New perspectives offered by overlaps to design transparent boundary conditions in waveguides

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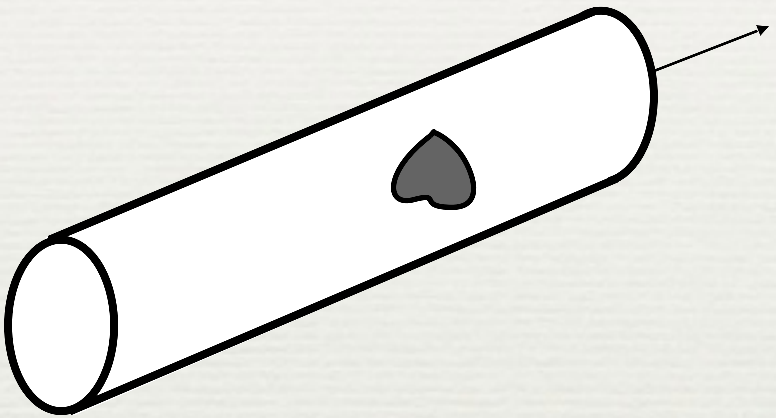
Context and motivation

Context: Numerical simulation of Non Destructive Testing (NDT) experiments in elastic waveguides.



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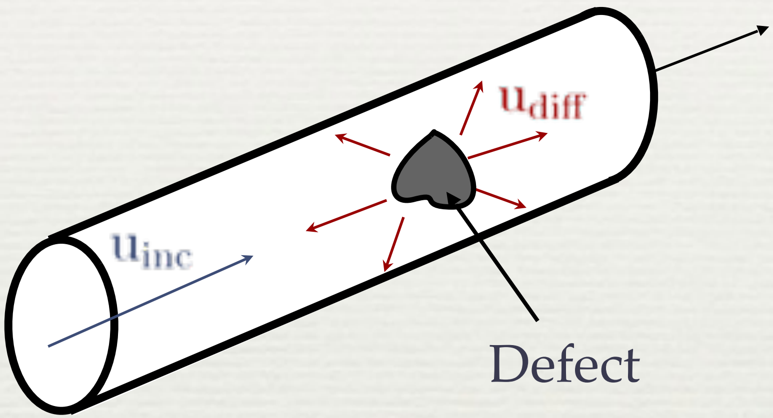
Basic NDT experiment:

By emitting an incident wave $\mathbf{u}_{\text{inc}}$

and by measuring the diffracted field $\mathbf{u}_{\text{diff}}$

we want to detect the defect



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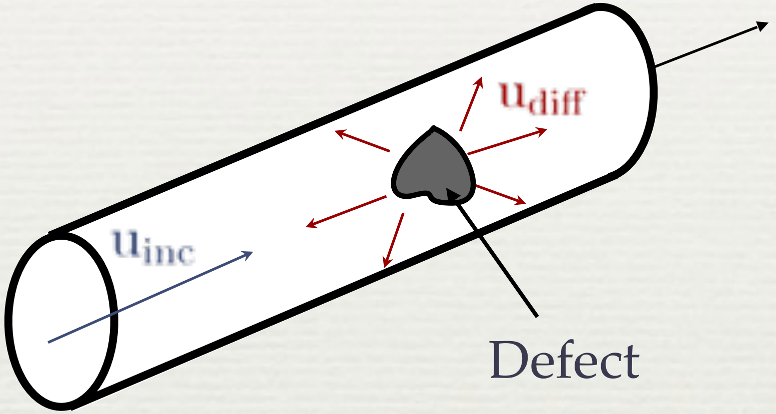

Guided waves for NDT:

- Inspection on wide distances without moving the sensors
- Inspection of inaccessible areas

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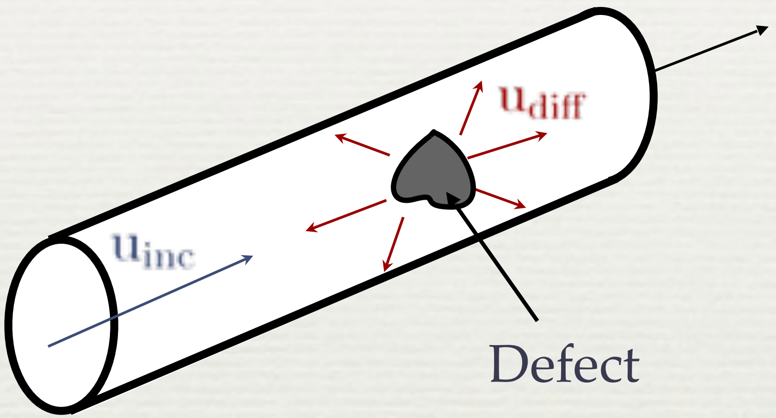

Guided waves for NDT:

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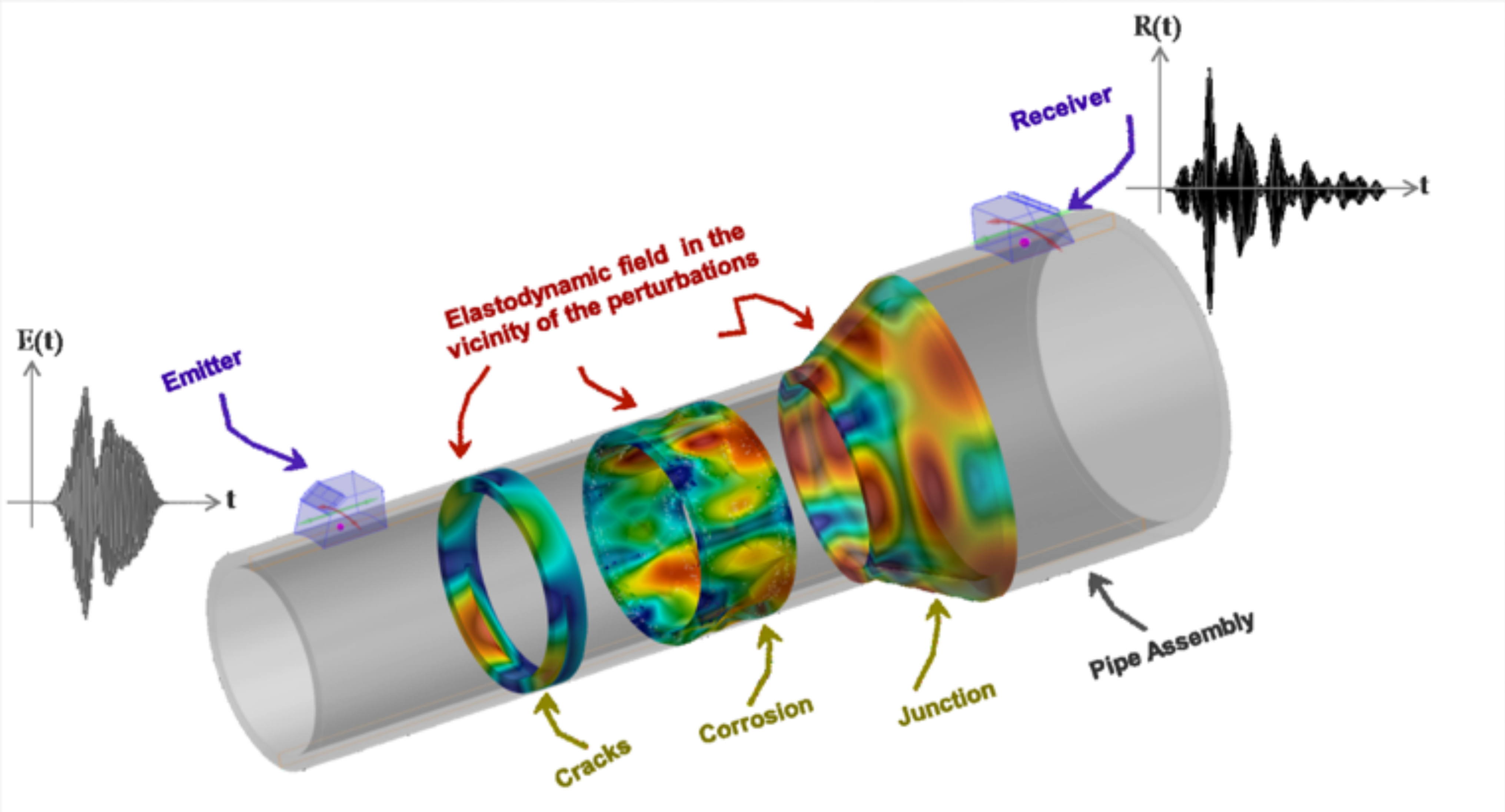
Issues:

- Analysis of the experimental results
- Design and optimize control configurations

Realistic simulations with CIVA

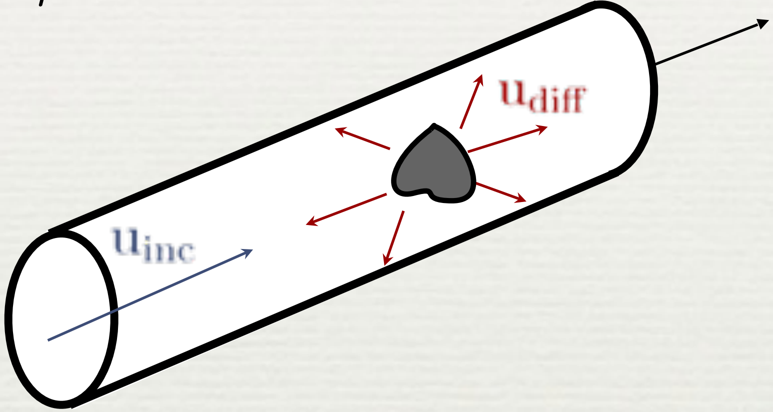


Model problems

Time-harmonic diffraction problems: $e^{-i\omega t}$

- Scalar equation (acoustic case): $\mathrm{div}\,(A\nabla \mathrm{u}) + \omega^2 \mathrm{u} = f$
- Vectorial equation (elastic case): $\mathbf{div}\sigma\,(\mathbf{u}) + \rho\omega^2 \mathbf{u} = \mathbf{f}$

  where $\mathbf{u} = (\mathrm{u}_x, \mathrm{u}_y, \mathrm{u}_z)$



Specifications:

- Localized defect of arbitrary shape
- Unbounded domain
- Anisotropic material

Goal: **Find a reformulation of the problem in a bounded domain which is:**
suitable for numerical computations

Absorbing layers (PML)

