delta} \end{pmatrix}.$$

→ We rearrange the terms :

$$\begin{pmatrix} \mathrm{I}_{op}[\boldsymbol{q}] & \partial_{\boldsymbol{q}}(\mathrm{I}_{op}[\boldsymbol{q}]\psi) - \partial_{\boldsymbol{q}}\boldsymbol{f}[\boldsymbol{q}] \\ \mathscr{F}^{\infty}[\boldsymbol{q}] & \partial_{\boldsymbol{q}}(\mathscr{F}^{\infty}[\boldsymbol{q}]\psi) \end{pmatrix} \begin{pmatrix} \nu \\ \xi \end{pmatrix} \approx \begin{pmatrix} \boldsymbol{f}[\boldsymbol{q}] - \mathrm{I}_{op}[\boldsymbol{q}]\psi \\ \boldsymbol{E}^{\infty}_{\delta} - \mathscr{F}^{\infty}[\boldsymbol{q}]\psi \end{pmatrix}.$$

O. IVANYSHYN YAMAN, Nonlinear Boundary Integral Equations in Inverse Scattering, PhD thesis (2007)

Recall $\boldsymbol{F}(\Gamma_{\boldsymbol{q}}) = \mathscr{F}^{\infty}[\boldsymbol{q}]\ \mathrm{I}_{op}[\boldsymbol{q}]^{-1}\ (-\boldsymbol{n_q} \times \boldsymbol{E}^{inc})_{|\Gamma_{\boldsymbol{q}}}$

(IP) Given noisy farfield data $\boldsymbol{E}_{\delta}^{\infty}$, solve $\begin{cases} \mathrm{I}_{op}[\boldsymbol{q}]\psi = (-\boldsymbol{n_q} \times \boldsymbol{E}^{inc})_{|\Gamma_{\boldsymbol{q}}} = \boldsymbol{f}[\boldsymbol{q}] \\ \mathscr{F}^{\infty}[\boldsymbol{q}]\psi = \boldsymbol{E}_{\delta}^{\infty} \end{cases}$

1. Choose an initial guess $\Gamma_0 := \boldsymbol{q}_0(\Gamma_{\mathrm{ref}})$ and compute ψ_0.

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Recall $\boldsymbol{F}(\Gamma_{\boldsymbol{q}}) = \mathscr{F}^{\infty}[\boldsymbol{q}]\,\mathrm{I}_{op}[\boldsymbol{q}]^{-1}\,(-\boldsymbol{n}_{\boldsymbol{q}} \times \boldsymbol{E}^{inc})_{|\Gamma_{\boldsymbol{q}}}$

(IP) Given noisy farfield data $\boldsymbol{E}_{\delta}^{\infty}$, solve $\left\{ \begin{array}{rcl} \mathrm{I}_{op}[\boldsymbol{q}]\psi & = & (-\boldsymbol{n}_{\boldsymbol{q}} \times \boldsymbol{E}^{inc})_{|\Gamma_{\boldsymbol{q}}} = \boldsymbol{f}[\boldsymbol{q}] \\ \mathscr{F}^{\infty}[\boldsymbol{q}]\psi & = & \boldsymbol{E}_{\delta}^{\infty} \end{array} \right.$

1. Choose an initial guess $\Gamma_0 := \boldsymbol{q}_0(\Gamma_{\mathrm{ref}})$ and compute ψ_0.
2. For the iterates ψ_N^{δ} and $\Gamma_N^{\delta} := \boldsymbol{q}_N^{\delta}(\Gamma_{\mathrm{ref}})$, evaluate $\boldsymbol{F}(\Gamma_N^{\delta}) = \mathscr{F}^{\infty}[\boldsymbol{q}_N^{\delta}]\psi_N^{\delta}$
   If the stopping rule is not satisfied at the Nth regularized Newton step :

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Recall $\quad \boldsymbol{F}(\Gamma_{\boldsymbol{q}}) = \mathscr{F}^{\infty}[\boldsymbol{q}]\, \mathrm{I}_{op}[\boldsymbol{q}]^{-1}\, (-\boldsymbol{n}_{\boldsymbol{q}} \times \boldsymbol{E}^{inc})_{|\Gamma_{\boldsymbol{q}}}$

(IP) Given noisy farfield data $\boldsymbol{E}^{\infty}_{\delta}$, solve $\begin{cases} \mathrm{I}_{op}[\boldsymbol{q}]\psi & = & (-\boldsymbol{n}_{\boldsymbol{q}} \times \boldsymbol{E}^{inc})_{|\Gamma_{\boldsymbol{q}}} = \boldsymbol{f}[\boldsymbol{q}] \\ \mathscr{F}^{\infty}[\boldsymbol{q}]\psi & = & \boldsymbol{E}^{\infty}_{\delta} \end{cases}$

1. Choose an initial guess $\Gamma_0 := \boldsymbol{q}_0(\Gamma_{\mathrm{ref}})$ and compute ψ_0.
2. For the iterates ψ^{δ}_N and $\Gamma^{\delta}_N := \boldsymbol{q}^{\delta}_N(\Gamma_{\mathrm{ref}})$, evaluate $\boldsymbol{F}(\Gamma^{\delta}_N) = \mathscr{F}^{\infty}[\boldsymbol{q}^{\delta}_N]\psi^{\delta}_N$
   If the stopping rule is not satisfied at the Nth regularized Newton step :
3. Compute the next iterates ψ^{δ}_{N+1} and $\boldsymbol{q}^{\delta}_{N+1}$ by minimizing the least square (IRGNM)

$$\begin{pmatrix} \psi^{\delta}_{N+1} \\ \boldsymbol{q}^{\delta}_{N+1} \end{pmatrix} := \mathrm{argmin}_{\psi,\boldsymbol{q}} \left[\left\| A^{\delta}_N \begin{pmatrix} \psi - \psi^{\delta}_N \\ \boldsymbol{q} - \boldsymbol{q}^{\delta}_N \end{pmatrix} - B^{\delta}_N \right\|^2 + \alpha_N \left\| \begin{pmatrix} \psi - \psi_0 \\ \boldsymbol{q} - \boldsymbol{q}_0 \end{pmatrix} \right\|^2 \right].$$

using conjugate gradient method where we have set

$$A^{\delta}_N = \begin{pmatrix} \mathrm{I}_{op}[\boldsymbol{q}^{\delta}_N] & \partial_{\boldsymbol{q}}(\mathrm{I}_{op}[\boldsymbol{q}^{\delta}_N]\psi^{\delta}_N) - \partial_{\boldsymbol{q}}\boldsymbol{f}[\boldsymbol{q}^{\delta}_N] \\ \mathscr{F}^{\infty}[\boldsymbol{q}^{\delta}_N] & \partial_{\boldsymbol{q}}(\mathscr{F}^{\infty}[\boldsymbol{q}^{\delta}_N]\psi^{\delta}_N) \end{pmatrix} \quad B^{\delta}_N = \begin{pmatrix} \boldsymbol{f}[\boldsymbol{q}^{\delta}_N] - \mathrm{I}_{op}[\boldsymbol{q}^{\delta}_N]\psi^{\delta}_N \\ \boldsymbol{E}^{\infty}_{\delta} - \mathscr{F}^{\infty}[\boldsymbol{q}^{\delta}_N]\psi^{\delta}_N \end{pmatrix}$$

O. IVANYSHYN YAMAN, Nonlinear Boundary Integral Equations in Inverse Scattering, PhD thesis (2007)

$$\text{Recall} \quad \boldsymbol{F}(\Gamma_{\boldsymbol{q}}) = \mathscr{F}^\infty[\boldsymbol{q}]\, \mathrm{I}_{op}[\boldsymbol{q}]^{-1} \, (-\boldsymbol{n_q} \times \boldsymbol{E}^{inc})_{|\Gamma_{\boldsymbol{q}}}$$

(IP) Given noisy farfield data $\boldsymbol{E}^\infty_\delta$, solve $\begin{cases} \mathrm{I}_{op}[\boldsymbol{q}]\psi & = & (-\boldsymbol{n_q} \times \boldsymbol{E}^{inc})_{|\Gamma_{\boldsymbol{q}}} = \boldsymbol{f}[\boldsymbol{q}] \\ \mathscr{F}^\infty[\boldsymbol{q}]\psi & = & \boldsymbol{E}^\infty_\delta \end{cases}$

1. Choose an initial guess $\Gamma_0 := \boldsymbol{q}_0(\Gamma_{\mathrm{ref}})$ and compute ψ_0.
2. For the iterates ψ^δ_N and $\Gamma^\delta_N := \boldsymbol{q}^\delta_N(\Gamma_{\mathrm{ref}})$, evaluate $\boldsymbol{F}(\Gamma^\delta_N) = \mathscr{F}^\infty[\boldsymbol{q}^\delta_N]\psi^\delta_N$
   If the stopping rule is not satisfied at the Nth regularized Newton step :
3. Compute the next iterates ψ^δ_{N+1} and $\boldsymbol{q}^\delta_{N+1}$ by minimizing the least square (IRGNM)

$$\begin{pmatrix} \psi^\delta_{N+1} \\ \boldsymbol{q}^\delta_{N+1} \end{pmatrix} := \mathrm{argmin}_{\psi,\boldsymbol{q}} \left[\left\| A^\delta_N \begin{pmatrix} \psi - \psi^\delta_N \\ \boldsymbol{q} - \boldsymbol{q}^\delta_N \end{pmatrix} - B^\delta_N \right\|^2 + \alpha_N \left\| \begin{pmatrix} \psi - \psi_0 \\ \boldsymbol{q} - \boldsymbol{q}_0 \end{pmatrix} \right\|^2 \right].$$

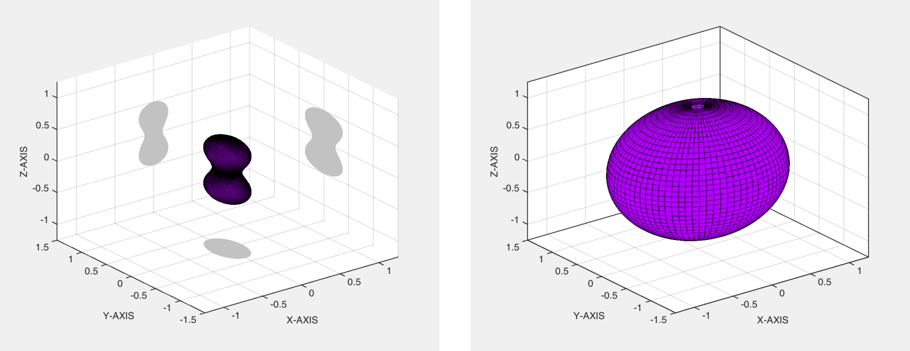
using conjugate gradient method where we have set

$$A^\delta_N = \begin{pmatrix} \mathrm{I}_{op}[\boldsymbol{q}^\delta_N] & \partial_{\boldsymbol{q}}(\mathrm{I}_{op}[\boldsymbol{q}^\delta_N]\psi^\delta_N) - \partial_{\boldsymbol{q}}\boldsymbol{f}[\boldsymbol{q}^\delta_N] \\ \mathscr{F}^\infty[\boldsymbol{q}^\delta_N] & \partial_{\boldsymbol{q}}(\mathscr{F}^\infty[\boldsymbol{q}^\delta_N]\psi^\delta_N) \end{pmatrix} \quad B^\delta_N = \begin{pmatrix} \boldsymbol{f}[\boldsymbol{q}^\delta_N] - \mathrm{I}_{op}[\boldsymbol{q}^\delta_N]\psi^\delta_N \\ \boldsymbol{E}^\infty_\delta - \mathscr{F}^\infty[\boldsymbol{q}^\delta_N]\psi^\delta_N \end{pmatrix}$$

→ Main advantage : The inversion of a BIE is replaced by one matrix-vector product.

