High-Order Explicit Local Time-Stepping Methods For Wave Propagation

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joint work with:
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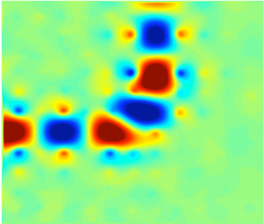
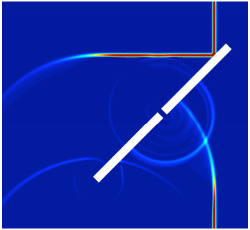
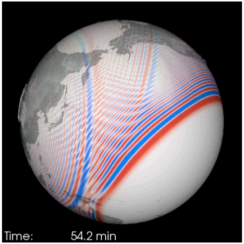
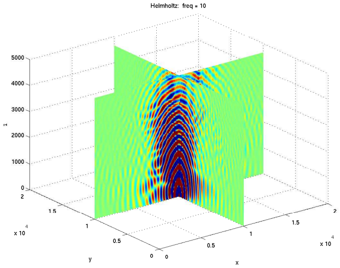
Wave Phenomena

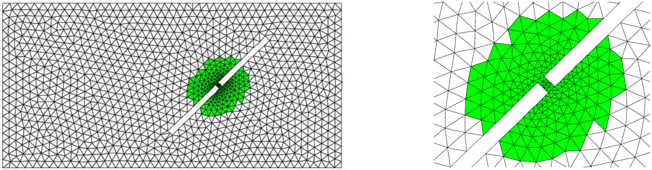




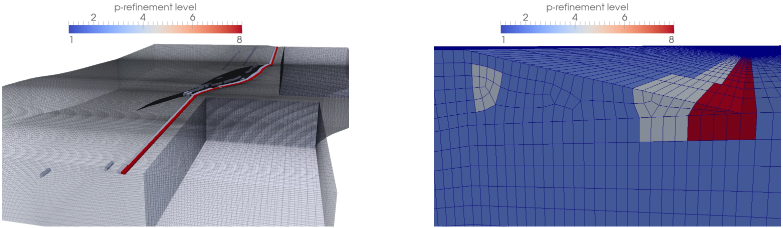




Adaptive Mesh Refinement



geometric features



Tohoku fault: mesh generation

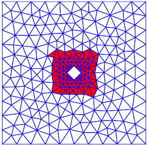
Overcoming Geometry Induced Stiffness

Problem

Locally refined meshes induce severe stability restrictions for explicit time-stepping schemes.

Solutions

- Locally implicit schemes e.g. Ascher 1995, Piperno 2006, Verwer 2009, Lantéri et al. 2010, 2013, Chabassier et al 2015
- Explicit local time-stepping (LTS) schemes in this talk!
- Local exponential integrators Hochbruck et al. 2011



High-order Local Time Stepping (LTS) Methods

Outline:

- The (damped) wave equation
- CG, IP-DG and nodal DG FE discretizations
- LTS methods: previous work
- Runge-Kutta based LTS methods
- Multi-level leap-frog based LTS methods
- Parallel performance
- Concluding remarks

The (Damped) Wave Equation

Model problem (second-order form)

$$
\begin{aligned}
u_{tt} + \sigma u_t - \nabla \cdot (\, c \, \nabla u) &= f && \text{in } \Omega \times (0, T) \\
u &= 0 && \text{on } \partial\Omega \times (0, T) \\
u\big|_{t=0} = u_0 \, , u_t\big|_{t=0} &= v_0 && \text{in } \Omega
\end{aligned}
$$

- $\Omega \subset \mathbb{R}^d$ bounded, $\sigma(\mathbf{x}) \geq 0$, $c(\mathbf{x}) > 0$

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- $\Omega \subset \mathbb{R}^d$ bounded, $\sigma(\mathbf{x}) \geq 0$, $c(\mathbf{x}) > 0$

Weak formulation Find $u \in C^0(0,T; H_0^1(\Omega)) \cap C^1(0,T; L^2(\Omega))$:

$$\langle u_{tt}, v\rangle_{(H^{-1},H_0^1)} + (\sigma u_t, v) + a(u,v) = (f,v), \quad \forall v \in H_0^1(\Omega),$$

$$a(u,v) = (c\,\nabla u, \nabla v)$$

The (Damped) Wave Equation

Model problem (second-order form)

$$\begin{aligned} u_{tt} + \sigma u_t - \nabla\cdot(c\,\nabla u) &= f && \text{in } \Omega\times(0,T) \\ u &= 0 && \text{on } \partial\Omega\times(0,T) \\ u|_{t=0} = u_0\,,\, u_t|_{t=0} &= v_0 && \text{in } \Omega \end{aligned}$$

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$$a(u,v) = (c\,\nabla u, \nabla v)$$

Energy conservation For $\sigma = 0$, $f = 0$ the energy

$$E[u](t) := \frac{1}{2}\left[\|u_t\|^2 + a(u,u)\right] \equiv \text{const.}$$

Second-order semi-discrete FE formulations

- Conforming mass-lumped FEM: (*Cohen-Joly-Roberts-Tordjman, SINUM, 2001*)

$$a(u,\varphi) := \sum_{K\in\mathcal{T}_h} \int_K c\nabla u\cdot\nabla\varphi\,dx$$

- IP-DG FEM: (*G.-Schneebeli-Schötzau, SINUM 2006*)

$$a_{DG}(u,\varphi) := \sum_{K\in\mathcal{T}_h} \int_K c\nabla u\cdot\nabla\varphi\,dx - \sum_{e\in\mathcal{E}_h} \int_e [\![\varphi]\!]\cdot\{\!\{c\nabla u\}\!\}\,dA$$
$$- \sum_{e\in\mathcal{E}_h} \int_e [\![u]\!]\cdot\{\!\{c\nabla\varphi\}\!\}\,dA + \sum_{e\in\mathcal{E}_h} \mathsf{a}[\![u]\!]\cdot[\![\varphi]\!]\,dA$$