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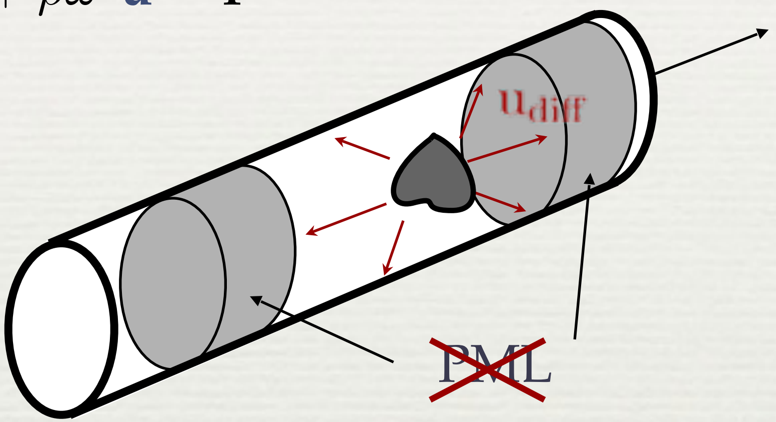

Perfectly matched layers:

Bérenger, *A perfectly matched layer for the absorption of electromagnetic waves*, (1994)

These layers produce a sparse FE matrix and are perfectly matched

But lead to a wrong solution in presence of backward waves

Bécache et al., Stability of *PML, group velocity and anisotropic waves*, (2003)

Craster et al., *Guided elastic waves and PML*, (2007)

Bonnet-Ben Dhia, Chambeyron and Legendre, *On the use of PML in presence of long or backward guided elastic waves*, (2007)

Hardy space infinite elements

Time-harmonic diffraction problems: $e^{-i\omega t}$

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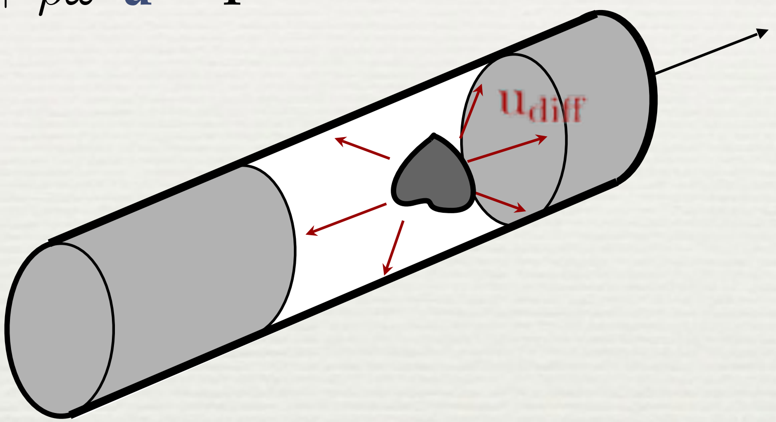

Using curved Hardy spaces to impose a pole radiation condition

Avoids the problem linked to the inverse modes

But intricate to implement

M. Halla and L. Nannen, *Hardy space infinite elements for time-harmonic two-dimensional elastic waveguide problems,* (2015)

Transparent Boundary Conditions (TBC)

Time-harmonic diffraction problems: $e^{-i\omega t}$

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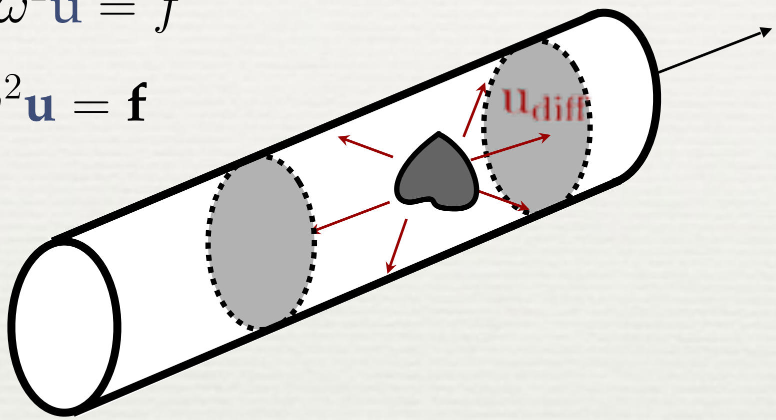

Transparent boundary conditions:

- Isotropic acoustic case:

Goldstein, *A Finite element method for solving Helmholtz type equations in waveguides and other unbounbed domain*, (1981)

- Isotropic elastic case:

Baronian, Bonnet-Ben Dhia and Lunéville, *TBC for the time-harmonic diffraction problem in an elastic waveguide*, (2010)

Exact condition, gives very accurate results

But applicable only for isotropic cases, and requires the use of non-local operators

Transparent Boundary Conditions (TBC)

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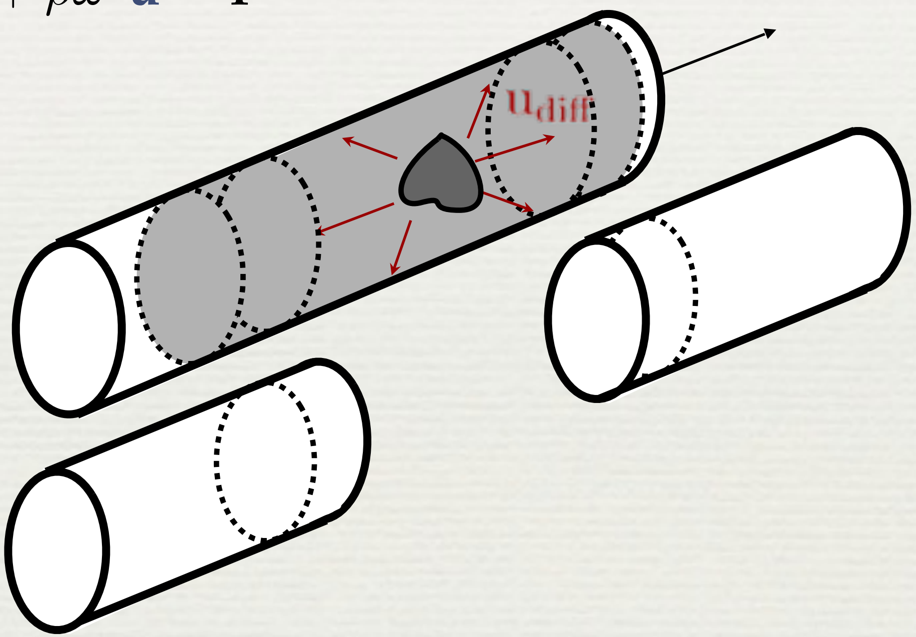

Iterative solvers have been developed

- Isotropic acoustic case:

Gmati et al., *Convergence bounds of GMRES with Schwarz's preconditioner for scattering problem*, (2009)

Gmati et al., *On schwarz algorithms for elliptic exterior boundary value problems*, (2005)

- Anisotropic elastic case:

The subject of my talk !

« Toy » problem for the presentation

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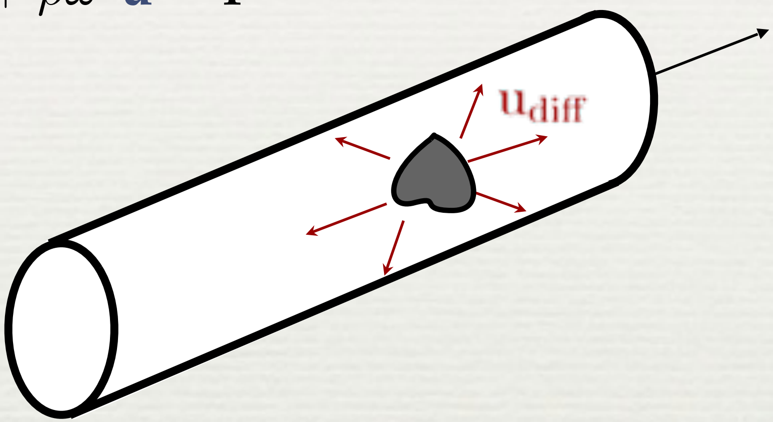
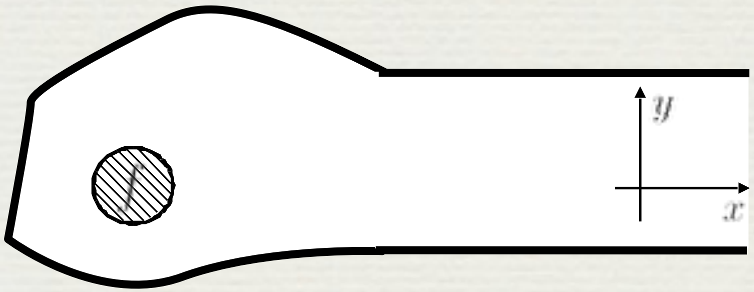


To simplify the presentation, we will consider an acoustic 2D configuration, with only one semi-infinite waveguide



« Toy » problem for the presentation

Find $\mathbf{u} \in H^1_{\text{loc}}(\Omega)$ the outgoing solution of

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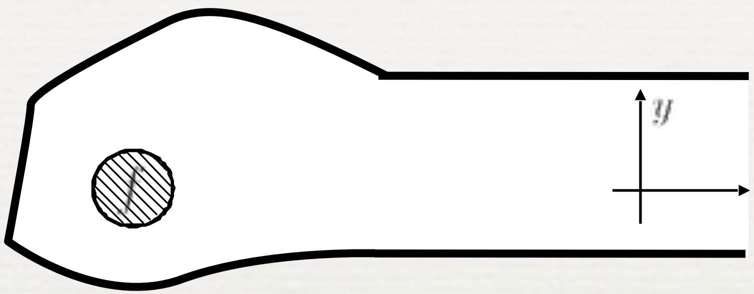

where:
- A is a symmetric positive definite matrix
- f is compactly supported

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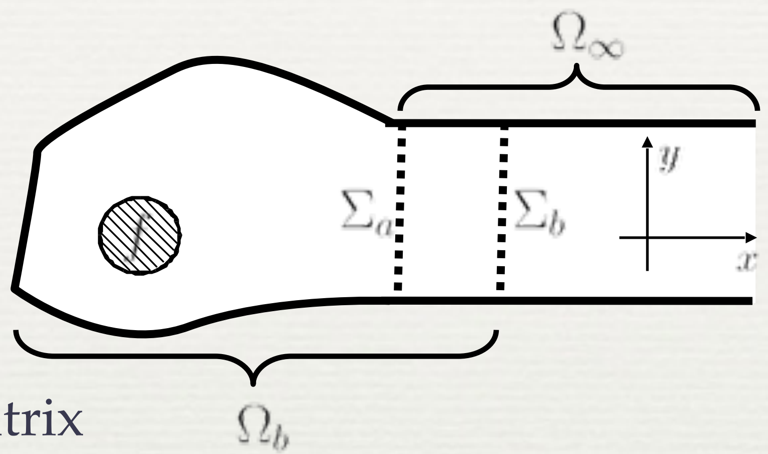