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ and $\beta_0 = 0.01$, $\beta_N = \beta_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

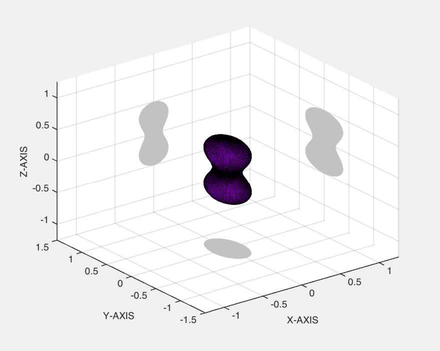
$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2\right)^{\frac{1}{2}} \leqslant \tau\delta$$



ORIGINAL INITIAL GUESS

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ and $\beta_0 = 0.01$, $\beta_N = \beta_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2\right)^{\frac{1}{2}} \leqslant \tau\delta$$



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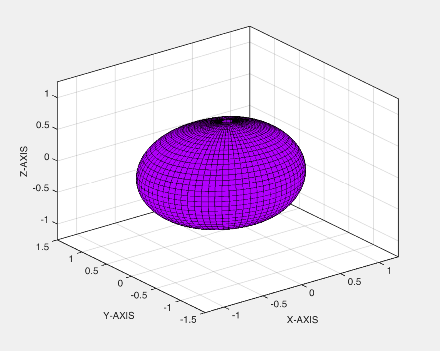
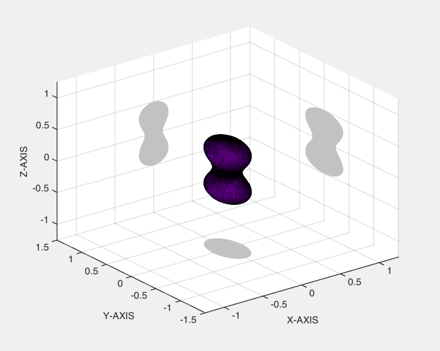


ITÉRATION $N = 1$

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1,0,0)$, $\boldsymbol{p}_1 = (0,0,1)$ and $\boldsymbol{d}_2 = (0,1,0)$, $\boldsymbol{p}_1 = (1,0,0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ and $\beta_0 = 0.01$, $\beta_N = \beta_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2\right)^{\frac{1}{2}} \leqslant \tau\delta$$



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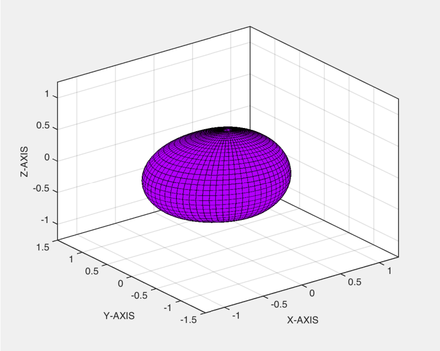
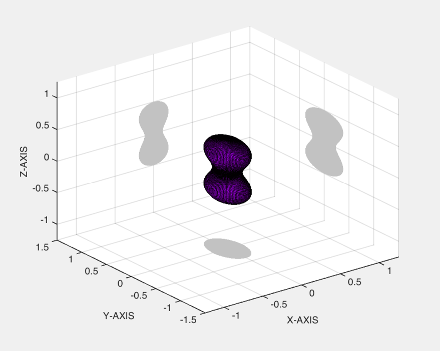


ITÉRATION $N = 2$

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ and $\beta_0 = 0.01$, $\beta_N = \beta_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2\right)^{\frac{1}{2}} \leqslant \tau\delta$$



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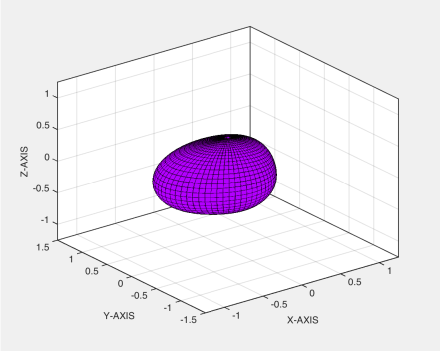
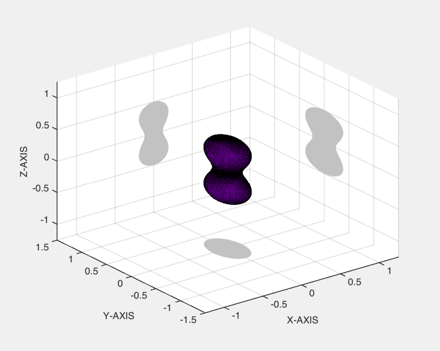


ITÉRATION $N = 3$

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ and $\beta_0 = 0.01$, $\beta_N = \beta_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2\right)^{\frac{1}{2}} \leqslant \tau\delta$$



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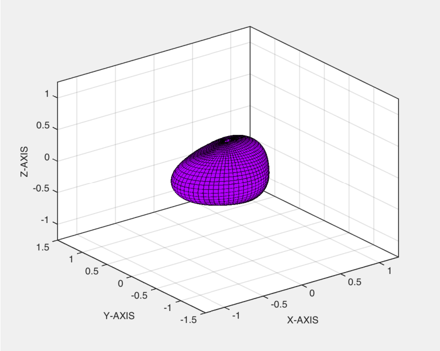
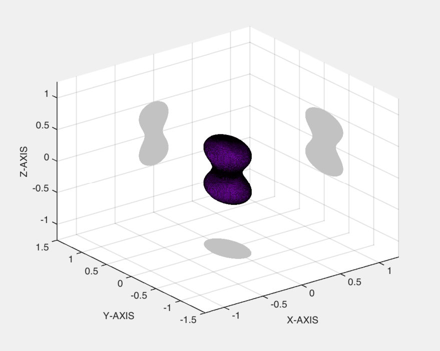


ITÉRATION $N = 4$

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ and $\beta_0 = 0.01$, $\beta_N = \beta_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2 \right)^{\frac{1}{2}} \leqslant \tau\delta$$



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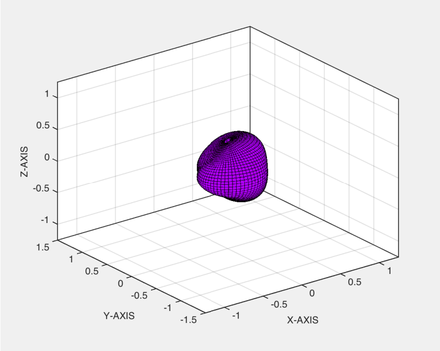
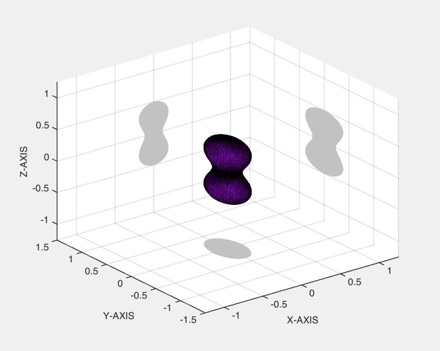


ITÉRATION $N = 5$

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ and $\beta_0 = 0.01$, $\beta_N = \beta_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2\right)^{\frac{1}{2}} \leqslant \tau\delta$$



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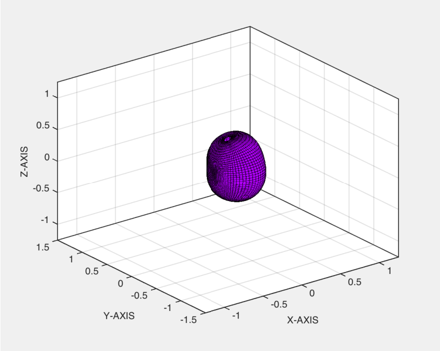
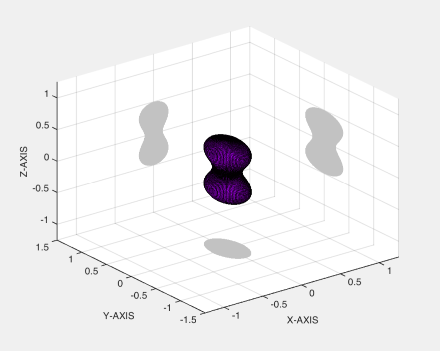


ITÉRATION $N = 6$

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ and $\beta_0 = 0.01$, $\beta_N = \beta_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||^2_{\boldsymbol{L}^2}\right)^{\frac{1}{2}} \leqslant \tau\delta$$



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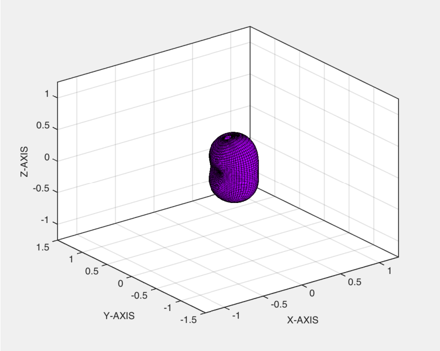
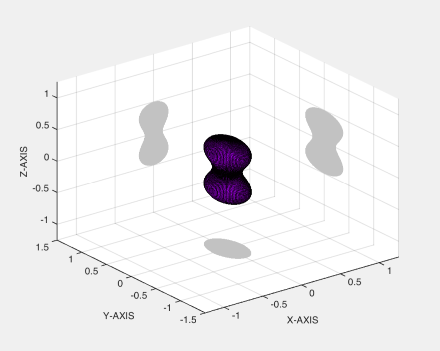


ITÉRATION $N = 7$

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ and $\beta_0 = 0.01$, $\beta_N = \beta_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2\right)^{\frac{1}{2}} \leqslant \tau\delta$$



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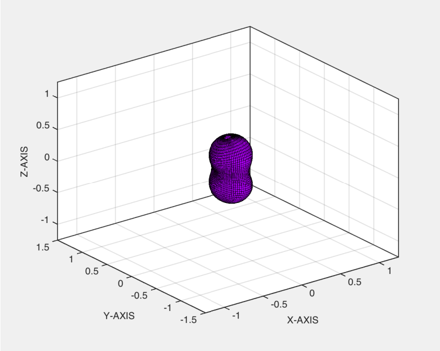
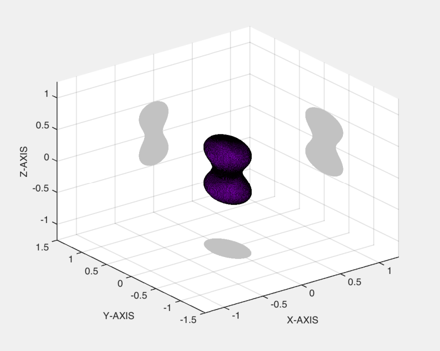