The (Damped) Wave Equation

Model problem (first-order form, $v := u_t$ and $\mathbf{w} := -\nabla u$)

$$
\begin{aligned}
v_t + \sigma v + \nabla \cdot (c\,\mathbf{w}) &= f && \text{in } \; \Omega \times (0,T) \\
\mathbf{w}_t + \nabla v &= \mathbf{0} && \text{in } \; \Omega \times (0,T) \\
v &= 0 && \text{on } \; \partial\Omega \times (0,T) \\
v|_{t=0} = v_0\,, \;\; \mathbf{w}|_{t=0} &= -\nabla u_0 && \text{in } \Omega
\end{aligned}
$$

$$
\mathbf{q}_t + \boldsymbol{\Sigma}\,\mathbf{q} + \nabla \cdot \mathcal{F}(\mathbf{q}) = \mathbf{S} \qquad \text{with} \;\; \mathbf{q} = (v, \mathbf{w})^t
$$

Nodal DG FE Formulation

Find $\mathbf{q}^h : [0,T] \times \mathbf{V}^h \to \mathbb{R}$ such that

$$(\mathbf{q}_t^h, \boldsymbol{\psi}) + (\boldsymbol{\Sigma}\, \mathbf{q}^h, \boldsymbol{\psi}) + a_{DG}(\mathbf{q}^h, \boldsymbol{\psi}) = (\mathbf{S}, \boldsymbol{\psi}) \quad \forall\, \boldsymbol{\psi} \in \mathbf{V}^h\,, \quad t \in (0,T)\,.$$

- Nodal DG FEM: (*Hesthaven-Warburton, Springer, 2008*)

$$a_{DG}(\mathbf{q}, \boldsymbol{\psi}) := \sum_{K \in \mathcal{T}_h} \int_K (\nabla \cdot \mathcal{F}(\mathbf{q})) \cdot \boldsymbol{\psi}\, dx - \sum_{e \in \mathcal{E}_h} \int_e (\mathbf{n} \cdot \mathcal{F}(\mathbf{q}) - (\mathbf{n} \cdot \mathcal{F}(\mathbf{q}))^*) \cdot \boldsymbol{\psi}\, dA$$

Here, $(\mathbf{n} \cdot \mathcal{F}(\mathbf{q}))^*$ denotes a suitable numerical flux in the unit normal direction $\mathbf{n}$.

Semi-Discrete Galerkin FE Formulations

The discretization in space leads to a system of ODE's

$$\mathbf{M}\,\frac{d^2\mathbf{U}}{dt^2}(t) + \mathbf{M}_\sigma\,\frac{d\mathbf{U}}{dt}(t) + \mathbf{K}\,\mathbf{U}(t) = \mathbf{R}(t)\,, \qquad t \in (0,T)$$

or

$$\mathbf{M}\frac{d\mathbf{Q}}{dt}(t) + \mathbf{M}_\sigma\,\mathbf{Q}(t) + \mathbf{K}\,\mathbf{Q}(t) = \mathbf{R}(t)\,, \qquad t \in (0,T)\,.$$

The stiffness matrix $\mathbf{K}$ and the mass matrix $\mathbf{M}$ are sparse. Moreover, the mass matrix $\mathbf{M}$ is SPD and (block-)diagonal

$\Rightarrow$ computing $\mathbf{M}^{-1}$ is cheap $\Rightarrow$ fully explicit time-stepping!

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adaptivity, small geometric features

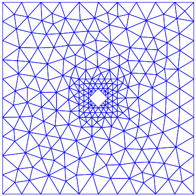
$\Downarrow$

locally refined meshes

$\Downarrow$

CFL condition for explicit time-stepping

$$\Delta t \le C\,h, \qquad h = \min_{T\in\mathcal{T}_h} h_T$$



Multirate Time-Stepping for ODEs / Previous Work

- *Rice, J. Res. Nat. Bureau Stand.-B 1960*
  - Split Runge-Kutta methods
- *Gear-Wells, BIT 1984*
  - Multirate linear multistep methods: "fast-first", "slow-first"
- *Günther-Kværnø-Rentrop, BIT 2001*
  - Multirate partitioned (IMEX) Runge-Kutta methods
- *Leimkuhler-Reich, JCP 2001*
  - The reversible averaging (RA) method
- *Hairer-Lubich-Wanner, Geometric Numerical Integration 2002*
  - Multiple time-stepping for ODEs
- *Savcenco-Hundsdorfer-Verwer, BIT, 2007*
  - Multirate (IMEX) time-stepping strategy for stiff ODEs
- *A. Klöckner, PhD thesis, 2010*
  - Multirate ABk time-stepping (Gear-Wells type)

Explicit LTS for PDEs / Previous Work

- *Berger and Oliger, JCP 1984*
  - AMR method, based on rectangular FD patches (AMROC)
- *Collino et al., Numer. Math. 2003, JCP 2006*; *Piperno, M2AN 2006*
  - Sympletic second-order Störmer-Verlet
- *Dumbser et al., Geophys. J. Int. 2007*; *Int. J. Numer. Model. 2009*
  - LTS ADER-DG schemes
- *Constantinescu-Sandu, J. Sc. Comp. 2007, 2009*
  - Multirate time integration, limited to second order accuracy
- *Diaz-G., SISC 2009, CMAME 2015*
  - $\sigma = 0$: LTS-LF of arbitrarily high accuracy, multi-level version
- *G.-Mitkova, JCAM 2010, 2013*
  - $\sigma \geq 0$: LTS-AB of arbitrarily high accuracy
- *Hochbruck-Ostermann, BIT 2011*
  - Exponential multistep methods of Adams type

RK Based Explicit LTS

Advantages of RK methods:

- One-step method, no starting procedure
- Time adaptivity straightforward
- Larger stability regions (but more work per step)
- Low storage (LSRK) versions available

- *Knoth et al., BIT 2009, JCAM 2009*
  - Multirate RK for advection equations, 3d order
- *Liu, Li, Hu, JCP 2010*
  - Non-uniform LDDRK-DG for CFD, linear coupling conditions

"...the availability of extrapolation from past values is an advantage for multistep methods over Runge-Kutta methods in the multirate context."