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Geometrical notations:

- $\Omega_b = \Omega \cap \{(x,y),\ x \leq b\}$
- $\Omega_\infty = \Omega \cap \{(x,y),\ x \geq a\}$
- $\Sigma_a = \Omega \cap \{(x,y),\ x = a\}$
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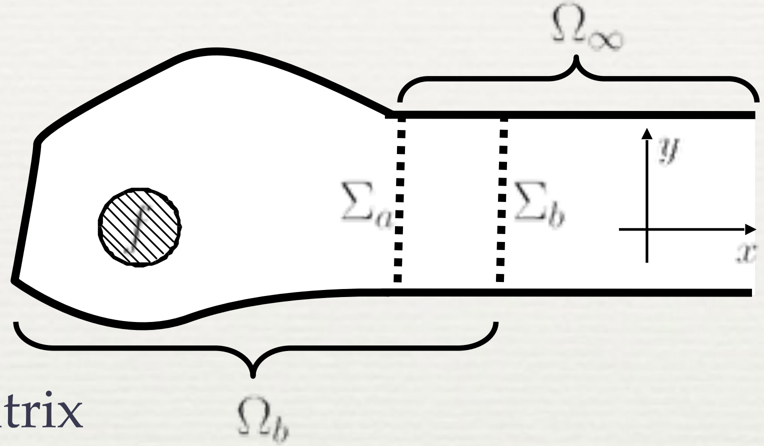

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Using a: ○ **Modal** representation in Ω_∞ ○ **FE** representation in Ω_b

we will give a multi-domain formulation of the diffraction problem.

In what follows...

Find $\mathrm{u} \in H^1_{\mathrm{loc}}(\Omega)$ the outgoing solution of

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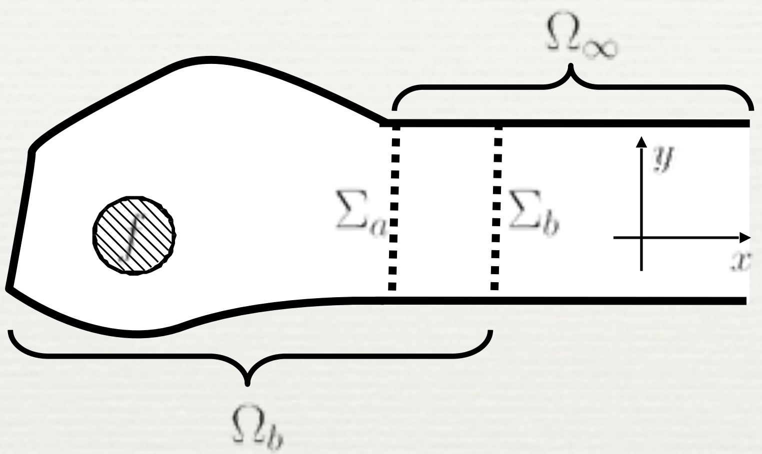

Outline:

- The modal representation in Ω_∞
- The classical « Dirichlet - Neumann » multi-domain formulation
  Isotropic case $A = Id$
- New multi-domain formulation « Transparent - Neumann »
  Anisotropic case $A \neq Id$
- Iterative resolution
  Isotropic case $A = Id$

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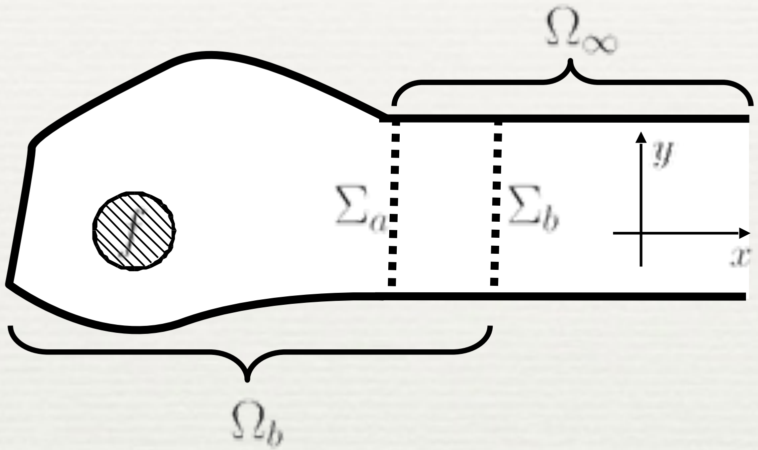

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Modes in a waveguide

Modes are non-trivial solutions u of

$$\left| \begin{array}{ll} \operatorname{div}(A\nabla \mathrm{u}) + \omega^2 \mathrm{u} = 0 & \text{in} \quad \mathbb{R} \times [0,1], \\ A\nabla \mathrm{u} \cdot \nu = 0 & \text{on} \quad \mathbb{R} \times \{0,1\}, \end{array} \right.$$

of the form $\mathrm{u}(x,y) = \varphi(y) e^{i\beta x} \qquad \beta \in \mathbb{C}$

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⟶ Leads to solve a quadratic eigenvalue problem

Among the countable set of solutions, we distinguish:

- The propagative modes with $\operatorname{Im}(\beta) = 0$
- The evanescent modes with $\operatorname{Im}(\beta) \neq 0$

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⟶ Leads to solve a quadratic eigenvalue problem

Among the countable set of solutions, we distinguish:

- The propagative modes with $\operatorname{Im}(\beta) = 0$ — Sign of $\partial\omega/\partial\beta$
- The evanescent modes with $\operatorname{Im}(\beta) \neq 0$ — Sign of $\operatorname{Im}(\beta)$

Moreover, we distinguish two families of modes:

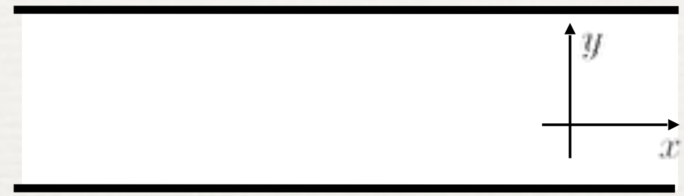
- The right-going modes: $e^{i\beta_k x}\varphi_k^+(y)$
- The left-going modes: $e^{-i\beta_k x}\varphi_k^-(y)$

⟶ Enables to define the outgoing solution

Modes in the isotropic case

Modes are non-trivial solutions u of

$$\begin{vmatrix} \operatorname{div}(A\nabla \mathrm{u}) + \omega^2 \mathrm{u} = 0 & \text{in} & \mathbb{R} \times [0,1], \\ A\nabla \mathrm{u} \cdot \nu = 0 & \text{on} & \mathbb{R} \times \{0,1\}, \end{vmatrix}$$



of the form $\mathrm{u}(x,y) = \varphi(y) e^{i\beta x}$ $\quad \beta \in \mathbb{C}$

Isotropic case: $\varphi_k^+(y) = \varphi_k^-(y) = \varphi_k(y) \propto \cos(k\pi y)$ $\quad \beta_k = \sqrt{\omega^2 - (k\pi)^2}$

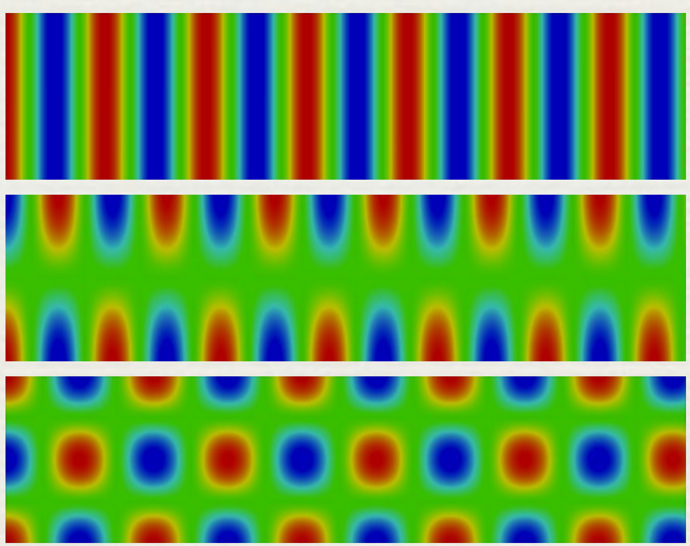
$A = Id$

$\mathrm{Im}\,(\beta_k) = 0$

$k = 0$

$k = 1$

$k = 2$



$\mathrm{Im}\,(\beta_k) > 0$

$k = 3$

$k = 4$

$k \geq 5$

$\vdots$

Modes in the anisotropic case

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of the form $\mathrm{u}(x,y) = \varphi(y)e^{i\beta x}$ $\quad \beta\in\mathbb{C}$

Anisotropic case: $\varphi_k^+ \neq \varphi_k^-$

$$A = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

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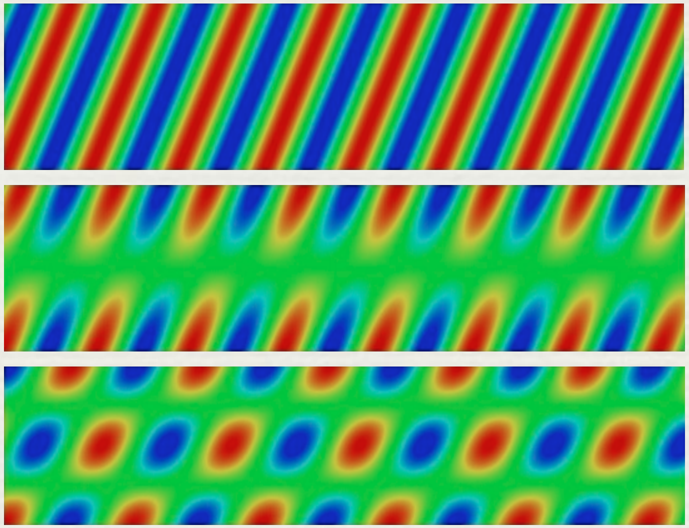

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Some properties on the modes

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Anisotropic case: $\varphi_k^+ \neq \varphi_k^-$

No orthogonality relations: $(\varphi_k^\pm, \varphi_m^\pm) \neq \delta_{km}$

but bi-orthogonality relations:

$$(\varphi_k^+, \psi_m^-) - (\psi_k^+, \varphi_m^-) = \delta_{km}$$

où $\psi_k^\pm(y)e^{\pm i\beta_k x} = A\nabla\left(\varphi_k^\pm(y)e^{\pm i\beta_k x}\right) \cdot e_x$