ITÉRATION $N = 8$

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ and $\beta_0 = 0.01$, $\beta_N = \beta_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2 \right)^{\frac{1}{2}} \leqslant \tau\delta$$



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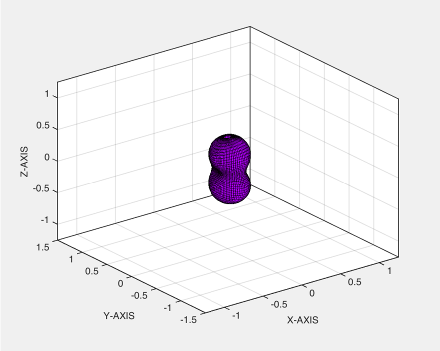
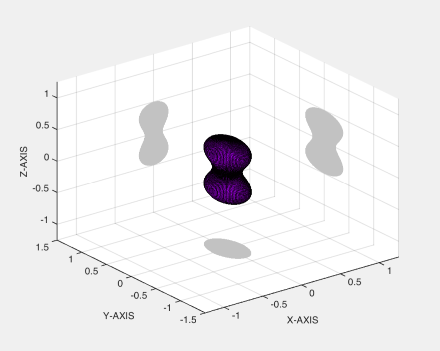


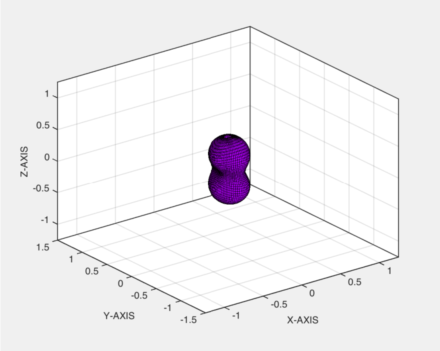
ITÉRATION $N = 9$

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ and $\beta_0 = 0.01$, $\beta_N = \beta_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2\right)^{\frac{1}{2}} \leqslant \tau\delta$$



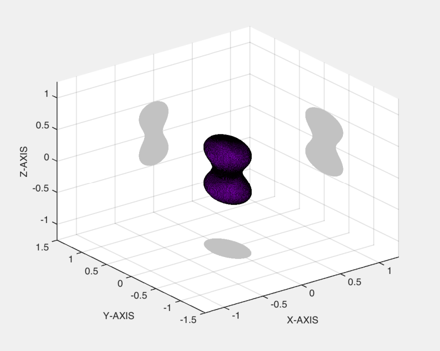
ORIGINAL



ITÉRATION $N = 10$

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ and $\beta_0 = 0.01$, $\beta_N = \beta_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2\right)^{\frac{1}{2}} \leqslant \tau\delta$$



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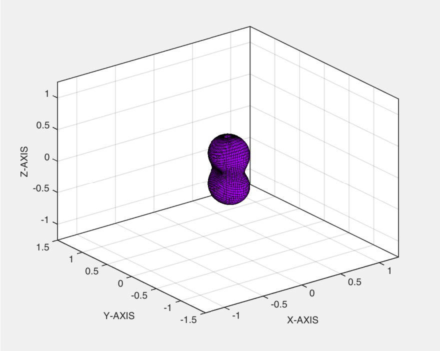
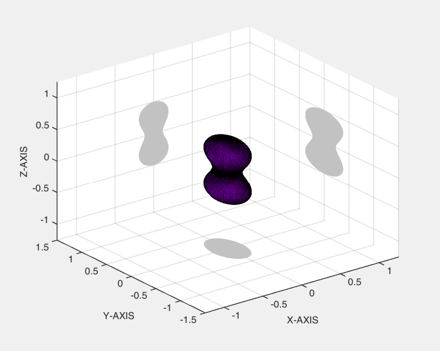


ITÉRATION $N = 11$

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ and $\beta_0 = 0.01$, $\beta_N = \beta_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2\right)^{\frac{1}{2}} \leqslant \tau\delta$$



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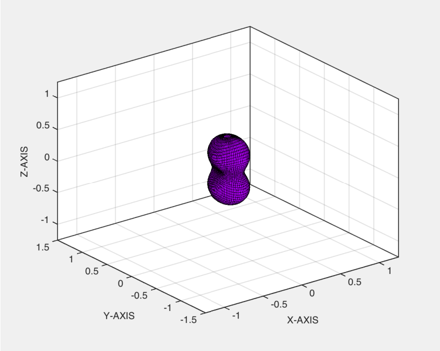
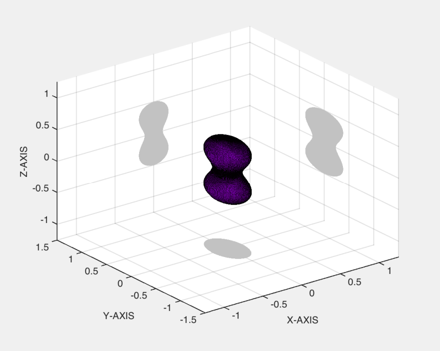


ITÉRATION $N = 12$

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1,0,0)$, $\boldsymbol{p}_1 = (0,0,1)$ and $\boldsymbol{d}_2 = (0,1,0)$, $\boldsymbol{p}_1 = (1,0,0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ and $\beta_0 = 0.01$, $\beta_N = \beta_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2\right)^{\frac{1}{2}} \leqslant \tau\delta$$



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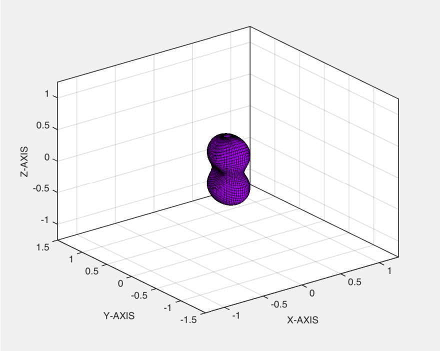
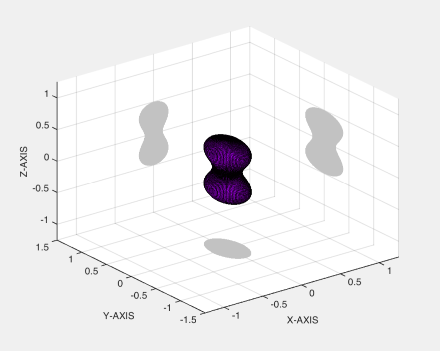


ITÉRATION $N = 13$

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ and $\beta_0 = 0.01$, $\beta_N = \beta_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2 \right)^{\frac{1}{2}} \leqslant \tau\delta$$



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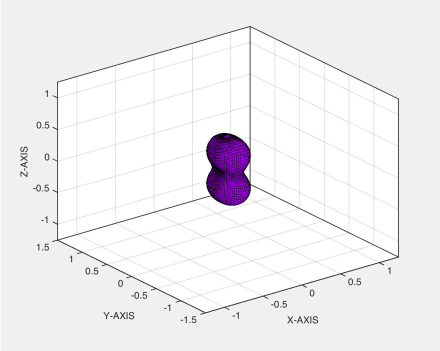
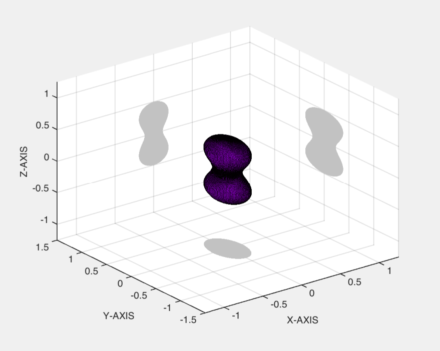


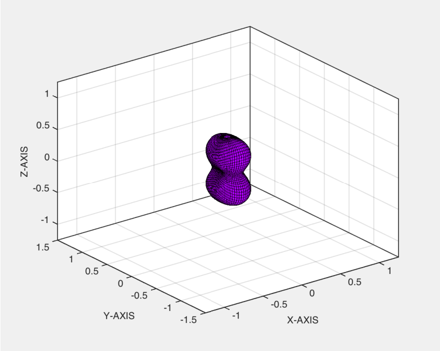
ITÉRATION $N = 14$

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ and $\beta_0 = 0.01$, $\beta_N = \beta_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2\right)^{\frac{1}{2}} \leqslant \tau\delta$$



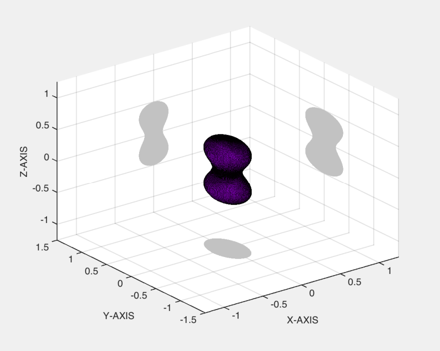
ORIGINAL



ITÉRATION $N = 15$

- $m = 2$ incident plane waves $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{p}_1 = (0, 0, 1)$ and $\boldsymbol{d}_2 = (0, 1, 0)$, $\boldsymbol{p}_1 = (1, 0, 0)$
- size of the unknown obstacle $\approx 0.5\lambda$ where $\lambda = 2\pi/\kappa$
- regularizing parameters : $\alpha_0 = 0.5$, $\alpha_N = \alpha_0\gamma^{-N}$ and $\beta_0 = 0.01$, $\beta_N = \beta_0\gamma^{-N}$ with $\gamma > 1$
- stopping rule : Morozov's discrepancy principle : $\delta = 2\%$ random noise and $\tau = 1.5$.

$$\left(\sum_{k=1}^{m} ||\mathcal{F}_k(\boldsymbol{q}_\delta^N) - \boldsymbol{E}_k^\infty||_{\boldsymbol{L}^2}^2 \right)^{\frac{1}{2}} \leqslant \tau\delta$$



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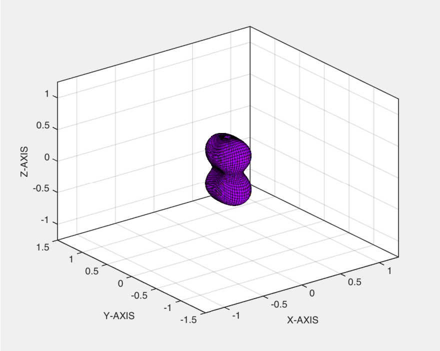