(Gear-Wells, BIT, 1984)

RK Based Explicit LTS Methods

Goal: Derive Runge-Kutta (RK) based explicit LTS methods for

$$\frac{d\mathbf{y}}{dt}(t) = \mathbf{B}\mathbf{y}(t) + \mathbf{F}(t)\,, \quad t \in (0, T)\,. \tag{1}$$

$\mathbf{B}$ involves the factor $\mathbf{M}^{-1}$. The mass matrix $\mathbf{M}$ is (block-)diagonal

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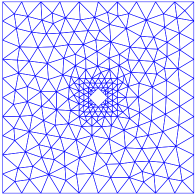

RK-methods and numerical integration

$$\boxed{y'(t) = f(y(t), t)), \; y(0) = y_0}$$

$$
\begin{aligned}
k_1 &= f(y_n, t_n), \\
k_2 &= f(y_n + \Delta t a_{21} k_1, t_n + c_2 \Delta t), \\
&\vdots \\
k_s &= f(y_n + \Delta t \sum_{i=1}^{s-1} a_{si} k_i, t_n + c_s \Delta t),
\end{aligned}
$$

$$y(t_{n+1}) \approx y_{n+1} = y_n + \Delta t \sum_{i=1}^{s} b_i k_i.$$

$$
\begin{array}{c|ccccc}
0 & & & & \\
c_2 & a_{21} & & & \\
c_3 & a_{31} & a_{32} & & \\
\vdots & \vdots & \vdots & \ddots & \\
c_s & a_{s1} & \dots & a_{s,s-1} & \\
\hline
 & b_1 & \dots & b_{s-1} & b_s
\end{array}
$$

Butcher-tableau of an explicit RKs scheme of order k.

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c_2 & a_{21} & & & \\
c_3 & a_{31} & a_{32} & & \\
\vdots & \vdots & \vdots & \ddots & \\
c_s & a_{s1} & \dots & a_{s,s-1} & \\
\hline
 & b_1 & \dots & b_{s-1} & b_s
\end{array}$$

Butcher-tableau of an explicit RKs scheme of order k.

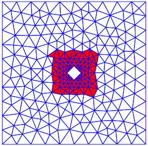
Underlying quadrature formula with weights $b_1, \dots, b_s$ and nodes $0, c_2, \dots, c_s$ has at least order k.

RK Based Explicit LTS

Let us now split $\mathbf{y}$ and $\mathbf{F}$ in two parts

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}^{\text{coarse}} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{y}^{\text{fine}} \end{bmatrix} = (\mathbf{I} - \mathbf{P})\mathbf{y} + \mathbf{P}\mathbf{y}, \;\; \mathbf{P}^2 = \mathbf{P},$$

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}^{\text{coarse}} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{F}^{\text{fine}} \end{bmatrix} = (\mathbf{I} - \mathbf{P})\mathbf{F} + \mathbf{P}\mathbf{F}, \;\; \mathbf{P}^2 = \mathbf{P}.$$



Then, we have

$$\frac{d}{dt}\mathbf{y} = \mathbf{B}\mathbf{y} + \mathbf{F} = \mathbf{B}(\mathbf{I} - \mathbf{P})\mathbf{y} + \mathbf{B}\mathbf{P}\mathbf{y} + (\mathbf{I} - \mathbf{P})\mathbf{F} + \mathbf{P}\mathbf{F}$$

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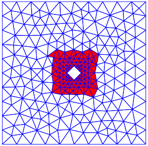

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or

$$\begin{aligned} \mathbf{y}(t_n + \xi\Delta t) = \mathbf{y}(t_n) &+ \int_{t_n}^{t_n+\xi\Delta t} \mathbf{B}(\mathbf{I} - \mathbf{P})\mathbf{y}(t) + (\mathbf{I} - \mathbf{P})\mathbf{F}(t)\, dt \\ &+ \int_{t_n}^{t_n+\xi\Delta t} \mathbf{B}\mathbf{P}\mathbf{y}(t) + \mathbf{P}\mathbf{F}(t)\, dt\,. \end{aligned}$$

RK Based Explicit LTS/ LTS-RK2(p)

Coarse part

$$\int_{t_n}^{t_n+\xi\Delta t} \mathbf{B}(\mathbf{I}-\mathbf{P})\mathbf{y}(t) + (\mathbf{I}-\mathbf{P})\mathbf{F}(t)\, dt$$

$$\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & 1/2 & 1/2 \end{array}$$

coeff. of RK2

RK Based Explicit LTS/ LTS-RK2(p)

Coarse part

$$\int_{t_n}^{t_n+\xi\Delta t} \mathbf{B}(\mathbf{I}-\mathbf{P})\mathbf{y}(t) + (\mathbf{I}-\mathbf{P})\mathbf{F}(t)\,dt$$

0		
1	1	
	1/2	1/2

coeff. of RK2

$$\int_{t_n}^{t_n+\xi\Delta t} \mathbf{B}(\mathbf{I}-\mathbf{P})\mathbf{y}(t)\,dt \approx \frac{\xi\Delta t}{2}\mathbf{B}(\mathbf{I}-\mathbf{P})\left(\mathbf{y}(t_n)+\mathbf{y}(t_n+\xi\Delta t)\right) \quad \text{(QF)}$$

$$\approx \frac{\xi\Delta t}{2}\Big[\mathbf{B}(\mathbf{I}-\mathbf{P})\mathbf{y}_n + \mathbf{B}(\mathbf{I}-\mathbf{P})\big(\mathbf{y}_n+\xi\Delta t\,(\mathbf{B}\mathbf{y}_n+\mathbf{F}_n)\big)\Big] \quad \text{(Taylor)}$$

$$= \xi\Delta t\mathbf{B}(\mathbf{I}-\mathbf{P})\Big[\mathbf{y}_n + \frac{\xi\Delta t}{2}(\mathbf{B}\mathbf{y}_n+\mathbf{F}_n)\Big]$$