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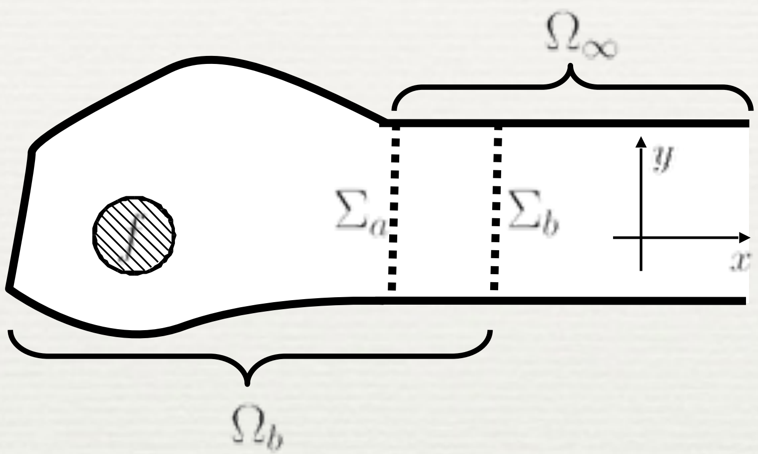

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Multi-domain formulation (isotropic)

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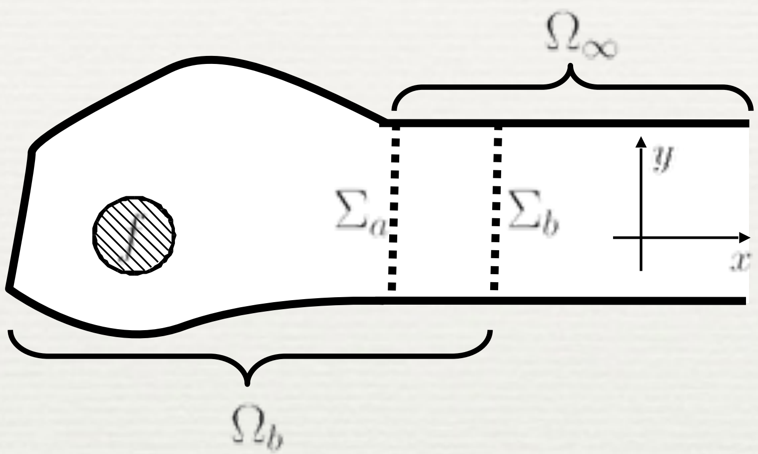

Let us define $\mathbf{u}_b = \mathbf{u}|_{\Omega_b}$ and $\mathbf{u}_\infty = \mathbf{u}|_{\Omega_\infty}$ such that

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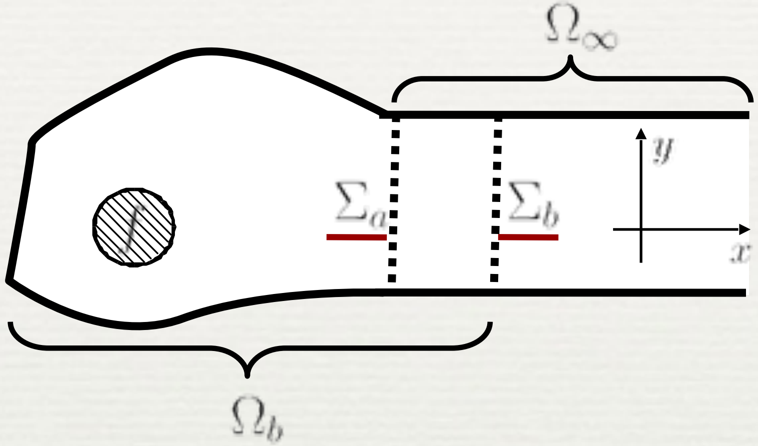

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To get the multi-domain formulation, we must choose transmission conditions (**TC**) on the interfaces Σ_a and Σ_b

Multi-domain formulation (isotropic case)

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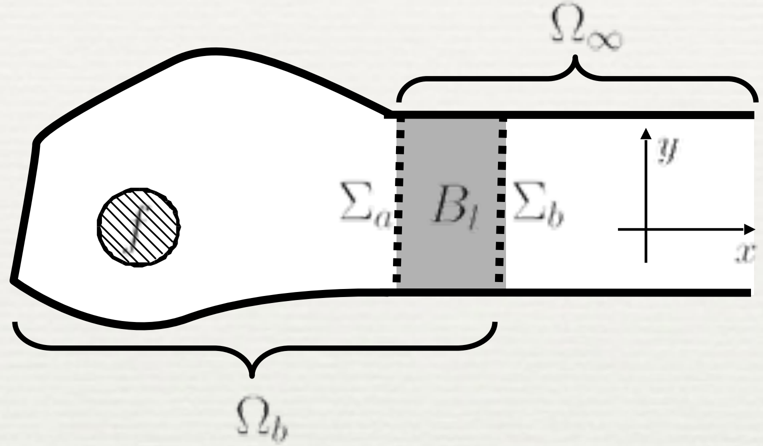
$$\left| \begin{array}{lll} \Delta \mathbf{u} + \omega^2 \mathbf{u} = f & \text{in} & \Omega, \\ \partial_\nu \mathbf{u} = 0 & \text{on} & \partial\Omega, \end{array} \right.$$



Let us define $\mathbf{u}_b = \mathbf{u}|_{\Omega_b}$ and $\mathbf{u}_\infty = \mathbf{u}|_{\Omega_\infty}$ such that

$$\left| \begin{array}{lll} \Delta \mathbf{u}_b + \omega^2 \mathbf{u}_b = f & \text{in} & \Omega_b, \\ \partial_\nu \mathbf{u}_b = 0 & \text{on} & \partial\Omega_b \cap \Omega, \end{array} \right.$$

$$\left| \begin{array}{lll} \Delta \mathbf{u}_\infty + \omega^2 \mathbf{u}_\infty = 0 & \text{in} & \Omega_\infty, \\ \partial_\nu \mathbf{u}_\infty = 0 & \text{on} & \partial\Omega_\infty \cap \Omega, \end{array} \right.$$

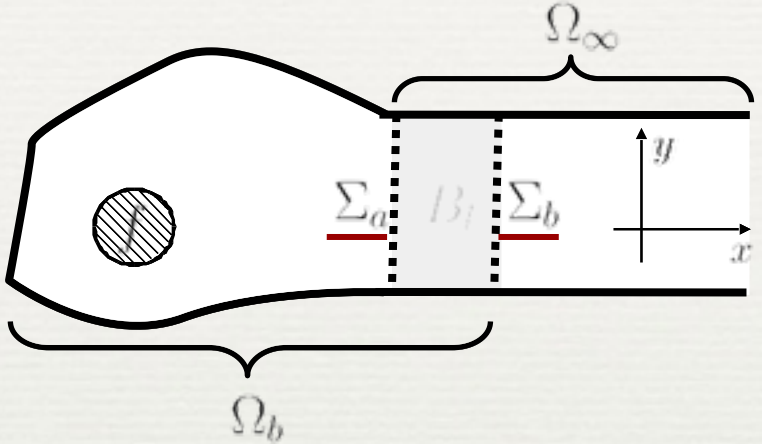
To get the multi-domain formulation, we must choose transmission conditions (**TC**) on the interfaces Σ_a and Σ_b

A necessary and sufficient condition to have the equivalence between the multi-domain formulation and the initial problem is that the **TC** ensure the compatibility:

$$\boxed{\mathbf{u}_b = \mathbf{u}_\infty \quad \text{in} \quad B_l = \Omega_b \cap \Omega_\infty} \qquad \text{where} \quad l = b - a$$

« Dirichlet-Neumann » formulation

Find $\mathbf{u} \in H^1_{\text{loc}}(\Omega)$ the outgoing solution of



$$\left| \begin{array}{ll} \Delta \mathbf{u} + \omega^2 \mathbf{u} = f & \text{in} \quad \Omega, \\ \partial_\nu \mathbf{u} = 0 & \text{on} \quad \partial\Omega, \end{array} \right.$$

Multi-domain formulation:

$$\begin{cases} \left| \begin{array}{ll} \Delta \mathbf{u}_b + \omega^2 \mathbf{u}_b = f & \text{in} \quad \Omega_b, \\ \partial_\nu \mathbf{u}_b = 0 & \text{on} \quad \partial\Omega_b \cap \Omega, \end{array} \right. \qquad \left| \begin{array}{ll} \Delta \mathbf{u}_\infty + \omega^2 \mathbf{u}_\infty = 0 & \text{in} \quad \Omega_\infty, \\ \partial_\nu \mathbf{u}_\infty = 0 & \text{on} \quad \partial\Omega_\infty \cap \Omega, \end{array} \right. \\ \textbf{(TC)} \quad \underline{\mathbf{u}_b = \mathbf{u}_\infty \text{ on } \Sigma_a} \quad \text{and} \quad \underline{\partial_\nu \mathbf{u}_b = \partial_\nu \mathbf{u}_\infty \text{ on } \Sigma_b} \end{cases}$$

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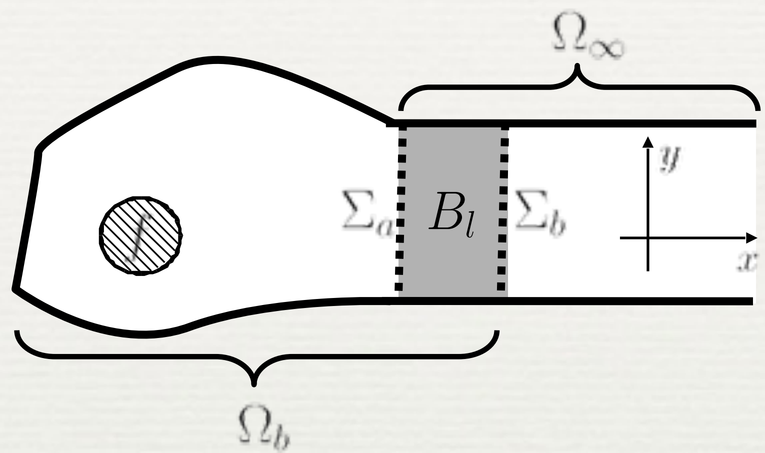


Multi-domain formulation:

$$\begin{cases} \left| \begin{array}{lll} \Delta \mathrm{u}_b + \omega^2 \mathrm{u}_b = f & \text{in} & \Omega_b, \\ \partial_\nu \mathrm{u}_b = 0 & \text{on} & \partial\Omega_b \cap \Omega, \end{array} \right. \qquad \left| \begin{array}{lll} \Delta \mathrm{u}_\infty + \omega^2 \mathrm{u}_\infty = 0 & \text{in} & \Omega_\infty, \\ \partial_\nu \mathrm{u}_\infty = 0 & \text{on} & \partial\Omega_\infty \cap \Omega, \end{array} \right. \\ \textbf{(TC)} \quad \mathrm{u}_b = \mathrm{u}_\infty \text{ on } \Sigma_a \quad \text{and} \quad \partial_\nu \mathrm{u}_b = \partial_\nu \mathrm{u}_\infty \text{ on } \Sigma_b \end{cases}$$