ITÉRATION $N = 16$

- $\boldsymbol{f}[\boldsymbol{q}] = (-\boldsymbol{n}_{\boldsymbol{q}} \times \boldsymbol{E}^{inc})_{|\Gamma_{\boldsymbol{q}}} \in \boldsymbol{L}^2_{\rm t}(\Gamma_{\boldsymbol{q}}) \cap \boldsymbol{H}^{-\frac{1}{2}}_{\rm div}(\Gamma_{\boldsymbol{q}})$.
- $\mathrm{I}_{op}[\boldsymbol{q}] = \frac{1}{2}\mathrm{I} + D_\kappa[\boldsymbol{q}] + i\eta S_\kappa[\boldsymbol{q}]\boldsymbol{\Lambda}[\boldsymbol{q}] : \boldsymbol{L}^2_{\rm t}(\Gamma_{\boldsymbol{q}}) \cap \boldsymbol{H}^{-\frac{1}{2}}_{\rm div}(\Gamma_{\boldsymbol{q}}) \to \boldsymbol{L}^2_{\rm t}(\Gamma_{\boldsymbol{q}}) \cap \boldsymbol{H}^{-\frac{1}{2}}_{\rm div}(\Gamma_{\boldsymbol{q}})$
- $\mathscr{F}^\infty[\boldsymbol{q}] = \mathscr{D}^\infty_\kappa[\boldsymbol{q}] + i\eta \mathscr{S}^\infty_\kappa[\boldsymbol{q}]\boldsymbol{\Lambda}[\boldsymbol{q}] : \boldsymbol{L}^2_{\rm t}(\Gamma_{\boldsymbol{q}}) \cap \boldsymbol{H}^{-\frac{1}{2}}_{\rm div}(\Gamma_{\boldsymbol{q}}) \to \boldsymbol{L}^2_{\rm t}(\mathbb{S}^2)$

- $\boldsymbol{f}[\boldsymbol{q}] = (-\boldsymbol{n_q} \times \boldsymbol{E}^{inc})_{|\Gamma_{\boldsymbol{q}}} \in \boldsymbol{L}^2_{\mathrm{t}}(\Gamma_{\boldsymbol{q}}) \cap \boldsymbol{H}^{-\frac{1}{2}}_{\mathrm{div}}(\Gamma_{\boldsymbol{q}})$.
- $\mathrm{I}_{op}[\boldsymbol{q}] = \frac{1}{2}\mathrm{I} + D_\kappa[\boldsymbol{q}] + i\eta S_\kappa[\boldsymbol{q}]\boldsymbol{\Lambda}[\boldsymbol{q}] : \boldsymbol{L}^2_{\mathrm{t}}(\Gamma_{\boldsymbol{q}}) \cap \boldsymbol{H}^{-\frac{1}{2}}_{\mathrm{div}}(\Gamma_{\boldsymbol{q}}) \to \boldsymbol{L}^2_{\mathrm{t}}(\Gamma_{\boldsymbol{q}}) \cap \boldsymbol{H}^{-\frac{1}{2}}_{\mathrm{div}}(\Gamma_{\boldsymbol{q}})$
- $\mathscr{F}^\infty[\boldsymbol{q}] = \mathscr{D}^\infty_\kappa[\boldsymbol{q}] + i\eta\mathscr{S}^\infty_\kappa[\boldsymbol{q}]\boldsymbol{\Lambda}[\boldsymbol{q}] : \boldsymbol{L}^2_{\mathrm{t}}(\Gamma_{\boldsymbol{q}}) \cap \boldsymbol{H}^{-\frac{1}{2}}_{\mathrm{div}}(\Gamma_{\boldsymbol{q}}) \to \boldsymbol{L}^2_{\mathrm{t}}(\mathbb{S}^2)$

Main difficulty : We have to compute Fréchet derivatives (with respect to $\boldsymbol{q}$) of functions and operators defined on the $\boldsymbol{q}$-dependent space $\boldsymbol{L}^2_{\mathrm{t}}(\Gamma_{\boldsymbol{q}})$ or $\boldsymbol{H}^{-\frac{1}{2}}_{\mathrm{div}}(\Gamma_{\boldsymbol{q}})$.

- $\boldsymbol{f}[\boldsymbol{q}] = (-\boldsymbol{n_q} \times \boldsymbol{E}^{inc})_{|\Gamma_{\boldsymbol{q}}} \in \boldsymbol{L}^2_{\mathrm{t}}(\Gamma_{\boldsymbol{q}}) \cap \boldsymbol{H}^{-\frac{1}{2}}_{\mathrm{div}}(\Gamma_{\boldsymbol{q}})$.
- $\mathrm{I}_{op}[\boldsymbol{q}] = \frac{1}{2}\mathrm{I} + D_\kappa[\boldsymbol{q}] + i\eta S_\kappa[\boldsymbol{q}]\boldsymbol{\Lambda}[\boldsymbol{q}] : \boldsymbol{L}^2_{\mathrm{t}}(\Gamma_{\boldsymbol{q}}) \cap \boldsymbol{H}^{-\frac{1}{2}}_{\mathrm{div}}(\Gamma_{\boldsymbol{q}}) \to \boldsymbol{L}^2_{\mathrm{t}}(\Gamma_{\boldsymbol{q}}) \cap \boldsymbol{H}^{-\frac{1}{2}}_{\mathrm{div}}(\Gamma_{\boldsymbol{q}})$
- $\mathscr{F}^\infty[\boldsymbol{q}] = \mathscr{D}^\infty_\kappa[\boldsymbol{q}] + i\eta\mathscr{S}^\infty_\kappa[\boldsymbol{q}]\boldsymbol{\Lambda}[\boldsymbol{q}] : \boldsymbol{L}^2_{\mathrm{t}}(\Gamma_{\boldsymbol{q}}) \cap \boldsymbol{H}^{-\frac{1}{2}}_{\mathrm{div}}(\Gamma_{\boldsymbol{q}}) \to \boldsymbol{L}^2_{\mathrm{t}}(\mathbb{S}^2)$

Main difficulty : We have to compute Fréchet derivatives (with respect to $\boldsymbol{q}$) of functions and operators defined on the $\boldsymbol{q}$-dependent space $\boldsymbol{L}^2_{\mathrm{t}}(\Gamma_{\boldsymbol{q}})$ or $\boldsymbol{H}^{-\frac{1}{2}}_{\mathrm{div}}(\Gamma_{\boldsymbol{q}})$.

Objective : We want to transport the integral equation system on the fixed boundary Γ_{ref}

R. Potthast, Fréchet differentiability of boundary integral operators in inverse acoustic scattering, (1994)

Ideas from acoustics : we can use changes of variable

$$\psi_{\boldsymbol{q}} \in L^2(\Gamma_{\boldsymbol{q}}) \underset{\tau_{\boldsymbol{q}}^{-1}}{\overset{\tau_{\boldsymbol{q}}}{\rightleftarrows}} \psi_{\boldsymbol{q}} \circ \boldsymbol{q} \in L^2(\Gamma_{\mathrm{ref}})$$

$$\begin{array}{ccccc}
L^2(\Gamma_{\boldsymbol{q}}) & \xrightarrow{\mathrm{I}_{op}[\boldsymbol{q}]^{-1}} & L^2(\Gamma_{\boldsymbol{q}}) & \xrightarrow{\mathscr{F}^\infty[\boldsymbol{q}]} & L^2(\mathbb{S}^2) \\
\tau_{\boldsymbol{q}} \downarrow & & \uparrow \tau_{\boldsymbol{q}}^{-1} & \nearrow \widehat{\mathscr{F}}^\infty[\boldsymbol{q}] & \\
L^2(\Gamma_{\mathrm{ref}}) & \xrightarrow{\hat{\mathrm{I}}_{op}[\boldsymbol{q}]^{-1}} & L^2(\Gamma_{\mathrm{ref}}) & &
\end{array}$$

Main difficulties of the whole analysis

- $\boldsymbol{f}[\boldsymbol{q}] = (-\boldsymbol{n}_{\boldsymbol{q}} \times \boldsymbol{E}^{inc})_{|\Gamma_{\boldsymbol{q}}} \in \boldsymbol{L}^2_{\mathrm{t}}(\Gamma_{\boldsymbol{q}}) \cap \boldsymbol{H}^{-\frac{1}{2}}_{\mathrm{div}}(\Gamma_{\boldsymbol{q}})$.
- $\mathrm{I}_{op}[\boldsymbol{q}] = \frac{1}{2}\mathrm{I} + D_\kappa[\boldsymbol{q}] + i\eta S_\kappa[\boldsymbol{q}]\boldsymbol{\Lambda}[\boldsymbol{q}] : \boldsymbol{L}^2_{\mathrm{t}}(\Gamma_{\boldsymbol{q}}) \cap \boldsymbol{H}^{-\frac{1}{2}}_{\mathrm{div}}(\Gamma_{\boldsymbol{q}}) \to \boldsymbol{L}^2_{\mathrm{t}}(\Gamma_{\boldsymbol{q}}) \cap \boldsymbol{H}^{-\frac{1}{2}}_{\mathrm{div}}(\Gamma_{\boldsymbol{q}})$
- $\mathscr{F}^\infty[\boldsymbol{q}] = \mathscr{D}^\infty_\kappa[\boldsymbol{q}] + i\eta \mathscr{S}^\infty_\kappa[\boldsymbol{q}]\boldsymbol{\Lambda}[\boldsymbol{q}] : \boldsymbol{L}^2_{\mathrm{t}}(\Gamma_{\boldsymbol{q}}) \cap \boldsymbol{H}^{-\frac{1}{2}}_{\mathrm{div}}(\Gamma_{\boldsymbol{q}}) \to \boldsymbol{L}^2_{\mathrm{t}}(\mathbb{S}^2)$

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$$\begin{array}{ccccc}
L^2(\Gamma_{\boldsymbol{q}}) & \xrightarrow{\mathrm{I}_{op}[\boldsymbol{q}]^{-1}} & L^2(\Gamma_{\boldsymbol{q}}) & \xrightarrow{\mathscr{F}^\infty[\boldsymbol{q}]} & L^2(\mathbb{S}^2) \\
\tau_{\boldsymbol{q}} \downarrow & & \uparrow \tau_{\boldsymbol{q}}^{-1} & \nearrow \widehat{\mathscr{F}}^\infty[\boldsymbol{q}] & \\
L^2(\Gamma_{\mathrm{ref}}) & \xrightarrow{\widehat{\mathrm{I}}_{op}[\boldsymbol{q}]^{-1}} & L^2(\Gamma_{\mathrm{ref}}) & &
\end{array}$$

Towards electromagnetism : The space $\tau_{\boldsymbol{q}}(\boldsymbol{L}^2_{\mathrm{t}}(\Gamma_{\boldsymbol{q}}))$ or $\tau_{\boldsymbol{q}}(\boldsymbol{H}^{-\frac{1}{2}}_{\mathrm{div}}(\Gamma_{\boldsymbol{q}}))$ still depends on $\boldsymbol{q}$