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$$\int_{t_n}^{t_n+\xi\Delta t} \mathbf{B}(\mathbf{I}-\mathbf{P})\mathbf{y}(t) + (\mathbf{I}-\mathbf{P})\mathbf{F}(t)\,dt$$

0		
1	1	
	1/2	1/2

coeff. of RK2

$$\int_{t_n}^{t_n+\xi\Delta t} \mathbf{B}(\mathbf{I}-\mathbf{P})\mathbf{y}(t)\,dt \approx \frac{\xi\Delta t}{2}\mathbf{B}(\mathbf{I}-\mathbf{P})\left(\mathbf{y}(t_n)+\mathbf{y}(t_n+\xi\Delta t)\right) \quad \text{(QF)}$$

$$\approx \frac{\xi\Delta t}{2}\Big[\mathbf{B}(\mathbf{I}-\mathbf{P})\mathbf{y}_n + \mathbf{B}(\mathbf{I}-\mathbf{P})\big(\mathbf{y}_n + \xi\Delta t\,(\mathbf{B}\mathbf{y}_n+\mathbf{F}_n)\big)\Big] \quad \text{(Taylor)}$$

$$= \xi\Delta t\mathbf{B}(\mathbf{I}-\mathbf{P})\Big[\mathbf{y}_n + \frac{\xi\Delta t}{2}(\mathbf{B}\mathbf{y}_n+\mathbf{F}_n)\Big]$$

We replace $(\mathbf{I}-\mathbf{P})\mathbf{F}(t)$ by $(\mathbf{I}-\mathbf{P})\mathbf{q}(t)$, where $\mathbf{q}(t)$ is the interpolation polynomial through the points $(t_n, \mathbf{F}(t_n))$, $(t_n+\Delta t, \mathbf{F}(t_n+\Delta t))$.

RK Based Explicit LTS / LTS-RK2(p)

Fine part

$$\int_{t_n}^{t_n+\xi\Delta t} \mathbf{BPy}(t) + \mathbf{PF}(t)\,dt \approx \int_0^{\xi\Delta t} \mathbf{BP}\widetilde{\mathbf{y}}(\tau) + \mathbf{PF}(t_n+\tau)\,d\tau$$

$$\begin{aligned}
\mathbf{y}(t_n+\xi\Delta t) &\approx \mathbf{y}_n \\
&+ \xi\Delta t\mathbf{B}(\mathbf{I}-\mathbf{P})\left[\mathbf{y}_n + \frac{\xi\,\Delta t}{2}(\mathbf{By}_n+\mathbf{F}_n)\right] + \int_0^{\xi\Delta t}(\mathbf{I}-\mathbf{P})\mathbf{q}(t_n+\tau)\,d\tau \\
&+ \int_0^{\xi\Delta t} \mathbf{BP}\widetilde{\mathbf{y}}(\tau) + \mathbf{PF}(t_n+\tau)\,d\tau + \mathcal{O}(\Delta t^3)
\end{aligned}$$

RK Based Explicit LTS / LTS-RK2(p)

Fine part

$$\int_{t_n}^{t_n+\xi\Delta t} \mathbf{BPy}(t) + \mathbf{PF}(t)\,dt \approx \int_0^{\xi\Delta t} \mathbf{BP}\widetilde{\mathbf{y}}(\tau) + \mathbf{PF}(t_n+\tau)\,d\tau$$

$$\begin{aligned}\mathbf{y}(t_n+\xi\Delta t) &\approx \mathbf{y}_n \\ &+ \xi\Delta t\mathbf{B}(\mathbf{I}-\mathbf{P})\Big[\mathbf{y}_n + \frac{\xi\,\Delta t}{2}\,(\mathbf{B}\mathbf{y}_n+\mathbf{F}_n)\Big] + \int_0^{\xi\Delta t}(\mathbf{I}-\mathbf{P})\mathbf{q}(t_n+\tau)\,d\tau \\ &+ \int_0^{\xi\Delta t}\mathbf{BP}\widetilde{\mathbf{y}}(\tau) + \mathbf{PF}(t_n+\tau)\,d\tau + \mathcal{O}(\Delta t^3)\end{aligned}$$

Where $\widetilde{\mathbf{y}}$ is the solution of

$$\left|\begin{aligned}&\widetilde{\mathbf{y}}(0) = \mathbf{y}_n \\ &\frac{d}{d\tau}\widetilde{\mathbf{y}}(\tau) = \mathbf{B}(\mathbf{I}-\mathbf{P})\Big[\mathbf{y}_n + \tau\,(\mathbf{B}\mathbf{y}_n+\mathbf{F}_n)\Big] \\ &\qquad\qquad\quad +(\mathbf{I}-\mathbf{P})\mathbf{q}(t_n+\tau) \\ &\qquad\qquad\quad +\mathbf{BP}\,\widetilde{\mathbf{y}}(\tau) + \mathbf{PF}(t_n+\tau)\end{aligned}\right.$$

$$\mathbf{y}(t_n+\xi\Delta t) \approx \widetilde{\mathbf{y}}(\xi\Delta t) \Longrightarrow \mathbf{y}_{n+1} := \widetilde{\mathbf{y}}(\Delta t)$$

RK Based Explicit LTS / LTS-RK2(p) Algorithm

We compute $\widetilde{\mathbf{y}}(\tau)$ for $0 \le \tau \le \Delta t$ again by using the RK2 method with the smaller time-step $\Delta\tau = \Delta t/p$.