Compatibility $\mathrm{u}_b = \mathrm{u}_\infty \quad \text{in} \quad B_l = \Omega_b \cap \Omega_\infty$

The difference $\mathrm{v} = \mathrm{u}_b - \mathrm{u}_\infty$ satisfies

$$\left| \begin{array}{lll} \Delta \mathrm{v} + \omega^2 \mathrm{v} = 0 & \text{in} & B_l, \\ \partial_\nu \mathrm{v} = 0 & \text{on} & \partial B_l \setminus \Sigma_a, \\ \mathrm{v} = 0 & \text{on} & \Sigma_a. \end{array} \right.$$

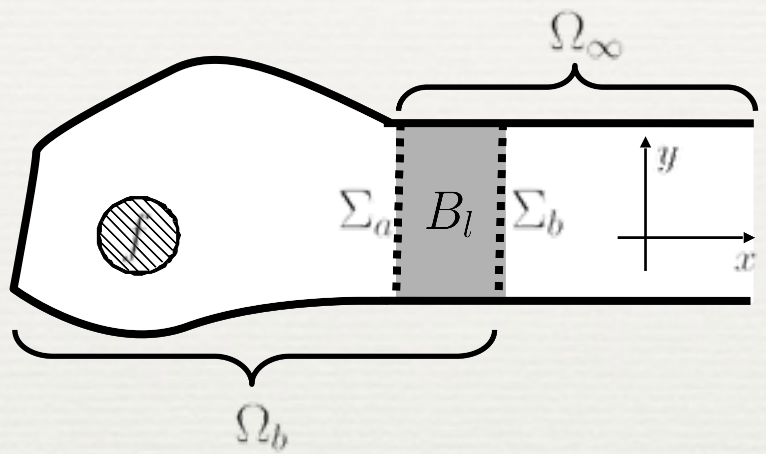
$$\boxed{\longrightarrow \ \mathrm{v} = 0}$$

except for a countable set of « box frequencies »

DtN formulation with « overlap »

Find $\mathbf{u} \in H^1_{\text{loc}}(\Omega)$ the outgoing solution of

$$\left| \begin{array}{lll} \Delta \mathbf{u} + \omega^2 \mathbf{u} = f & \text{in} & \Omega, \\ \partial_\nu \mathbf{u} = 0 & \text{on} & \partial\Omega, \end{array} \right.$$



Equivalent multi-domain formulation:

$$\left\{ \begin{array}{l} \left| \begin{array}{lll} \Delta \mathbf{u}_b + \omega^2 \mathbf{u}_b = f & \text{in} & \Omega_b, \\ \partial_\nu \mathbf{u}_b = 0 & \text{on} & \partial\Omega_b \cap \Omega, \end{array} \right. \qquad \left| \begin{array}{lll} \Delta \mathbf{u}_\infty + \omega^2 \mathbf{u}_\infty = 0 & \text{in} & \Omega_\infty, \\ \partial_\nu \mathbf{u}_\infty = 0 & \text{on} & \partial\Omega_\infty \cap \Omega, \end{array} \right. \\ \textbf{(TC)} \quad \mathbf{u}_b = \mathbf{u}_\infty \text{ on } \Sigma_a \quad \text{and} \quad \partial_\nu \mathbf{u}_b = \partial_\nu \mathbf{u}_\infty \text{ on } \Sigma_b \end{array} \right.$$

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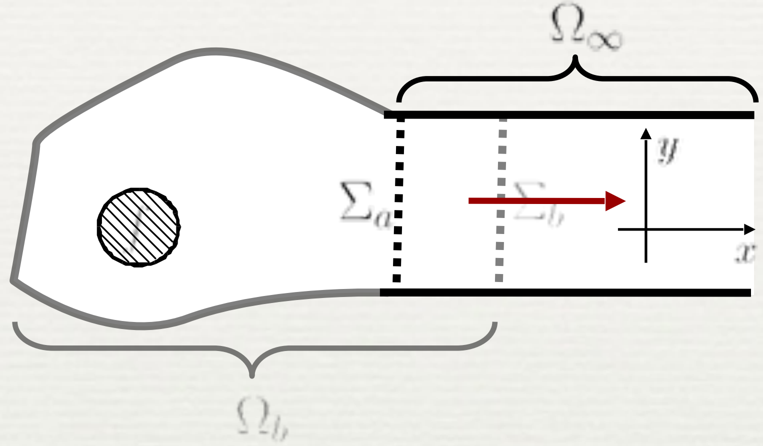

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$$\mathrm{u}_\infty(x, y) = \sum_{k \geq 0} a_k^\infty e^{i\beta_k(x-a)} \varphi_k(y),$$

DtN formulation with « overlap »

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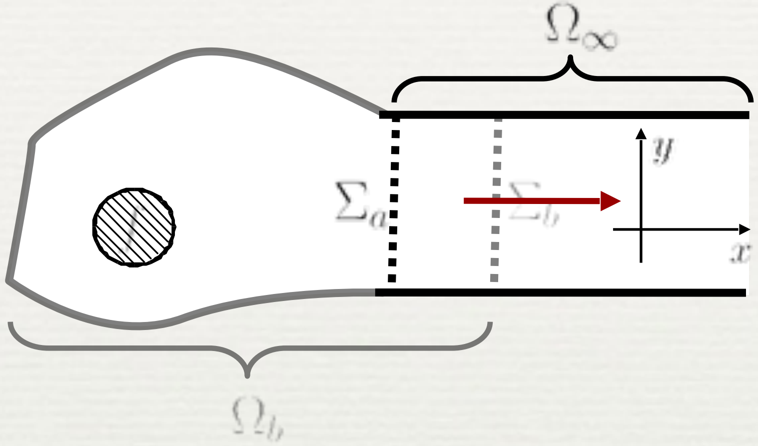

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Elimination of u_∞

Thanks to the orthogonality of the modes, it comes: $(\mathrm{u}_b, \varphi_k)_{\Sigma_a} = (\mathrm{u}_\infty, \varphi_k)_{\Sigma_a} = a_k^\infty$

DtN formulation with « overlap »

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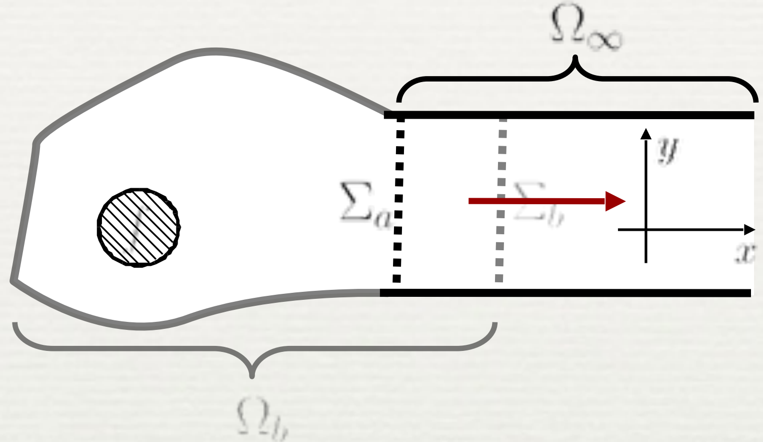

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$$\left\{ \begin{array}{l} \left| \begin{array}{lll} \Delta \mathrm{u}_b + \omega^2 \mathrm{u}_b = f & \text{in} & \Omega_b, \\ \partial_\nu \mathrm{u}_b = 0 & \text{on} & \partial\Omega_b \cap \Omega, \end{array} \right. \\ \textbf{(TC)} \quad \mathrm{u}_b = \mathrm{u}_\infty \text{ on } \Sigma_a \quad \text{and} \quad \partial_\nu \mathrm{u}_b = \partial_\nu \mathrm{u}_\infty \text{ on } \Sigma_b \end{array} \right.$$

$$\mathrm{u}_\infty(x,y) = \sum_{k \geq 0} e^{i\beta_k(x-a)} (\mathrm{u}_b|_{\Sigma_a}, \varphi_k)_{\Sigma_a} \, \varphi_k(y)$$

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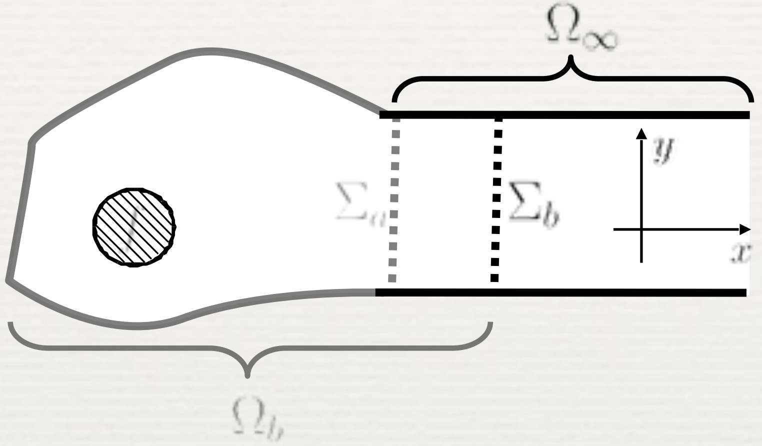

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$$\mathrm{u}_\infty(x,y) = \sum_{k \geq 0} e^{i\beta_k(x-a)} (\mathrm{u}_b|_{\Sigma_a}, \varphi_k)_{\Sigma_a} \, \varphi_k(y)$$

Elimination of u_∞

Thanks to the orthogonality of the modes, it comes: $(\mathrm{u}_b, \varphi_k)_{\Sigma_a} = (\mathrm{u}_\infty, \varphi_k)_{\Sigma_a} = a_k^\infty$

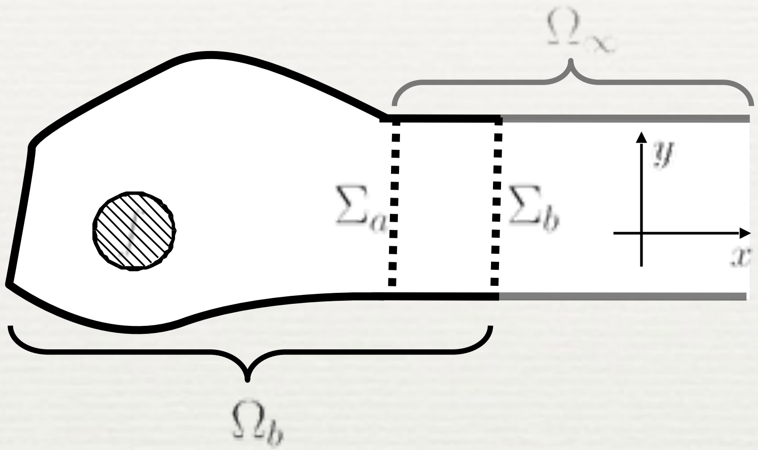
We then deduce the following TBC:

$$\partial_\nu \mathrm{u}_b|_{\Sigma_b} = \sum_{k \geq 0} i\beta_k e^{i\beta_k l} (\mathrm{u}_b|_{\Sigma_a}, \varphi_k)_{\Sigma_a} \, \varphi_k$$