- We can use projectors on the tangent plane to Γ_{ref}

R. POTTHAST, *Domain derivatives in electromagnetic scattering*, (1996)

$$\begin{array}{rcl} \Pi_{\boldsymbol{q}} : \boldsymbol{L}^2_{\text{t}}(\Gamma_{\boldsymbol{q}}) & \to & \boldsymbol{L}^2_{\text{t}}(\Gamma_{\text{ref}}) \\ \boldsymbol{\psi}_{\boldsymbol{q}} & \mapsto & \boldsymbol{\psi}_{\boldsymbol{q}} \circ \boldsymbol{q} - \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{\psi}_{\boldsymbol{q}}) \end{array} \quad \text{and} \quad \begin{array}{rcl} \Pi^{-1}_{\boldsymbol{q}} : \boldsymbol{L}^2_{\text{t}}(\Gamma_{\text{ref}}) & \to & \boldsymbol{L}^2_{\text{t}}(\Gamma_{\boldsymbol{q}}) \\ \boldsymbol{\psi} & \mapsto & \boldsymbol{\psi} - \boldsymbol{n} \dfrac{\boldsymbol{n}_{\boldsymbol{q}} \cdot \boldsymbol{\psi}}{\boldsymbol{n}_{\boldsymbol{q}} \cdot \boldsymbol{n}} \end{array}$$

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  R. POTTHAST, *Domain derivatives in electromagnetic scattering*, (1996)

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- We can use the differential geometry of a surface

  M. PIEPER, Nonlinear integral equations for an inverse electromagnetic scattering problem, (2008)

$$\begin{array}{rcl} \Pi_{\boldsymbol{q}}^{-1} : \boldsymbol{L}^2_{\text{t}}(\Gamma_{\text{ref}}) & \rightarrow & \boldsymbol{L}^2_{\text{t}}(\Gamma_{\boldsymbol{q}}) \\ \boldsymbol{\psi} & \mapsto & ([\mathbf{D}_{\Gamma_{\text{ref}}}\boldsymbol{q}]\boldsymbol{\psi}) \circ \boldsymbol{q}^{-1} \end{array} \quad \text{and} \quad \begin{array}{rcl} \Pi_{\boldsymbol{q}} : \boldsymbol{L}^2_{\text{t}}(\Gamma_{\boldsymbol{q}}) & \rightarrow & \boldsymbol{L}^2_{\text{t}}(\Gamma_{\text{ref}}) \\ \boldsymbol{\psi}_{\boldsymbol{q}} & \mapsto & \left([\mathbf{D}_{\Gamma_{\boldsymbol{q}}}\boldsymbol{q}^{-1}]\boldsymbol{\psi}_{\boldsymbol{q}}\right) \circ \boldsymbol{q} \end{array}$$

- We can use projectors on the tangent plane to Γ_{ref}

  R. POTTHAST, *Domain derivatives in electromagnetic scattering*, (1996)

$$\begin{array}{rcl} \Pi_{\boldsymbol{q}} : \boldsymbol{L}^2_t(\Gamma_{\boldsymbol{q}}) & \rightarrow & \boldsymbol{L}^2_t(\Gamma_{\text{ref}}) \\ \boldsymbol{\psi}_{\boldsymbol{q}} & \mapsto & \boldsymbol{\psi}_{\boldsymbol{q}} \circ \boldsymbol{q} - \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{\psi}_{\boldsymbol{q}}) \end{array} \quad \text{and} \quad \begin{array}{rcl} \Pi_{\boldsymbol{q}}^{-1} : \boldsymbol{L}^2_t(\Gamma_{\text{ref}}) & \rightarrow & \boldsymbol{L}^2_t(\Gamma_{\boldsymbol{q}}) \\ \boldsymbol{\psi} & \mapsto & \boldsymbol{\psi} - \boldsymbol{n}\dfrac{\boldsymbol{n}_{\boldsymbol{q}} \cdot \boldsymbol{\psi}}{\boldsymbol{n}_{\boldsymbol{q}} \cdot \boldsymbol{n}} \end{array}$$

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$$\begin{array}{ccccc} \boldsymbol{L}^2_t(\Gamma_{\boldsymbol{q}}) & \xrightarrow{\mathrm{I}op[\boldsymbol{q}]^{-1}} & \boldsymbol{L}^2_t(\Gamma_{\boldsymbol{q}}) & \xrightarrow{\mathscr{F}^{\infty}[\boldsymbol{q}]} & \boldsymbol{L}^2_t(\mathbb{S}^2) \\ \Pi_{\boldsymbol{q}} \downarrow & & \uparrow \Pi_{\boldsymbol{q}}^{-1} & \nearrow \widehat{\mathscr{F}}^{\infty}[\boldsymbol{q}] & \\ \boldsymbol{L}^2_t(\Gamma_{\text{ref}}) & \xrightarrow{\hat{\mathrm{I}}op[\boldsymbol{q}]^{-1}} & \boldsymbol{L}^2_t(\Gamma_{\text{ref}}) & & \end{array}$$

$$\begin{array}{ccccc} \boldsymbol{H}^{-\frac{1}{2}}_{\text{div}}(\Gamma_{\boldsymbol{q}}) & \xrightarrow{\mathrm{I}op[\boldsymbol{q}]^{-1}} & \boldsymbol{H}^{-\frac{1}{2}}_{\text{div}}(\Gamma_{\boldsymbol{q}}) & \xrightarrow{\mathscr{F}^{\infty}[\boldsymbol{q}]} & \boldsymbol{L}^2_t(\mathbb{S}^2) \\ \Pi_{\boldsymbol{q}} \downarrow & & \uparrow \Pi_{\boldsymbol{q}}^{-1} & \nearrow \widehat{\mathscr{F}}^{\infty}[\boldsymbol{q}] & \\ \boldsymbol{H}^{-\frac{1}{2}}_{\text{div}}(\Gamma_{\text{ref}}) & \xrightarrow{\hat{\mathrm{I}}op[\boldsymbol{q}]^{-1}} & \boldsymbol{H}^{-\frac{1}{2}}_{\text{div}}(\Gamma_{\text{ref}}) & & \end{array}$$

- We can use the Helmholtz decomposition of the spaces $\boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma_{\boldsymbol{q}})$

A. DE LA BOURDONNAYE, Décomposition de $H_{\mathrm{div}}^{-1/2}(\Gamma)$ et nature de l'opérateur de Steklov-Poincaré du problème extérieur de l'électromagnétisme, (1993)

$$\boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma_{\boldsymbol{q}}) = \nabla_{\Gamma_{\boldsymbol{q}}}\left(H^{\frac{3}{2}}(\Gamma_{\boldsymbol{q}})/\mathbb{R}\right) \bigoplus \overrightarrow{\mathbf{curl}}_{\Gamma_{\boldsymbol{q}}}\left(H^{\frac{1}{2}}(\Gamma_{\boldsymbol{q}})/\mathbb{R}\right)$$

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M. COSTABEL & F. LE LOUËR, Shape derivatives of boundary integral operators in electromagnetic scattering, (2012)

$$\begin{array}{llcl} \Pi_{\boldsymbol{q}} : & \boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma_{\boldsymbol{q}}) & \rightarrow & \boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma_{\mathrm{ref}}) \\ & \nabla_{\Gamma_{\boldsymbol{q}}} g_{\boldsymbol{q}} + \mathbf{curl}_{\Gamma_{\boldsymbol{q}}} h_{\boldsymbol{q}} & \mapsto & \nabla_{\Gamma_{\mathrm{ref}}}(g_{\boldsymbol{q}} \circ \boldsymbol{q}) + \overrightarrow{\mathbf{curl}}_{\Gamma_{\mathrm{ref}}}(h_{\boldsymbol{q}} \circ \boldsymbol{q}) \end{array}$$

- We can use the Helmholtz decomposition of the spaces $\boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma_{\boldsymbol{q}})$

A. DE LA BOURDONNAYE, Décomposition de $H_{\mathrm{div}}^{-1/2}(\Gamma)$ et nature de l'opérateur de Steklov-Poincaré du problème extérieur de l'électromagnétisme, (1993)

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M. COSTABEL & F. LE LOUËR, Shape derivatives of boundary integral operators in electromagnetic scattering, (2012)

$$\begin{array}{rrcl} \Pi_{\boldsymbol{q}}: & \boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma_{\boldsymbol{q}}) & \rightarrow & \boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma_{\mathrm{ref}}) \\ & \nabla_{\Gamma_{\boldsymbol{q}}} g_{\boldsymbol{q}} + \mathbf{curl}_{\Gamma_{\boldsymbol{q}}} h_{\boldsymbol{q}} & \mapsto & \nabla_{\Gamma_{\mathrm{ref}}}(g_{\boldsymbol{q}} \circ \boldsymbol{q}) + \overrightarrow{\mathbf{curl}}_{\Gamma_{\mathrm{ref}}}(h_{\boldsymbol{q}} \circ \boldsymbol{q}) \\ & \psi_{\boldsymbol{q}} & \mapsto & \nabla_{\Gamma_{\mathrm{ref}}}((\Delta_{\Gamma_{\boldsymbol{q}}}^{-1} \mathrm{div}_{\Gamma_{\boldsymbol{q}}} \psi_{\boldsymbol{q}}) \circ \boldsymbol{q}) - \overrightarrow{\mathbf{curl}}_{\Gamma_{\mathrm{ref}}}((\Delta_{\Gamma_{\boldsymbol{q}}}^{-1} \mathrm{curl}_{\Gamma_{\boldsymbol{q}}} \psi_{\boldsymbol{q}}) \circ \boldsymbol{q}) \end{array}$$

- We can use the Helmholtz decomposition of the spaces $\boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma_{\boldsymbol{q}})$

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\end{array}$$

$$\begin{array}{ccccc}
\boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma_{\boldsymbol{q}}) & \xrightarrow{\mathrm{I}op[\boldsymbol{q}]^{-1}} & \boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma_{\boldsymbol{q}}) & \xrightarrow{\mathscr{F}^{\infty}[\boldsymbol{q}]} & \boldsymbol{L}_{\mathrm{t}}^{2}(\mathbb{S}^2) \\
\Pi_{\boldsymbol{q}} \downarrow & & \uparrow \Pi_{\boldsymbol{q}}^{-1} & \nearrow \widehat{\mathscr{F}}^{\infty}[\boldsymbol{q}] & \\
\boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma_{\mathrm{ref}}) & \xrightarrow{\hat{\mathrm{I}}op[\boldsymbol{q}]^{-1}} & \boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma_{\mathrm{ref}}) & &
\end{array}$$

Main difficulties of the whole analysis

- We can use the Helmholtz decomposition of the spaces $\boldsymbol{H}_{\text{div}}^{-\frac{1}{2}}(\Gamma_{\boldsymbol{q}})$

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\end{array}$$

→ Drawback : We need to compute the Fréchet derivatives of $\tau_{\boldsymbol{q}} \nabla_{\Gamma_{\boldsymbol{q}}} \tau_{\boldsymbol{q}}^{-1}$, $\tau_{\boldsymbol{q}} \operatorname{div}_{\Gamma_{\boldsymbol{q}}} \tau_{\boldsymbol{q}}^{-1}$, $\tau_{\boldsymbol{q}} \overrightarrow{\textbf{curl}}_{\Gamma_{\boldsymbol{q}}} \tau_{\boldsymbol{q}}^{-1}$, $\tau_{\boldsymbol{q}} \operatorname{curl}_{\Gamma_{\boldsymbol{q}}} \tau_{\boldsymbol{q}}^{-1}$, $\tau_{\boldsymbol{q}} \Delta_{\Gamma_{\boldsymbol{q}}} \tau_{\boldsymbol{q}}^{-1}$ and its inverse.