$$\mathbf{w}_{n,0} := \mathbf{B}(\mathbf{I}-\mathbf{P})\mathbf{y}_n + (\mathbf{I}-\mathbf{P})\mathbf{F}_n$$

$$\mathbf{w}_{n,1} := \mathbf{B}(\mathbf{I}-\mathbf{P})(\mathbf{B}\mathbf{y}_n + \mathbf{F}_n) + (\mathbf{I}-\mathbf{P})\frac{\mathbf{F}_{n+1}-\mathbf{F}_n}{\Delta t}$$

$$i = 0, \ldots, p-1$$

$$\mathbf{k}_{1,\frac{i+1}{p}} := \mathbf{w}_{n,0} + i\Delta\tau\,\mathbf{w}_{n,1} + \mathbf{B}\mathbf{P}\,\widetilde{\mathbf{y}}_{\frac{i}{p}} + \mathbf{P}\mathbf{F}_{n,i}$$

$$\mathbf{k}_{2,\frac{i+1}{p}} := \mathbf{w}_{n,0} + (i+1)\,\Delta\tau\mathbf{w}_{n,1} + \mathbf{B}\mathbf{P}\left(\widetilde{\mathbf{y}}_{\frac{i}{p}} + \Delta\tau\mathbf{k}_{1,\frac{i+1}{p}}\right) + \mathbf{P}\mathbf{F}_{n,i+1}$$

$$\widetilde{\mathbf{y}}_{\frac{i+1}{p}} := \widetilde{\mathbf{y}}_{\frac{i}{p}} + \frac{\Delta\tau}{2}\left(\mathbf{k}_{1,\frac{i+1}{p}} + \mathbf{k}_{2,\frac{i+1}{p}}\right)$$

$$\mathbf{y}_{n+1} := \widetilde{\mathbf{y}}(\Delta t) \Longrightarrow \mathbf{y}_{n+1} := \widetilde{\mathbf{y}}_1$$

RK Based Explicit LTS / LTS-RK2(p) Algorithm

We compute $\widetilde{\mathbf{y}}(\tau)$ for $0 \le \tau \le \Delta t$ again by using the RK2 method with the smaller time-step $\Delta\tau = \Delta t/p$.

$$
\begin{aligned}
&\mathbf{w}_{n,0} := \mathbf{B}(\mathbf{I}-\mathbf{P})\mathbf{y}_n + (\mathbf{I}-\mathbf{P})\mathbf{F}_n\\
&\mathbf{w}_{n,1} := \mathbf{B}(\mathbf{I}-\mathbf{P})(\mathbf{B}\mathbf{y}_n + \mathbf{F}_n) + (\mathbf{I}-\mathbf{P})\frac{\mathbf{F}_{n+1}-\mathbf{F}_n}{\Delta t}\\
&i = 0, \dots, p-1\\
&\quad \mathbf{k}_{1,\frac{i+1}{p}} := \mathbf{w}_{n,0} + i\Delta\tau\,\mathbf{w}_{n,1} + \mathbf{B}\mathbf{P}\,\widetilde{\mathbf{y}}_{\frac{i}{p}} + \mathbf{P}\mathbf{F}_{n,i}\\
&\quad \mathbf{k}_{2,\frac{i+1}{p}} := \mathbf{w}_{n,0} + (i+1)\,\Delta\tau\mathbf{w}_{n,1} + \mathbf{B}\mathbf{P}\left(\widetilde{\mathbf{y}}_{\frac{i}{p}} + \Delta\tau\mathbf{k}_{1,\frac{i+1}{p}}\right) + \mathbf{P}\mathbf{F}_{n,i+1}\\
&\quad \widetilde{\mathbf{y}}_{\frac{i+1}{p}} := \widetilde{\mathbf{y}}_{\frac{i}{p}} + \frac{\Delta\tau}{2}\left(\mathbf{k}_{1,\frac{i+1}{p}} + \mathbf{k}_{2,\frac{i+1}{p}}\right)
\end{aligned}
$$

$$
\mathbf{y}_{n+1} := \widetilde{\mathbf{y}}(\Delta t) \Longrightarrow \mathbf{y}_{n+1} := \widetilde{\mathbf{y}}_1
$$

The LTS-RK2 scheme requires two multiplications by $\mathbf{B}(\mathbf{I}-\mathbf{P})$ and $2p$ multiplications by $\mathbf{BP}$ per time-step Δt.

For $\mathbf{P} = \mathbf{0}$ or $p = 1$, the LTS-RK2 scheme coincides with the RK2 scheme.

High-order explicit LTS-RK and LTS-LSRK

The previous derivation can be extended to any explicit RK scheme of order k, including Low-Storage schemes.

High-order explicit LTS-RK and LTS-LSRK

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Proposition:

Consider an explicit RK method of order k (with at least $k-1$ different coefficients $c_1, \ldots, c_s$).
Then the corresponding LTS-RK$k(p)$ scheme has order k.

Theorem:

For $s = k = 2, 3, 4$ the LTS-RK$s(p)$ scheme is convergent of order k (in the ODE sense).

Remark:

Like standard RK methods, the LTS-RK$s(p)$ inherently conserve linear invariants.