DtN formulation with « overlap »

Find $\mathrm{u} \in H^1_{\mathrm{loc}}(\Omega)$ the outgoing solution of

$$\left| \begin{array}{lll} \Delta \mathrm{u} + \omega^2 \mathrm{u} = f & \text{in} & \Omega, \\ \partial_\nu \mathrm{u} = 0 & \text{on} & \partial\Omega, \end{array} \right.$$



Equivalent multi-domain formulation:

$$\begin{cases} \left| \begin{array}{lll} \Delta \mathrm{u}_b + \omega^2 \mathrm{u}_b = f & \text{in} & \Omega_b, \\ \partial_\nu \mathrm{u}_b = 0 & \text{on} & \partial\Omega_b \cap \Omega, \end{array} \right. \\ \\ \textbf{(TC)} \quad \mathrm{u}_b = \mathrm{u}_\infty \text{ on } \Sigma_a \quad \text{and} \quad \underline{\partial_\nu \mathrm{u}_b = \partial_\nu \mathrm{u}_\infty \text{ on } \Sigma_b} \end{cases}$$

$$\mathrm{u}_\infty(x,y) = \sum_{k\geq 0} e^{i\beta_k(x-a)} (\mathrm{u}_b|_{\Sigma_a}, \varphi_k)_{\Sigma_a}\, \varphi_k(y)$$

Formulation in a bounded domain

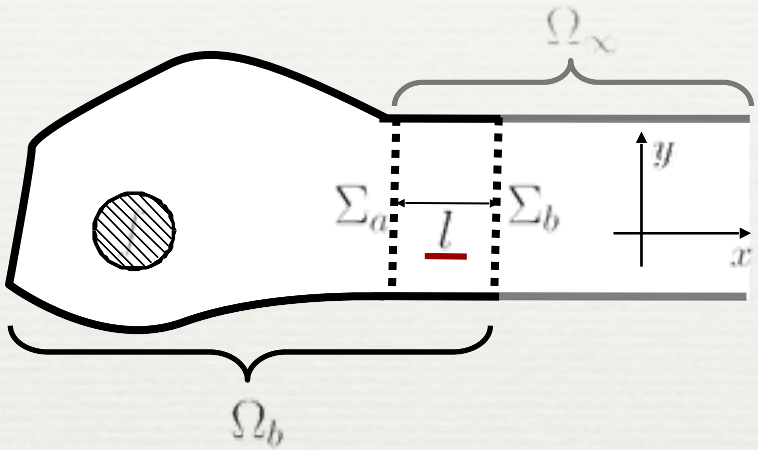
$$\left| \begin{array}{lll} \Delta \mathrm{u}_b + \omega^2 \mathrm{u}_b = f & \text{in} & \Omega_b, \\ \partial_\nu \mathrm{u}_b = 0 & \text{on} & \partial\Omega_b, \cap\Omega, \\ \partial_\nu \mathrm{u}_b = T^l_{D\to N}(\mathrm{u}_b|_{\Sigma_a}) & \text{on} & \Sigma_b \end{array} \right.$$

where $T^l_{D\to N}(\mathrm{u}_b|_{\Sigma_a}) \equiv \sum_{k\geq 0} i\beta_k e^{i\beta_k l} (\mathrm{u}_b|_{\Sigma_a}, \varphi_k)_{\Sigma_a}\, \varphi_k$

DtN formulation with « overlap »

Find $\mathrm{u} \in H^1_{\mathrm{loc}}(\Omega)$ the outgoing solution of

$$\left| \begin{array}{lll} \Delta \mathrm{u} + \omega^2 \mathrm{u} = f & \text{in} & \Omega, \\ \partial_\nu \mathrm{u} = 0 & \text{on} & \partial\Omega, \end{array} \right.$$



Equivalent multi-domain formulation:

$$\begin{cases} \left| \begin{array}{lll} \Delta \mathrm{u}_b + \omega^2 \mathrm{u}_b = f & \text{in} & \Omega_b, \\ \partial_\nu \mathrm{u}_b = 0 & \text{on} & \partial\Omega_b \cap \Omega, \end{array} \right. \\ \textbf{(TC)} \quad \mathrm{u}_b = \mathrm{u}_\infty \text{ on } \Sigma_a \quad \text{and} \quad \partial_\nu \mathrm{u}_b = \partial_\nu \mathrm{u}_\infty \text{ on } \Sigma_b \end{cases}$$

$$\mathrm{u}_\infty(x,y) = \sum_{k \geq 0} e^{i\beta_k(x-a)} (\mathrm{u}_b|_{\Sigma_a}, \varphi_k)_{\Sigma_a} \, \varphi_k(y)$$

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Remark:

$\forall l > 0,\ T^l_{D\to N}(\cdot)$ is compact.

where $T^l_{D\to N}(\mathrm{u}_b|_{\Sigma_a}) \equiv \sum_{k\geq 0} i\beta_k \underline{e^{i\beta_k l}} (\mathrm{u}_b|_{\Sigma_a}, \varphi_k)_{\Sigma_a} \, \varphi_k$

DtN formulation with « overlap »

Formulation in a bounded domain

$$\left|\begin{array}{lll}\Delta \mathrm{u}_b + \omega^2 \mathrm{u}_b = f & \text{in} & \Omega_b, \\ \partial_\nu \mathrm{u}_b = 0 & \text{on} & \partial\Omega_b, \cap \Omega, \\ \partial_\nu \mathrm{u}_b = T^l_{D\to N}(\mathrm{u}_b|_{\Sigma_a}) & \text{on} & \Sigma_b \end{array}\right.$$

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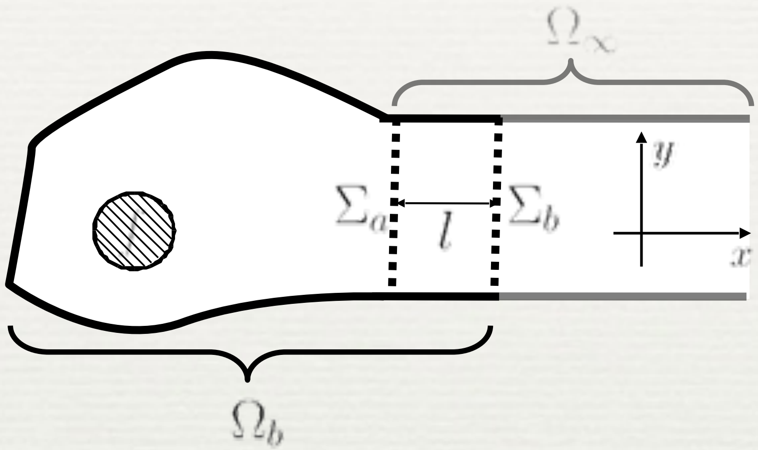


Advantage:

- Compactness of the DtN operator for iterative resolution

Limitations:

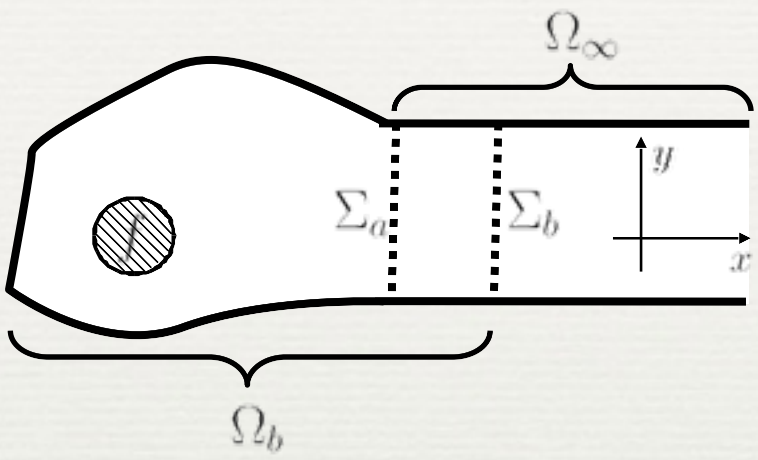
- Equivalence not always ensured (« box frequencies»)
- Requires the orthogonality of the modes

Goal: **Derive a new formulation for the anisotropic case that does not require the orthogonality of the modes**

In what follows...

Find $\mathbf{u} \in H^1_{\mathrm{loc}}(\Omega)$ the outgoing solution of

$$\left| \begin{array}{lll} \operatorname{div}(A\nabla \mathbf{u}) + \omega^2 \mathbf{u} = f & \text{in} & \Omega, \\ A\nabla \mathbf{u} \cdot \nu = 0 & \text{on} & \partial\Omega, \end{array} \right.$$



Outline:

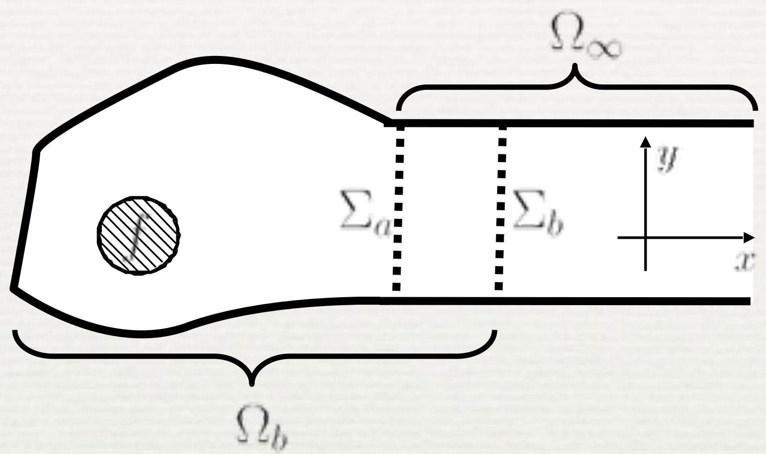
- The modal representation in Ω_∞
- The classical « Dirichlet - Neumann » multi-domain formulation
  Isotropic case $A = Id$
- New « Transparent - Neumann » multi-domain formulation
  Anisotropic case $A \neq Id$
- Iterative resolution
  Isotropic case $A = Id$