J.-C. Nédélec, *Acoustic and electromagnetic equations* ...

- $(\boldsymbol{e}_1(x), \boldsymbol{e}_2(x))$ contravariant basis of the tangent plane $\mathbf{T}_x$ to Γ_{ref} at the point x
- We assume $\|\boldsymbol{e}_1 \times \boldsymbol{e}_2\| = 1 = \sqrt{\det G}$ where $G = (\boldsymbol{e}_i \cdot \boldsymbol{e}_j)_{1 \leqslant i,j \leqslant 2}$
- $\boldsymbol{n} = \boldsymbol{e}_1 \times \boldsymbol{e}_2$.

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- $f \in \mathscr{C}^1(\Gamma_{\mathrm{ref}})$: $f_{,1} = \nabla_{\Gamma_{\mathrm{ref}}} f \cdot \boldsymbol{e}_1$ and $f_{,2} = \nabla_{\Gamma_{\mathrm{ref}}} f \cdot \boldsymbol{e}_2 \Rightarrow \boxed{\nabla_{\Gamma_{\mathrm{ref}}} f = f_{,1}\boldsymbol{e}^1 + f_{,2}\boldsymbol{e}^2}$

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- $[\mathbf{D}_{\Gamma_{\text{ref}}}\boldsymbol{q}(x)]$ maps $\boldsymbol{T}_x$ onto the tangent plane $\mathbf{T}_{\boldsymbol{q}(x)}$ to $\Gamma_{\boldsymbol{q}}$ at the point $\boldsymbol{q}(x)$
- $\boldsymbol{q} : \Gamma_{\text{ref}} \to \Gamma_{\boldsymbol{q}}$ is a diffeomorphism $\Rightarrow$ let set $[\mathbf{D}_{\Gamma_{\text{ref}}}\boldsymbol{q}(x)]^{-1} = [\mathbf{D}_{\Gamma_{\boldsymbol{q}}}\boldsymbol{q}^{-1}] \circ \boldsymbol{q}(x)$
- ${}^{\mathsf{T}}[\mathbf{D}_{\Gamma_{\text{ref}}}\boldsymbol{q}(x)]^{-1}$ maps the cotangent plane $\boldsymbol{T}^*_x$ onto the cotangent plane $\mathbf{T}^*_{\boldsymbol{q}(x)}$

- Contravariant basis of the tangent plane $\mathbf{T}_{\boldsymbol{q}(x)}$: $\boldsymbol{e}_i[\boldsymbol{q}](x) = [\mathbf{D}_{\Gamma_{\text{ref}}}\boldsymbol{q}(x)]\boldsymbol{e}_i(x)$
- Covariant basis of the cotangent plane $\mathbf{T}^*_{\boldsymbol{q}(x)}$: $\boldsymbol{e}^i[\boldsymbol{q}](x) = {}^{\mathsf{T}}[\mathbf{D}_{\Gamma_{\text{ref}}}\boldsymbol{q}(x)]^{-1}\boldsymbol{e}^i(x)$
- Normal vector : $\boldsymbol{n}_{\boldsymbol{q}} \circ \boldsymbol{q} = \dfrac{\boldsymbol{e}^1[\boldsymbol{q}] \times \boldsymbol{e}_2[\boldsymbol{q}]}{\|\boldsymbol{e}^1[\boldsymbol{q}] \times \boldsymbol{e}_2[\boldsymbol{q}]\|} = \dfrac{{}^{\mathsf{T}}[\mathbf{D}_{\Gamma_{\text{ref}}}\boldsymbol{q}]^{-1}\boldsymbol{n}}{\|{}^{\mathsf{T}}[\mathbf{D}_{\Gamma_{\text{ref}}}\boldsymbol{q}]^{-1}\boldsymbol{n}\|}$

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- $\boxed{\tau_{\boldsymbol{q}}\nabla_{\Gamma_{\boldsymbol{q}}}(\tau_{\boldsymbol{q}}^{-1}f) = (\nabla_{\Gamma_{\boldsymbol{q}}}(f \circ \boldsymbol{q}^{-1})) \circ \boldsymbol{q} = f_{,1}\boldsymbol{e}^1[\boldsymbol{q}] + f_{,2}\boldsymbol{e}^2[\boldsymbol{q}] = {}^{\mathsf{T}}[\mathbf{D}_{\Gamma_{\mathrm{ref}}}\boldsymbol{q}(x)]^{-1}\nabla_{\Gamma_{\mathrm{ref}}}f}$

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- $\boxed{\tau_{\boldsymbol{q}}\nabla_{\Gamma_{\boldsymbol{q}}}(\tau_{\boldsymbol{q}}^{-1}f) = (\nabla_{\Gamma_{\boldsymbol{q}}}(f \circ \boldsymbol{q}^{-1})) \circ \boldsymbol{q} = f_{,1}\boldsymbol{e}^1[\boldsymbol{q}] + f_{,2}\boldsymbol{e}^2[\boldsymbol{q}] = {}^{\mathsf{T}}[\mathbf{D}_{\Gamma_{\mathrm{ref}}}\boldsymbol{q}(x)]^{-1}\nabla_{\Gamma_{\mathrm{ref}}}f}$
- $\tau_{\boldsymbol{q}}\overrightarrow{\mathbf{curl}}_{\Gamma_{\boldsymbol{q}}}(\tau_{\boldsymbol{q}}^{-1}f) = \tau_{\boldsymbol{q}}\nabla_{\Gamma_{\boldsymbol{q}}}(\tau_{\boldsymbol{q}}^{-1}f) \times \tau_{\boldsymbol{q}}\boldsymbol{n}_{\boldsymbol{q}} = J_{\boldsymbol{q}}^{-1}\left(f_{,2}\boldsymbol{e}_1[\boldsymbol{q}] - f_{,1}\boldsymbol{e}_2[\boldsymbol{q}]\right)$ **

  $\boxed{\tau_{\boldsymbol{q}}\overrightarrow{\mathbf{curl}}_{\Gamma_{\boldsymbol{q}}}(\tau_{\boldsymbol{q}}^{-1}f) = J_{\boldsymbol{q}}^{-1}[\mathbf{D}_{\Gamma_{\mathrm{ref}}}\boldsymbol{q}(x)]\overrightarrow{\mathbf{curl}}_{\Gamma_{\mathrm{ref}}}f}$ **
- $\boldsymbol{u} = u^1\boldsymbol{e}_1 + u^2\boldsymbol{e}_2 \Rightarrow \boxed{J_{\boldsymbol{q}}\tau_{\boldsymbol{q}}\,\mathrm{div}_{\Gamma_{\boldsymbol{q}}}(\tau_{\boldsymbol{q}}^{-1}J_{\boldsymbol{q}}^{-1}[\mathbf{D}_{\Gamma_{\mathrm{ref}}}\boldsymbol{q}(x)]\boldsymbol{u}) = u^1_{,1} + u^1_{,2} = \mathrm{div}_{\Gamma_{\mathrm{ref}}}\boldsymbol{u}}$ **
- $\boldsymbol{v} = v_1\boldsymbol{e}^1 + v_2\boldsymbol{e}^2 \Rightarrow \boxed{J_{\boldsymbol{q}}\tau_{\boldsymbol{q}}\,\mathrm{curl}_{\Gamma_{\boldsymbol{q}}}\tau_{\boldsymbol{q}}^{-1}({}^{\mathsf{T}}[\mathbf{D}_{\Gamma_{\mathrm{ref}}}\boldsymbol{q}(x)]^{-1}\boldsymbol{v}) = -\left(v_{1,2} - v_{2,1}\right) = \mathrm{curl}_{\Gamma_{\mathrm{ref}}}\boldsymbol{v}}$ **

** $J_{\boldsymbol{q}} = \|\boldsymbol{e}_1[\boldsymbol{q}] \times \boldsymbol{e}_2[\boldsymbol{q}]\| = \sqrt{\det G}$ *where* $G = (\boldsymbol{e}_i[\boldsymbol{q}] \cdot \boldsymbol{e}_j[\boldsymbol{q}])_{1 \leqslant i,j \leqslant 2}$

- Piola transform of $\boldsymbol{q}$: (a bicontinuous invertible operator)

$$\begin{array}{rccc} \mathcal{P}_{\boldsymbol{q}}: & \boldsymbol{H}_{\text{div}}^{-\frac{1}{2}}(\Gamma_{\boldsymbol{q}}) & \longrightarrow & \boldsymbol{H}_{\text{div}}^{-\frac{1}{2}}(\Gamma_{\text{ref}}) \\ & \psi_{\boldsymbol{q}} & \mapsto & \psi = J_{\boldsymbol{q}}[\mathbf{D}_{\Gamma_{\text{ref}}}\boldsymbol{q}]^{-1}(\psi_{\boldsymbol{q}} \circ \boldsymbol{q}), \end{array}$$

$$\begin{array}{ccccc} \boldsymbol{H}_{\text{div}}^{-\frac{1}{2}}(\Gamma_{\boldsymbol{q}}) & \xrightarrow{\mathrm{I}op[\boldsymbol{q}]^{-1}} & \boldsymbol{H}_{\text{div}}^{-\frac{1}{2}}(\Gamma_{\boldsymbol{q}}) & \xrightarrow{\mathscr{F}^{\infty}[\boldsymbol{q}]} & \boldsymbol{L}_{\text{t}}^{2}(\mathbb{S}^2) \\ \mathcal{P}_{\boldsymbol{q}}\downarrow & & \uparrow \mathcal{P}_{\boldsymbol{q}}^{-1} & \nearrow \widehat{\mathscr{F}}^{\infty}[\boldsymbol{q}] & \\ \boldsymbol{H}_{\text{div}}^{-\frac{1}{2}}(\Gamma_{\text{ref}}) & \xrightarrow{\hat{\mathrm{I}}op[\boldsymbol{q}]^{-1}} & \boldsymbol{H}_{\text{div}}^{-\frac{1}{2}}(\Gamma_{\text{ref}}) & & \end{array}$$

- $\boldsymbol{L}_{\text{t}}^{2}$ adjoint operator of the Piola transform :

$$\begin{array}{rccc} \mathcal{P}_{\boldsymbol{q}}^{*}: & \boldsymbol{H}_{\text{curl}}^{-\frac{1}{2}}(\Gamma_{\text{ref}}) & \longrightarrow & \boldsymbol{H}_{\text{curl}}^{-\frac{1}{2}}(\Gamma_{\boldsymbol{q}}) \\ & \psi & \mapsto & \psi_{\boldsymbol{q}} = \left({}^{\mathsf{T}}[\mathbf{D}_{\Gamma_{\text{ref}}}\boldsymbol{q}]^{-1}\psi\right) \circ \boldsymbol{q}^{-1}. \end{array}$$

$$\begin{array}{rcl} \mathcal{P}_{\boldsymbol{q}}\left(\text{Ker}[\text{div}_{\Gamma_{\boldsymbol{q}}}(\boldsymbol{H}_{\text{t}}^{-\frac{1}{2}}(\Gamma_{\boldsymbol{q}})]\right) & = & \text{Ker}[\text{div}_{\Gamma_{\text{ref}}}(\boldsymbol{H}_{\text{t}}^{-\frac{1}{2}}(\Gamma_{\text{ref}})] \,, \\ \text{Ker}[\text{curl}_{\Gamma_{\boldsymbol{q}}}(\boldsymbol{H}_{\text{t}}^{-\frac{1}{2}}(\Gamma_{\boldsymbol{q}})] & = & \mathcal{P}_{\boldsymbol{q}}^{*}\left(\text{Ker}[\text{curl}_{\Gamma_{\text{ref}}}(\boldsymbol{H}_{\text{t}}^{-\frac{1}{2}}(\Gamma_{\text{ref}})]\right) . \end{array}$$