G.-Mehlin-Mitkova, SISC 2015

Numerical Experiments 1D: LTS-RK3(p)

- Computational domain:

  $\Omega = [0,6]\,,\ \Omega^{\text{coarse}} = [0,2] \cup [4,6]\,,\ \Omega^{\text{fine}} = [2,4]\,,\ p = 2, 5, 11$

- Exact solution: $(c \equiv 1)$

$$u(x,t) = \cos(t) \cdot \sin(\pi x)$$

- Source data

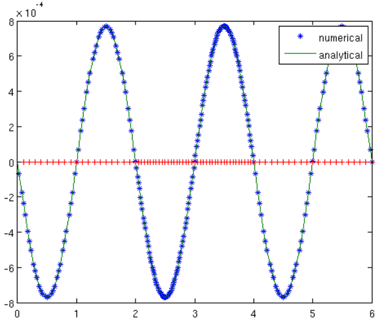
$$f(x,t) = \sin(\pi x)\Big[(\pi^2 - 1)\cos(t) - \sigma \sin(t)\Big]$$

- Homogeneous Dirichlet boundary condition
- Damping parameter: $\sigma \equiv 0.1$
- Space discretization: $\mathcal{P}^2$ mass-lumped and nodal DG FEM

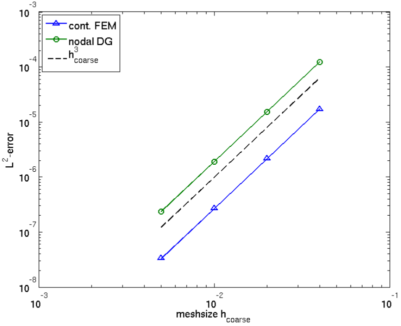
Numerical Experiments 1D: LTS-RK3(p)

local refinement $p = 2$, $\Delta t = \Delta t_{\mathrm{RK3}}$ (optimal CFL condition)

- Δt_{RK3} = max. time-step for standard non-LTS RK3



solution at $T = 10$

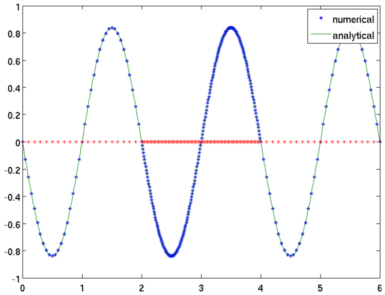


L^2-errors at $T = 10$

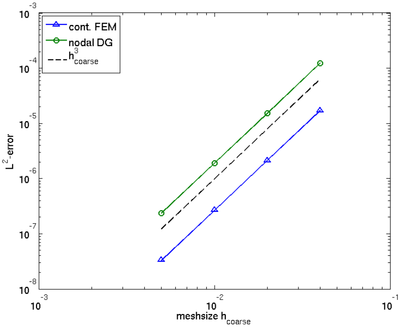
Numerical Experiments 1D: LTS-RK3(p)

local refinement $p = 5$, $\Delta t = \Delta t_{\mathrm{RK3}}$ (optimal CFL condition)

- Δt_{RK3} = max. time-step for standard non-LTS RK3



solution at $T = 10$

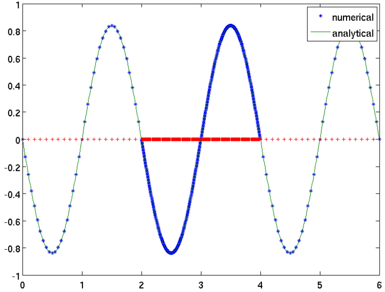


L^2-errors at $T = 10$

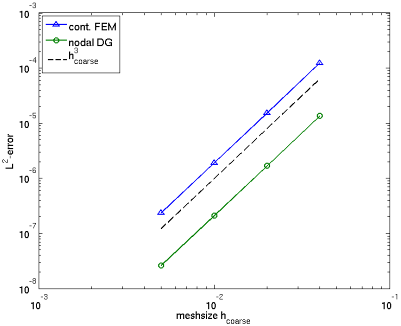
Numerical Experiments 1D: LTS-RK3(p)

local refinement $p = 11$, $\Delta t = \Delta t_{\mathrm{RK3}}$ (optimal CFL condition)

- Δt_{RK3} = max. time-step for standard non-LTS RK3



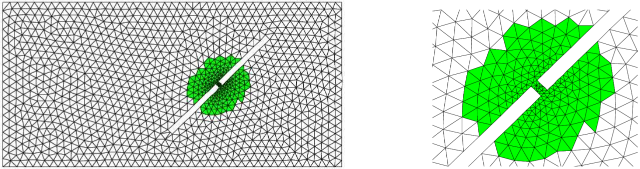
solution at $T = 10$



L^2-errors at $T = 10$

Numerical Experiments 2D: Narrow Gap

- Computational domain: Ω is rectangular of size $[0,2] \times [0,1]$ with two rectangular barriers inside forming a narrow gap
- Mesh with ratio of local refinement $p = 7$ resolves the small geometric features of the gap

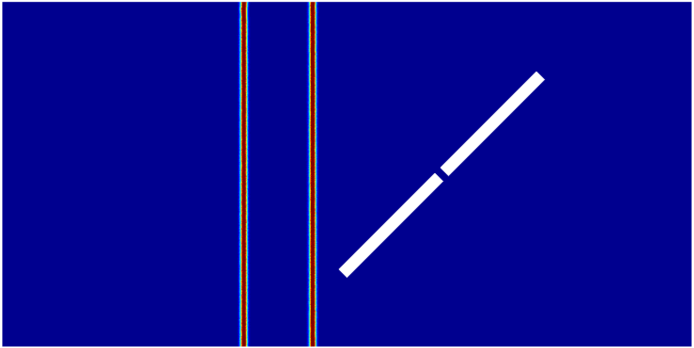


The initial triangular mesh : $h_{fine} \approx h_{coarse}/7$

- Homogeneous source data and Neumann BC
- Model parameters: $c \equiv 1$, $\sigma = 0.1$
- A plane wave is excited through the initial conditions
- LTS-LSRK3(7) combined with $\mathcal{P}^2$ mass-lumped FEM