« Transparent-Neumann » formulation

Multi-domain formulation:

$$
\left|\begin{array}{lr}
\operatorname{div}(A\nabla \mathrm{u}_b) + \omega^2 \mathrm{u}_b = f & \text{in} \quad \Omega_b, \\
A\nabla \mathrm{u}_b \cdot \nu = 0 & \text{on} \quad \partial\Omega_b \cap \Omega, \\
\mathrm{u}_\infty(x,y) = \sum_{k\geq 0} a_k^\infty e^{i\beta_k(x-a)} \varphi_k^+(y) & \text{in} \quad \Omega_\infty, \\
\quad ?? & \text{on} \quad \Sigma_a, \\
A\nabla \mathrm{u}_b \cdot \nu = A\nabla \mathrm{u}_\infty \cdot \nu & \text{on} \quad \Sigma_b.
\end{array}\right.
$$

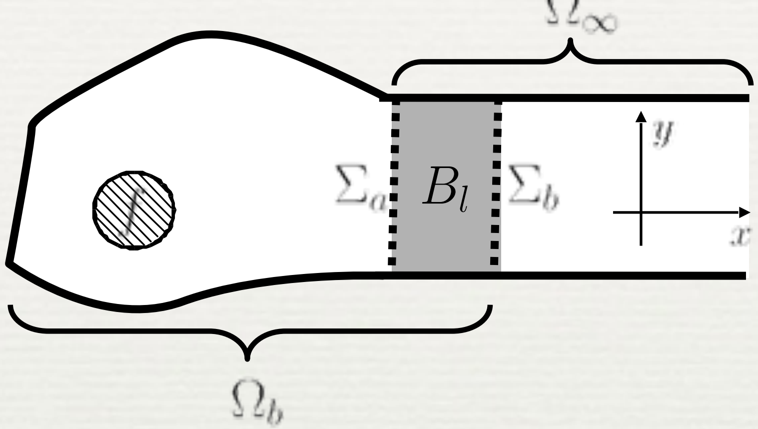
(TC)



« Transparent-Neumann » formulation

Multi-domain formulation:

$$\left|\begin{array}{lll} \operatorname{div}(A\nabla \mathbf{u}_b) + \omega^2 \mathbf{u}_b = f & \text{in} & \Omega_b, \\ A\nabla \mathbf{u}_b \cdot \nu = 0 & \text{on} & \partial\Omega_b \cap \Omega, \\ \mathbf{u}_\infty(x,y) = \sum_{k\geq 0} a_k^\infty e^{i\beta_k(x-a)} \varphi_k^+(y) & \text{in} & \Omega_\infty, \\ \quad ?? & \text{on} & \Sigma_a, \\ A\nabla \mathbf{u}_b \cdot \nu = A\nabla \mathbf{u}_\infty \cdot \nu & \text{on} & \Sigma_b. \end{array}\right. \Bigg\} \textbf{(TC)}$$



In B_l, we have *a priori* the following modal decomposition:

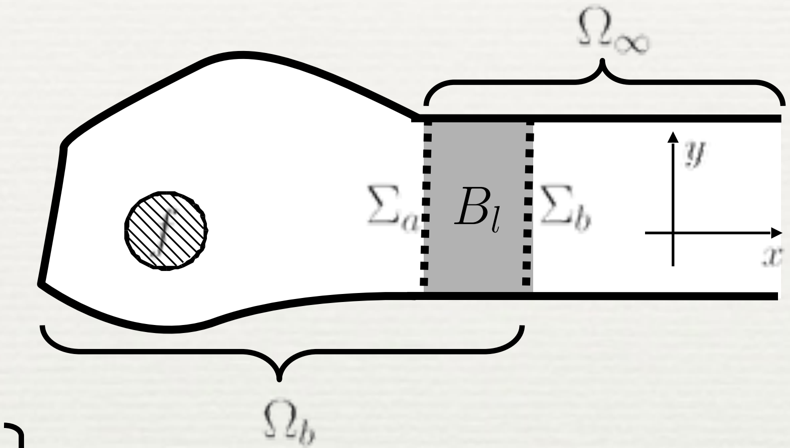
$$\mathbf{u}_b(x,y) = \sum_{k\geq 0} a_k^+ e^{i\beta_k(x-a)} \varphi_k^+(y) + a_k^- e^{-i\beta_k(x-a)} \varphi_k^-(y)$$

« Transparent-Neumann » formulation

Multi-domain formulation:

$$\left|\begin{array}{lll} \mathrm{div}\,(A\nabla \mathbf{u}_b) + \omega^2 \mathbf{u}_b = f & \text{in} & \Omega_b, \\ A\nabla \mathbf{u}_b \cdot \nu = 0 & \text{on} & \partial\Omega_b \cap \Omega, \\ \mathbf{u}_\infty(x,y) = \sum_{k\geq 0} a_k^\infty e^{i\beta_k(x-a)} \varphi_k^+(y) & \text{in} & \Omega_\infty, \\ \quad ?? & \text{on} & \Sigma_a, \\ A\nabla \mathbf{u}_b \cdot \nu = A\nabla \mathbf{u}_\infty \cdot \nu & \text{on} & \Sigma_b. \end{array}\right.$$

(TC)



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On Σ_a, we propose to match the outgoing modal amplitudes:

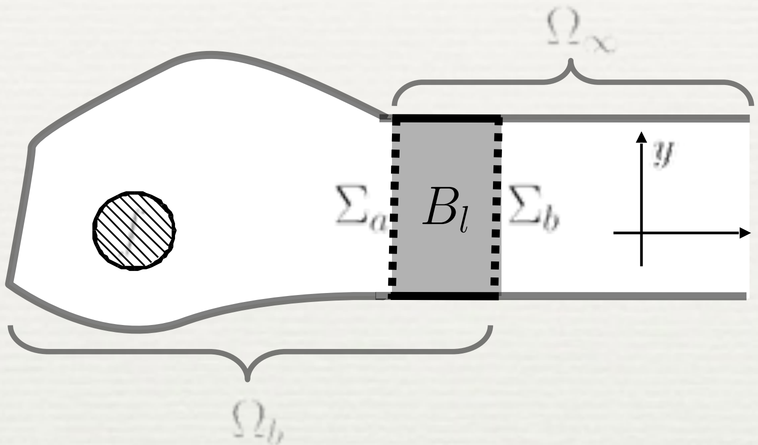
$$\forall k \in \mathbb{N}, \quad a_k^\infty = a_k^+$$

Remark: This modal condition is not local

T-N formulation: Compatibility

Multi-domain formulation:

$$
\left|
\begin{array}{lll}
\operatorname{div}(A\nabla \mathrm{u}_b) + \omega^2 \mathrm{u}_b = f & \text{in} & \Omega_b, \\
A\nabla \mathrm{u}_b \cdot \nu = 0 & \text{on} & \partial\Omega_b \cap \Omega, \\
\mathrm{u}_\infty(x,y) = \sum_{k\geq 0} a_k^\infty e^{i\beta_k(x-a)} \varphi_k^+(y) & \text{in} & \Omega_\infty, \\
\forall k \in \mathbb{N},\ a_k^+ = a_k^\infty & \text{on} & \Sigma_a, \\
A\nabla \mathrm{u}_b \cdot \nu = A\nabla \mathrm{u}_\infty \cdot \nu & \text{on} & \Sigma_b.
\end{array}
\right.
\quad \text{(the last two lines: } \textbf{(TC)}\text{)}
$$



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Compatibility $\mathrm{u}_b = \mathrm{u}_\infty$ in $B_l = \Omega_b \cap \Omega_\infty$

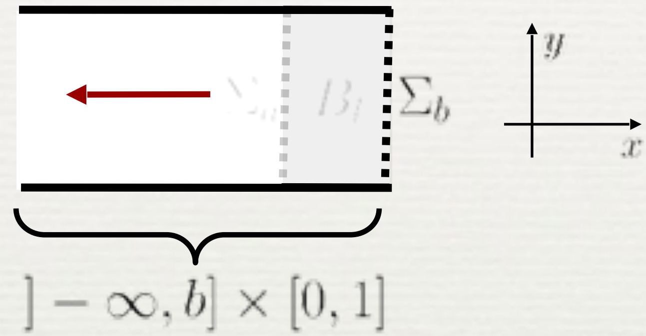
The difference $\mathrm{v} = \mathrm{u}_b - \mathrm{u}_\infty$ satisfies

$$\mathrm{v}(x,y) = \sum_{k\geq 0} a_k^- e^{-i\beta_k(x-a)} \varphi_k^-(y) \quad \text{in} \quad B_l$$

T-N formulation: Compatibility

Multi-domain formulation:

$$\left|\begin{array}{lll} \operatorname{div}(A\nabla u_b) + \omega^2 u_b = f & \text{in} & \Omega_b, \\ A\nabla u_b \cdot \nu = 0 & \text{on} & \partial\Omega_b \cap \Omega, \\ u_\infty(x,y) = \sum_{k\geq 0} a_k^\infty e^{i\beta_k(x-a)} \varphi_k^+(y) & \text{in} & \Omega_\infty, \\ \forall k \in \mathbb{N},\ a_k^+ = a_k^\infty & \text{on} & \Sigma_a, \\ A\nabla u_b \cdot \nu = A\nabla u_\infty \cdot \nu & \text{on} & \Sigma_b. \end{array}\right. \quad \left.\begin{array}{l} \\ \\ \end{array}\right\} \textbf{(TC)}$$



Compatibility $u_b = u_\infty$ in $B_l = \Omega_b \cap \Omega_\infty$

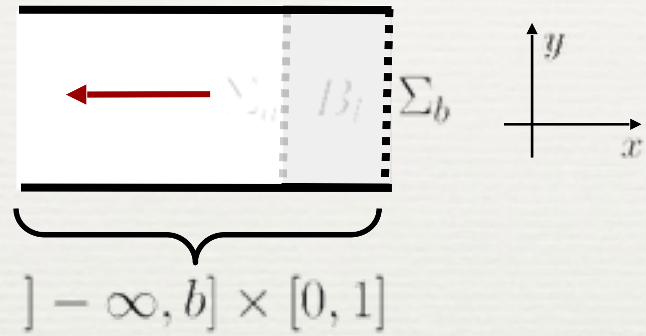
The difference $v = u_b - u_\infty$ verifies

$$v(x,y) = \sum_{k\geq 0} a_k^- e^{-i\beta_k(x-a)} \varphi_k^-(y) \quad \text{in} \quad \overbrace{]-\infty, b] \times [0,1]}^{\text{«Fictitious» half-guide}}$$