- $\widehat{\boldsymbol{f}}[\boldsymbol{q}] = \mathcal{P}_{\boldsymbol{q}}(-\boldsymbol{n}_{\boldsymbol{q}} \times \boldsymbol{E}^{inc})_{|\Gamma_{\boldsymbol{q}}} = \boldsymbol{e}_1(\boldsymbol{E}^{inc} \cdot \boldsymbol{e}_2[\boldsymbol{q}]) - \boldsymbol{e}_2(\boldsymbol{E}^{inc} \cdot \boldsymbol{e}_1[\boldsymbol{q}]) \in \boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma_{\mathrm{ref}})$.
- $\widehat{\mathrm{I}}_{op}[\boldsymbol{q}] = \frac{1}{2}\mathrm{I} + \mathcal{P}_{\boldsymbol{q}} D_\kappa[\boldsymbol{q}]\mathcal{P}_{\boldsymbol{q}}^{-1} + i\eta\left(\mathcal{P}_{\boldsymbol{q}} S_\kappa[\boldsymbol{q}]\mathcal{P}_{\boldsymbol{q}}^{-1}\right)\left(\mathcal{P}_{\boldsymbol{q}}\boldsymbol{\Lambda}[\boldsymbol{q}]\mathcal{P}_{\boldsymbol{q}}^{-1}\right) : \boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma_{\mathrm{ref}}) \to \boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma_{\mathrm{ref}})$
- $\widehat{\mathscr{F}}^{\infty}[\boldsymbol{q}] = \mathscr{D}_\kappa^\infty[\boldsymbol{q}]\mathcal{P}_{\boldsymbol{q}}^{-1} + i\eta\mathscr{S}_\kappa^\infty[\boldsymbol{q}]\mathcal{P}_{\boldsymbol{q}}^{-1}\left(\mathcal{P}_{\boldsymbol{q}}\boldsymbol{\Lambda}[\boldsymbol{q}]\mathcal{P}_{\boldsymbol{q}}^{-1}\right) : \boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma_{\mathrm{ref}}) \to \boldsymbol{L}_{\mathrm{t}}^2(\mathbb{S}^2)$

- $\widehat{\boldsymbol{f}}[\boldsymbol{q}] = \mathcal{P}_{\boldsymbol{q}}(-\boldsymbol{n}_{\boldsymbol{q}} \times \boldsymbol{E}^{inc})_{|\Gamma_{\boldsymbol{q}}} = \boldsymbol{e}_1(\boldsymbol{E}^{inc} \cdot \boldsymbol{e}_2[\boldsymbol{q}]) - \boldsymbol{e}_2(\boldsymbol{E}^{inc} \cdot \boldsymbol{e}_1[\boldsymbol{q}]) \in \boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma_{\mathrm{ref}})$.
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- $\widehat{\mathscr{F}}^\infty[\boldsymbol{q}] = \mathscr{D}_\kappa^\infty[\boldsymbol{q}]\mathcal{P}_{\boldsymbol{q}}^{-1} + i\eta \mathscr{S}_\kappa^\infty[\boldsymbol{q}]\mathcal{P}_{\boldsymbol{q}}^{-1}\left(\mathcal{P}_{\boldsymbol{q}} \boldsymbol{\Lambda}[\boldsymbol{q}]\mathcal{P}_{\boldsymbol{q}}^{-1}\right) : \boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma_{\mathrm{ref}}) \to \boldsymbol{L}_{\mathrm{t}}^2(\mathbb{S}^2)$

- A result from the differential geometry : $\boldsymbol{n} \times (\mathcal{P}_{\boldsymbol{q}}\psi_{\boldsymbol{q}}) = \mathcal{P}_{\boldsymbol{q}}^{*\,-1}(\boldsymbol{n}_{\boldsymbol{q}} \times \psi_{\boldsymbol{q}})$.
- Let set $\widehat{\boldsymbol{\Lambda}}[\boldsymbol{q}] = \mathcal{P}_{\boldsymbol{q}}\boldsymbol{\Lambda}[\boldsymbol{q}]\mathcal{P}_{\boldsymbol{q}}^{-1}$ and $\psi = \mathcal{P}_{\boldsymbol{q}}\psi_{\boldsymbol{q}}$

$$\int_{\Gamma_{\boldsymbol{q}}} \boldsymbol{\Lambda}[\boldsymbol{q}]\psi_{\boldsymbol{q}} \cdot \left(\boldsymbol{n}_{\boldsymbol{q}} \times \overline{\psi}_{\boldsymbol{q}}\right) d\sigma = \int_{\Gamma_{\boldsymbol{q}}} (\mathcal{P}_{\boldsymbol{q}}^{-1}\widehat{\boldsymbol{\Lambda}}[\boldsymbol{q}]\mathcal{P}_{\boldsymbol{q}}\psi_{\boldsymbol{q}}) \cdot \left(\boldsymbol{n}_{\boldsymbol{q}} \times \overline{\psi}_{\boldsymbol{q}}\right) d\sigma$$
$$= \int_{\Gamma_{\mathrm{ref}}} (\widehat{\boldsymbol{\Lambda}}[\boldsymbol{q}]\mathcal{P}_{\boldsymbol{q}}\psi_{\boldsymbol{q}}) \cdot \mathcal{P}_{\boldsymbol{q}}^{*\,-1}\left(\boldsymbol{n}_{\boldsymbol{q}} \times \overline{\psi}_{\boldsymbol{q}}\right) d\sigma = \int_{\Gamma_{\mathrm{ref}}} (\widehat{\boldsymbol{\Lambda}}[\boldsymbol{q}]\psi) \cdot \left(\boldsymbol{n} \times \overline{\psi}\right) d\sigma$$

Alternative choice for the regularizing operator

- $\widehat{\boldsymbol{f}}[\boldsymbol{q}] = \mathcal{P}_{\boldsymbol{q}}(-\boldsymbol{n}_{\boldsymbol{q}} \times \boldsymbol{E}^{inc})_{|\Gamma_{\boldsymbol{q}}} = \boldsymbol{e}_1(\boldsymbol{E}^{inc} \cdot \boldsymbol{e}_2[\boldsymbol{q}]) - \boldsymbol{e}_2(\boldsymbol{E}^{inc} \cdot \boldsymbol{e}_1[\boldsymbol{q}]) \in \boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma_{\mathrm{ref}})$.
- $\widehat{\mathrm{I}}_{op}[\boldsymbol{q}] = \frac{1}{2}\mathrm{I} + \mathcal{P}_{\boldsymbol{q}} D_\kappa[\boldsymbol{q}]\mathcal{P}_{\boldsymbol{q}}^{-1} + i\eta\left(\mathcal{P}_{\boldsymbol{q}} S_\kappa[\boldsymbol{q}]\mathcal{P}_{\boldsymbol{q}}^{-1}\right)\left(\mathcal{P}_{\boldsymbol{q}}\boldsymbol{\Lambda}[\boldsymbol{q}]\mathcal{P}_{\boldsymbol{q}}^{-1}\right) : \boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma_{\mathrm{ref}}) \to \boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma_{\mathrm{ref}})$
- $\widehat{\mathscr{F}}^\infty[\boldsymbol{q}] = \mathscr{D}_\kappa^\infty[\boldsymbol{q}]\mathcal{P}_{\boldsymbol{q}}^{-1} + i\eta\mathscr{S}_\kappa^\infty[\boldsymbol{q}]\mathcal{P}_{\boldsymbol{q}}^{-1}\left(\mathcal{P}_{\boldsymbol{q}}\boldsymbol{\Lambda}[\boldsymbol{q}]\mathcal{P}_{\boldsymbol{q}}^{-1}\right) : \boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\Gamma_{\mathrm{ref}}) \to \boldsymbol{L}_{\mathrm{t}}^2(\mathbb{S}^2)$

- A result from the differential geometry : $\boldsymbol{n} \times (\mathcal{P}_{\boldsymbol{q}}\psi_{\boldsymbol{q}}) = \mathcal{P}_{\boldsymbol{q}}^{*\,-1}(\boldsymbol{n}_{\boldsymbol{q}} \times \psi_{\boldsymbol{q}})$.
- Let set $\widehat{\boldsymbol{\Lambda}}[\boldsymbol{q}] = \mathcal{P}_{\boldsymbol{q}}\boldsymbol{\Lambda}[\boldsymbol{q}]\mathcal{P}_{\boldsymbol{q}}^{-1}$ and $\psi = \mathcal{P}_{\boldsymbol{q}}\psi_{\boldsymbol{q}}$

$$\int_{\Gamma_{\boldsymbol{q}}} \boldsymbol{\Lambda}[\boldsymbol{q}]\psi_{\boldsymbol{q}} \cdot \left(\boldsymbol{n}_{\boldsymbol{q}} \times \overline{\psi_{\boldsymbol{q}}}\right) d\sigma = \int_{\Gamma_{\boldsymbol{q}}} (\mathcal{P}_{\boldsymbol{q}}^{-1}\widehat{\boldsymbol{\Lambda}}[\boldsymbol{q}]\mathcal{P}_{\boldsymbol{q}}\psi_{\boldsymbol{q}}) \cdot \left(\boldsymbol{n}_{\boldsymbol{q}} \times \overline{\psi_{\boldsymbol{q}}}\right) d\sigma$$

$$= \int_{\Gamma_{\mathrm{ref}}} (\widehat{\boldsymbol{\Lambda}}[\boldsymbol{q}]\mathcal{P}_{\boldsymbol{q}}\psi_{\boldsymbol{q}}) \cdot \mathcal{P}_{\boldsymbol{q}}^{*\,-1}\left(\boldsymbol{n}_{\boldsymbol{q}} \times \overline{\psi_{\boldsymbol{q}}}\right) d\sigma = \int_{\Gamma_{\mathrm{ref}}} (\widehat{\boldsymbol{\Lambda}}[\boldsymbol{q}]\psi) \cdot \left(\boldsymbol{n} \times \overline{\psi}\right) d\sigma$$