Numerical Experiments 2D: Narrow Gap

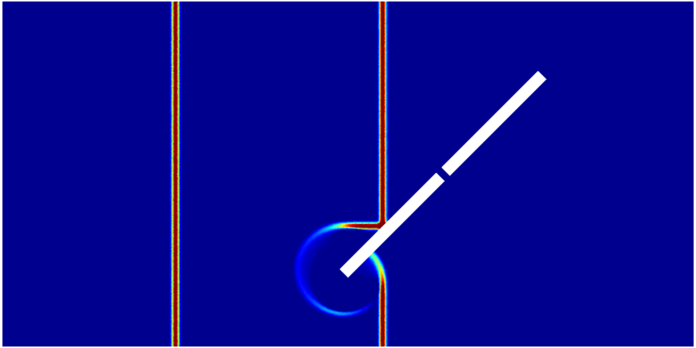
$t = 0.1$



Numerical solution on the global refinement level 4

Numerical Experiments 2D: Narrow Gap

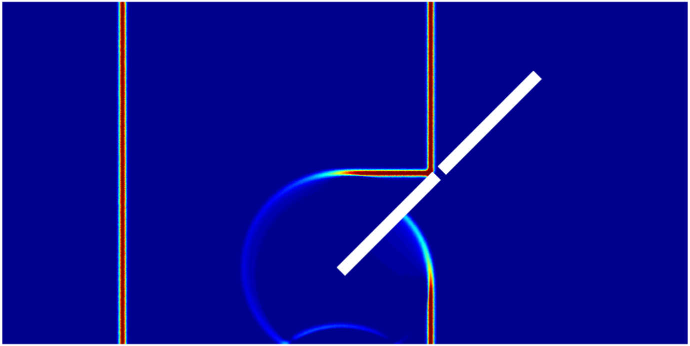
$t = 0.3$



Numerical solution on the global refinement level 4

Numerical Experiments 2D: Narrow Gap

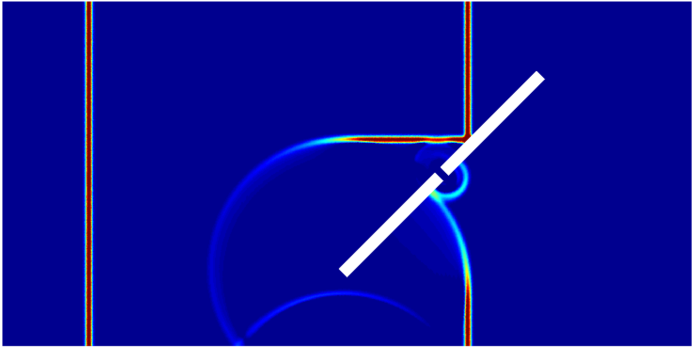
$t = 0.45$



Numerical solution on the global refinement level 4

Numerical Experiments 2D: Narrow Gap

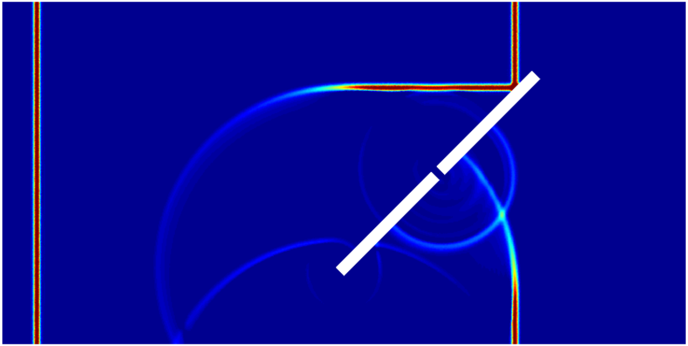
$t = 0.55$



Numerical solution on the global refinement level 4

Numerical Experiments 2D: Narrow Gap

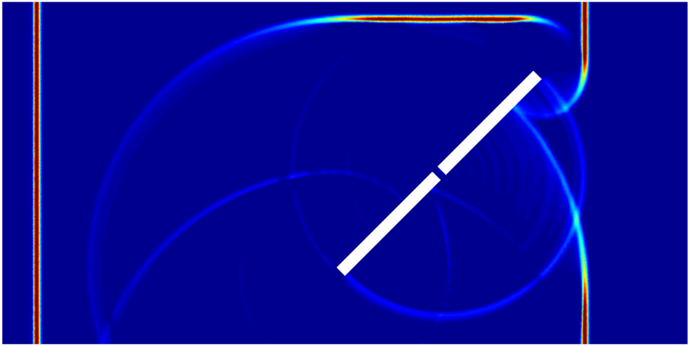
$t = 0.7$



Numerical solution on the global refinement level 4

Numerical Experiments 2D: Narrow Gap

$t = 0.9$



Numerical solution on the global refinement level 4

Leap-Frog based LTS methods

Semi-discrete Galerkin FE of the wave equation with ($\sigma = 0$):

$$\frac{d^2}{dt^2}\mathbf{y}(t) + \mathbf{A}\mathbf{y}(t) = F$$

where $\mathbf{A} = \mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}$.

- conforming FE (with mass lumping) or IP-DG
- $\mathbf{M}$ and $\mathbf{K}$ are symmetric positive definite
- $\mathbf{M}$ is (block-) diagonal

$\Rightarrow$ computing $\mathbf{M}^{-1}$ is cheap $\Rightarrow$ fully explicit time-stepping!

LTS-LF methods

Following similar ideas, one can derive an LTS-LF method:

- second-order accurate
- energy conserving (for Δt sufficiently small)
- can be extended to arbitrary (even) order
- can be extended recursively to multiple levels: MLTS-LF

$$\mathbf{y}_{n+1} = -\mathbf{y}_{n-1} + 2\,\mathrm{LTS}_2(\mathbf{y}_n, -\mathbf{A}(\mathbf{I}-\mathbf{P})\mathbf{y}_n),$$

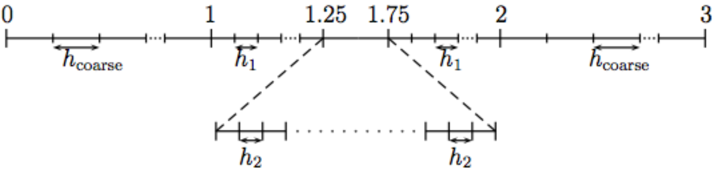
where the function $\mathbf{z}_{new} = \mathrm{LTS}_2(\mathbf{z},\mathbf{w})$ is defined as:

1. $\mathbf{z}_{new} := \mathbf{z} + \frac{1}{2}\Delta\tau^2\,(\mathbf{w} - \mathbf{APz}), \qquad \Delta\tau = \Delta t/p$
2. For $m = 1, ..., p-1$
   (i) $\mathbf{z}_{old} := \mathbf{z};\ \mathbf{z} := \mathbf{z}_{new}$
   (ii) $\mathbf{z}_{new} := 2\mathbf{z} - \mathbf{z}_{old} + \Delta\tau^2\,(\mathbf{w} - \mathbf{APz})$

Diaz and G., SISC (2009), CMAME (2015)

Multi-level local time stepping (MLTS)

Consider now a sequence of nested grids, such as from iterative refinement (hp adaptivity).