- We can choose $\widehat{\boldsymbol{\Lambda}}[\boldsymbol{q}] = \widehat{\boldsymbol{\Lambda}}$ independent on $\Gamma_{\boldsymbol{q}}$ but $\int_{\Gamma_{\mathrm{ref}}} \widehat{\boldsymbol{\Lambda}}\psi \cdot (\boldsymbol{n} \times \overline{\psi})\, d\sigma \geqslant c||\psi||^2$
- Example : If $\Gamma_{\mathrm{ref}} = \mathbb{S}^2$: for $t \geqslant 0$,

$$\widehat{\boldsymbol{\Lambda}} = \mathbf{curl}_{\mathbb{S}^2}(-\Delta_{\mathbb{S}^2})^{-(\frac{1}{2}+t)}\,\mathrm{div}_{\mathbb{S}^2} + \nabla_{\mathbb{S}^2}(-\Delta_{\mathbb{S}^2})^{-(\frac{3}{2}+t)}\,\mathrm{curl}_{\mathbb{S}^2}$$

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- $\widehat{\boldsymbol{f}}[\boldsymbol{q}] = \mathcal{P}_{\boldsymbol{q}}(-\boldsymbol{n}_{\boldsymbol{q}} \times \boldsymbol{E}^{inc})_{|\Gamma_{\boldsymbol{q}}} = \boldsymbol{e}_1(\boldsymbol{E}^{inc} \cdot \boldsymbol{e}_2[\boldsymbol{q}]) - \boldsymbol{e}_2(\boldsymbol{E}^{inc} \cdot \boldsymbol{e}_1[\boldsymbol{q}]) \in \boldsymbol{H}_{\text{div}}^{-\frac{1}{2}}(\Gamma_{\text{ref}})$.
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- $\widehat{\mathscr{F}^{\infty}}[\boldsymbol{q}] = \mathscr{D}_{\kappa}^{\infty}[\boldsymbol{q}]\mathcal{P}_{\boldsymbol{q}}^{-1} + i\eta\mathscr{S}_{\kappa}^{\infty}[\boldsymbol{q}]\mathcal{P}_{\boldsymbol{q}}^{-1}\widehat{\boldsymbol{\Lambda}} : \boldsymbol{H}_{\text{div}}^{-\frac{1}{2}}(\Gamma_{\text{ref}}) \to \boldsymbol{L}_{t}^{2}(\mathbb{S}^2)$

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J.-C. Nédélec, *Acoustic and electromagnetic equations* ...

- Integral representation of $\widehat{S}_\kappa[\boldsymbol{q},\psi] = \mathcal{P}_{\boldsymbol{q}} S_\kappa[\boldsymbol{q}]\mathcal{P}_{\boldsymbol{q}}^{-1}\psi$

$$\begin{aligned}\widehat{S}_\kappa[\boldsymbol{q},\psi](x) \quad = \quad & \kappa\, \boldsymbol{n}(x) \times \int_{\Gamma_{\text{ref}}} {}^{\mathsf{T}}[\mathbf{D}_{\Gamma_{\text{ref}}}\boldsymbol{q}(\mathbf{x})]\{G(\kappa, \boldsymbol{q}(x) - \boldsymbol{q}(y))[\mathbf{D}_{\Gamma_{\text{ref}}}\boldsymbol{q}(y)]\psi(y)\}d\sigma(y) \\ & -\frac{1}{\kappa}\left(\overrightarrow{\mathbf{curl}}_{\Gamma_{\text{ref}}}\int_{\Gamma_{\text{ref}}} G(\kappa, \boldsymbol{q}(\cdot) - \boldsymbol{q}(y))\, \operatorname{div}_{\Gamma_{\text{ref}}} \psi(y)d\sigma(y)\right)(x)\end{aligned}$$

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- We have $G(\kappa, \boldsymbol{q}(x) - \boldsymbol{q}(y)) \underset{\|x-y\|\to 0}{\rightsquigarrow} O(\|x-y\|^{-1})$

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$$\begin{aligned}\widehat{S}_\kappa[\boldsymbol{q},\psi](x) &= \kappa\, \boldsymbol{n}(x) \times \int_{\Gamma_{\text{ref}}} {}^{T}[\mathbf{D}_{\Gamma_{\text{ref}}}\boldsymbol{q}(\mathbf{x})]\{G(\kappa,\boldsymbol{q}(x)-\boldsymbol{q}(y))[\mathbf{D}_{\Gamma_{\text{ref}}}\boldsymbol{q}(y)]\psi(y)\}d\sigma(y)\\ &\quad -\frac{1}{\kappa}\left(\overrightarrow{\textbf{curl}}_{\Gamma_{\text{ref}}}\int_{\Gamma_{\text{ref}}} G(\kappa,\boldsymbol{q}(\cdot)-\boldsymbol{q}(y))\,\mathrm{div}_{\Gamma_{\text{ref}}}\psi(y)d\sigma(y)\right)(x)\end{aligned}$$

- We have $G(\kappa,\boldsymbol{q}(x)-\boldsymbol{q}(y)) \underset{\|x-y\|\to 0}{\rightsquigarrow} O(\|x-y\|^{-1})$

- Computing $\partial_{\boldsymbol{q}}\widehat{S}_\kappa[\boldsymbol{q},\psi]\boldsymbol{\xi} \Leftrightarrow$ Computing the Fréchet derivative of its kernel
  1. $\boldsymbol{e}_i[\boldsymbol{q}] = [\mathbf{D}_{\Gamma_{\text{ref}}}\boldsymbol{q}]\boldsymbol{e}_i \Rightarrow \partial_{\boldsymbol{q}}\boldsymbol{e}_i[\boldsymbol{q}]\boldsymbol{\xi} = [\mathbf{D}_{\Gamma_{\text{ref}}}\boldsymbol{\xi}]\boldsymbol{e}_i$
  2. R. Potthast, *Domain derivatives in electromagnetic scattering*, (1996)

$$\partial_{\boldsymbol{q}}\{G(\kappa,\boldsymbol{q}(x)-\boldsymbol{q}(y))\}\boldsymbol{\xi} = \nabla^{\boldsymbol{q}(x)}\{G(\kappa,\boldsymbol{q}(x)-\boldsymbol{q}(y))\}\cdot(\boldsymbol{\xi}(x)-\boldsymbol{\xi}(y)) \rightsquigarrow O(\|x-y\|^{-1})$$

- Integral representation of $\widehat{S}_\kappa[\boldsymbol{q},\psi] = \mathcal{P}_{\boldsymbol{q}} S_\kappa[\boldsymbol{q}]\mathcal{P}_{\boldsymbol{q}}^{-1}\psi$

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- $\mathrm{div}_{\Gamma_{\mathrm{ref}}}\partial_{\boldsymbol{q}}\widehat{S}_\kappa[\boldsymbol{q},\psi]\boldsymbol{\xi} = \partial_{\boldsymbol{q}}(\mathrm{div}_{\Gamma_{\mathrm{ref}}}\widehat{S}_\kappa[\boldsymbol{q},\psi])\boldsymbol{\xi}$

$$\mathrm{div}_{\Gamma_{\mathrm{ref}}}\widehat{S}_\kappa[\boldsymbol{q},\psi] = -\kappa\ \mathrm{curl}_{\Gamma_{\mathrm{ref}}}\int_{\Gamma_{\mathrm{ref}}} {}^{\mathsf{T}}[\mathbf{D}_{\Gamma_{\mathrm{ref}}}\boldsymbol{q}(\mathbf{x})]\{G(\kappa, \boldsymbol{q}(x)-\boldsymbol{q}(y))[\mathbf{D}_{\Gamma_{\mathrm{ref}}}\boldsymbol{q}(y)]\psi(y)\}d\sigma(y)$$

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$$\begin{aligned}\widehat{S}_\kappa[\boldsymbol{q},\psi](x) \quad = \quad & \kappa\, \boldsymbol{n}(x) \times \int_{\Gamma_{\text{ref}}} {}^{\mathsf{T}}[\mathbf{D}_{\Gamma_{\text{ref}}}\boldsymbol{q}(\mathbf{x})]\{G(\kappa,\boldsymbol{q}(x)-\boldsymbol{q}(y))[\mathbf{D}_{\Gamma_{\text{ref}}}\boldsymbol{q}(y)]\psi(y)\}d\sigma(y)\\ & -\frac{1}{\kappa}\left(\overrightarrow{\textbf{curl}}_{\Gamma_{\text{ref}}}\int_{\Gamma_{\text{ref}}} G(\kappa,\boldsymbol{q}(\cdot)-\boldsymbol{q}(y))\,\operatorname{div}_{\Gamma_{\text{ref}}}\psi(y)d\sigma(y)\right)(x)\end{aligned}$$

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- Computing $\partial_{\boldsymbol{q}}\widehat{S}_\kappa[\boldsymbol{q},\psi]\boldsymbol{\xi} \Leftrightarrow$ Computing the Fréchet derivative of its kernel
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- $\operatorname{div}_{\Gamma_{\text{ref}}}\partial_{\boldsymbol{q}}\widehat{S}_\kappa[\boldsymbol{q},\psi]\boldsymbol{\xi} = \partial_{\boldsymbol{q}}(\operatorname{div}_{\Gamma_{\text{ref}}}\widehat{S}_\kappa[\boldsymbol{q},\psi])\boldsymbol{\xi}$

$$\operatorname{div}_{\Gamma_{\text{ref}}}\widehat{S}_\kappa[\boldsymbol{q},\psi] = -\kappa\,\operatorname{curl}_{\Gamma_{\text{ref}}}\int_{\Gamma_{\text{ref}}} {}^{\mathsf{T}}[\mathbf{D}_{\Gamma_{\text{ref}}}\boldsymbol{q}(\mathbf{x})]\{G(\kappa,\boldsymbol{q}(x)-\boldsymbol{q}(y))[\mathbf{D}_{\Gamma_{\text{ref}}}\boldsymbol{q}(y)]\psi(y)\}d\sigma(y)$$

- We conclude $\partial_{\boldsymbol{q}}\widehat{S}_\kappa[\boldsymbol{q},\psi]\boldsymbol{\xi} \in \boldsymbol{H}_{\text{div}}^{-\frac{1}{2}}(\Gamma_{\text{ref}})$

- Done :
  1. Numerical implementation of the forward problem :
     - M. Ganesh & I. G. Graham, *A high-order algorithm for obstacle scattering in three dimensions*, (2004)
     - M. Ganesh & S. C. Hawkins, *A high-order tangential basis algorithm for electromagnetic scattering by curved surfaces*, (2007)

     → The Piola transform renders possible the implementation of hypersingular integral equations

     - T. Hohage & F. Le Louër, *A spectrally accurate method for the dielectric obstacle scattering problem and applications to the inverse problem*, (2013)
     - F. Le Louër, *Spectrally accurate numerical solution of hypersingular boundary integral equations for three-dimensional electromagnetic wave scattering problems*, (2014)
  2. Fréchet differentiability analysis :
     → Using the Piola transform is the best way to tackle the problem
  3. Numerical implementation of the whole algorithm 2
     - O. Ivanyshyn Yaman & F. Le Louër, *Material derivatives of boundary integral operators and application to inverse problems*, soumis (2016)
- In progress
  1. Extension to multiple obstacles.
  2. Srategies to find initial guesses.
     → geometric optimization tools + topological optimization tools
- Future work
  1. Other boundary conditions
  2. Elastodynamics