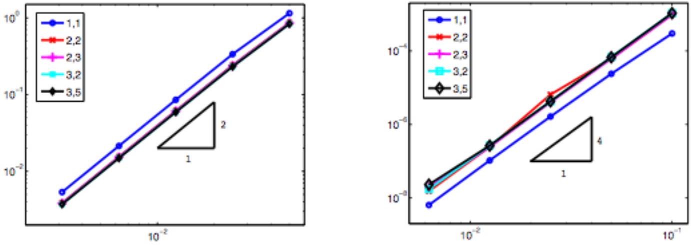


$$h_2 = \frac{h_1}{p_2} = \frac{h_{\text{coarse}}}{p_1 p_2}$$

Apply LTS-LF method recursively: $\Delta\theta = \dfrac{\Delta\tau}{p_2} = \dfrac{\Delta t}{p_1 p_2}$

Two-level local time stepping (p_1, p_2)

Convergence as $h_{\text{coarse}}, \Delta t \to 0$



MLTS-2, IP-DG P^1 FE   MLTS-4, IP-DG P^3 FE

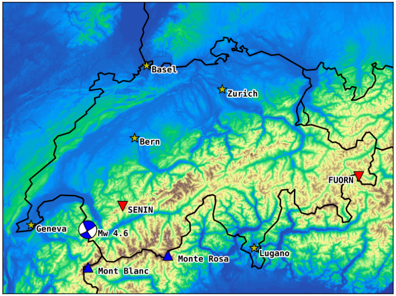
MLTS-2: Energy conservation proved regardless of $p_1, p_2, \ldots$

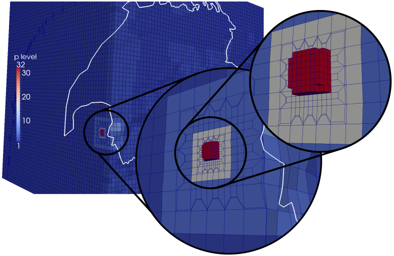
Numerical Experiments: SPECFEM3D-LTS

SPECFEM3D: SEM for the viscoelastic wave equation
SCOTCH: parallel graph partitioning software

Rietmann, G., Peter, Schenk, Uçar "Load-balanced local time stepping for large-scale wave propagation", IPDPS 2015.

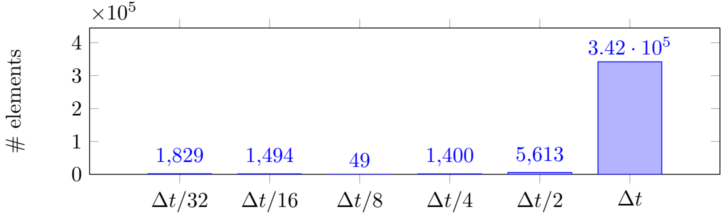
Numerical Experiments: 3D seismology





Mesh: 350’000 elements (23 Mio. dof’s), $T = 180$ [s],
single 8-core CPU-time: 16 [h] (instead of 40 [min]) due to local refinement)

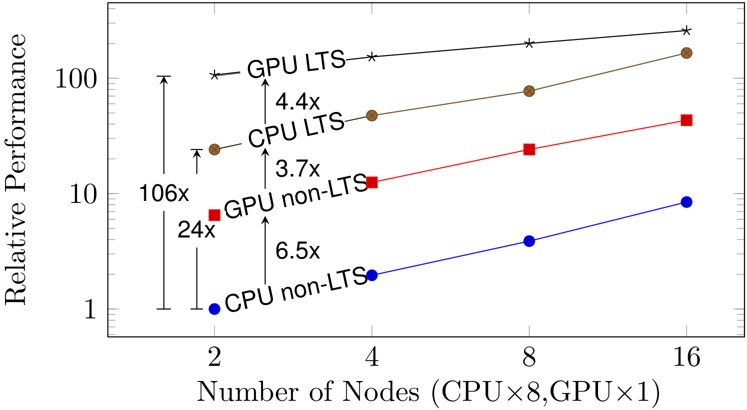
Numerical Experiments: Expected Speed-up



Expected speedup:

$$
\begin{aligned}
(32 \times 352482)/(32 \times 1829 \\
+16 \times 1494 \\
+ \ \ 8 \times 49 \\
+ \ \ 4 \times 1400 \\
+ \ \ 2 \times 5613 \\
+ \quad 342097) = 25.5.
\end{aligned}
$$

Numerical Experiments: Parallel performance



Concluding Remarks

- Wave equation with (or without) damping
- Mass-lumped/IP-DG/nodal DG FE discretization $\Rightarrow$ block-diagonal mass matrix $\Rightarrow$ explicit time integration
- Proposed explicit LTS schemes of arbitrarily high accuracy:
  - Without damping ($\sigma = 0$): LTS L-F type methods
    $\Rightarrow$ discrete energy conserved
  - With damping ($\sigma \leq 0$): LTS AB(k) methods
    $\Rightarrow$ optimal CFL for $k \geq 3$
  - With damping ($\sigma \leq 0$): LTS RK and LSRK methods
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THANK YOU FOR YOUR ATTENTION!