T-N formulation: Compatibility

Multi-domain formulation:

$$\left|\begin{array}{lll} \operatorname{div}(A\nabla u_b) + \omega^2 u_b = f & \text{in} & \Omega_b, \\ A\nabla u_b \cdot \nu = 0 & \text{on} & \partial\Omega_b \cap \Omega, \\ u_\infty(x,y) = \sum\limits_{k\geq 0} a_k^\infty e^{i\beta_k(x-a)} \varphi_k^+(y) & \text{in} & \Omega_\infty, \\ \forall k \in \mathbb{N},\ a_k^+ = a_k^\infty & \text{on} & \Sigma_a, \\ A\nabla u_b \cdot \nu = A\nabla u_\infty \cdot \nu & \text{on} & \Sigma_b. \end{array}\right. \quad \left.\begin{array}{l} \\ \\ \end{array}\right\} \textbf{(TC)}$$



Compatibility $u_b = u_\infty$ in $B_l = \Omega_b \cap \Omega_\infty$

The difference $v = u_b - u_\infty$ verifies

$$v(x,y) = \sum_{k\geq 0} a_k^- e^{-i\beta_k(x-a)} \varphi_k^-(y) \quad \text{in} \quad \overbrace{]-\infty, b] \times [0,1]}^{\text{«Fictitious» half-guide}}$$

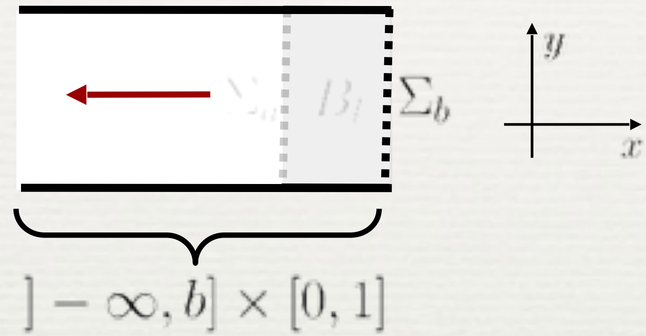
In other words, v is an outgoing solution of:

$$\left|\begin{array}{lll} \operatorname{div}(A\nabla v) + \omega^2 v = 0 & \text{in} &]-\infty, b] \times [0,1], \\ A\nabla v \cdot \nu = 0 & \text{on} & \Sigma_b \cup]-\infty, b] \times \{0,1\}, \end{array}\right.$$

T-N formulation: Compatibility

Multi-domain formulation:

$$\left|\begin{array}{lll} \operatorname{div}(A\nabla \mathrm{u}_b) + \omega^2 \mathrm{u}_b = f & \text{in} & \Omega_b, \\ A\nabla \mathrm{u}_b \cdot \nu = 0 & \text{on} & \partial\Omega_b \cap \Omega, \\ \mathrm{u}_\infty(x,y) = \displaystyle\sum_{k\geq 0} a_k^\infty e^{i\beta_k(x-a)} \varphi_k^+(y) & \text{in} & \Omega_\infty, \\ \forall k \in \mathbb{N},\ a_k^+ = a_k^\infty & \text{on} & \Sigma_a, \\ A\nabla \mathrm{u}_b \cdot \nu = A\nabla \mathrm{u}_\infty \cdot \nu & \text{on} & \Sigma_b. \end{array}\right. \quad \left.\begin{array}{l} \\ \\ \end{array}\right\} \textbf{(TC)}$$



Compatibility $\mathrm{u}_b = \mathrm{u}_\infty$ in $B_l = \Omega_b \cap \Omega_\infty$

The difference $\mathrm{v} = \mathrm{u}_b - \mathrm{u}_\infty$ verifies

$$\mathrm{v}(x,y) = \sum_{k\geq 0} a_k^- e^{-i\beta_k(x-a)} \varphi_k^-(y) \quad \text{in} \quad \overbrace{]-\infty, b] \times [0,1]}^{\text{«Fictitious» half-guide}}$$

In other words, v is an outgoing solution of:

$$\left|\begin{array}{lll} \operatorname{div}(A\nabla \mathrm{v}) + \omega^2 \mathrm{v} = 0 & \text{in} &]-\infty, b] \times [0,1], \\ A\nabla \mathrm{v} \cdot \nu = 0 & \text{on} & \Sigma_b \cup]-\infty, b] \times \{0,1\}, \end{array}\right.$$

$$\boxed{\longrightarrow \quad \mathrm{v} = 0}$$

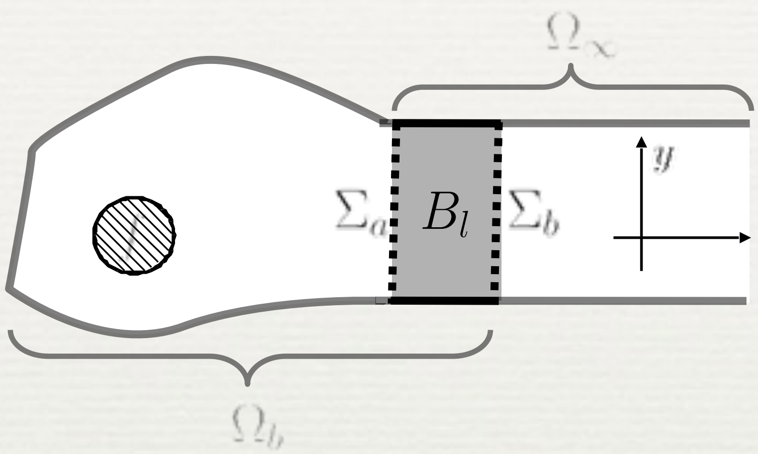
except for exceptional frequencies associated to « edge modes »

Remark: In the isotropic case, v = 0

T-N formulation: Modal amplitudes

Equivalent multi-domain formulation:

$$\left|\begin{array}{lll} \operatorname{div}(A\nabla \mathrm{u}_b) + \omega^2 \mathrm{u}_b = f & \text{in} & \Omega_b, \\ A\nabla \mathrm{u}_b \cdot \nu = 0 & \text{on} & \partial\Omega_b \cap \Omega, \\ \mathrm{u}_\infty(x,y) = \sum_{k\geq 0} a_k^\infty e^{i\beta_k(x-a)} \varphi_k^+(y) & \text{in} & \Omega_\infty, \\ \forall k \in \mathbb{N},\ a_k^+ = a_k^\infty & \text{on} & \Sigma_a, \\ A\nabla \mathrm{u}_b \cdot \nu = A\nabla \mathrm{u}_\infty \cdot \nu & \text{on} & \Sigma_b. \end{array}\right. \quad \left.\begin{array}{l} \\ \\ \end{array}\right\} \textbf{(TC)}$$



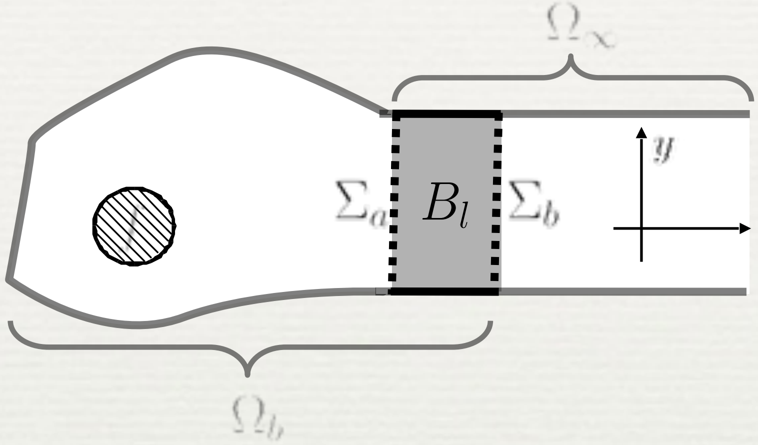
How to compute the modal amplitudes a_k^+ given u_b ?

$$\mathrm{u}_b(x,y) = \sum_{k\geq 0} a_k^+ e^{i\beta_k(x-a)} \varphi_k^+(y) + a_k^- e^{-i\beta_k(x-a)} \varphi_k^-(y)$$

T-N formulation: Modal amplitudes

Equivalent multi-domain formulation:

$$\left|\begin{array}{lll} \mathrm{div}\,(A\nabla u_b) + \omega^2 u_b = f & \text{in} & \Omega_b, \\ A\nabla u_b \cdot \nu = 0 & \text{on} & \partial\Omega_b \cap \Omega, \\ u_\infty(x,y) = \sum\limits_{k\geq 0} a_k^\infty e^{i\beta_k(x-a)} \varphi_k^+(y) & \text{in} & \Omega_\infty, \\ \forall k \in \mathbb{N},\ a_k^+ = a_k^\infty & \text{on} & \Sigma_a, \\ A\nabla u_b \cdot \nu = A\nabla u_\infty \cdot \nu & \text{on} & \Sigma_b. \end{array}\right. \quad \text{(TC)}$$



How to compute the modal amplitudes a_k^+ given u_b ?

$$u_b(x,y) = \sum_{k\geq 0} a_k^+ e^{i\beta_k(x-a)} \varphi_k^+(y) + a_k^- e^{-i\beta_k(x-a)} \varphi_k^-(y)$$

Thanks to the bi-orthogonality relations, it comes:

$$a_k^+ = \left(u_b, \psi_k^-\right)_{\Sigma_a} - \left(A\nabla u_b \cdot \nu, \varphi_k^-\right)_{\Sigma_a}$$

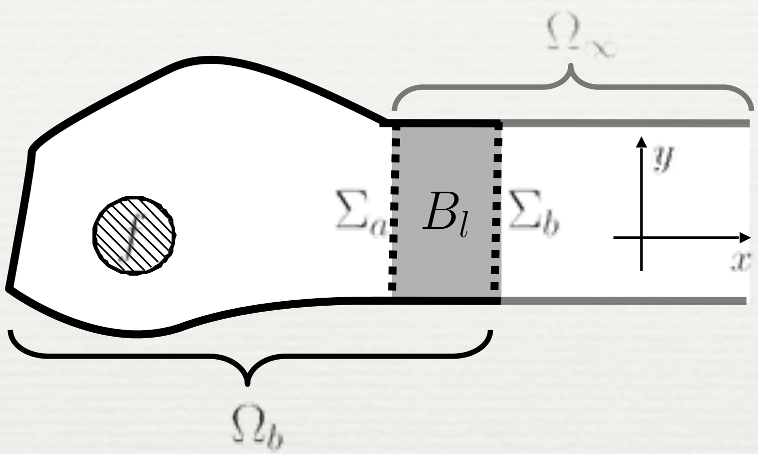
Remark: This method works only with overlap !

TtN formulation

Equivalent multi-domain formulation:

$$\left|\begin{array}{lll} \operatorname{div}(A\nabla u_b) + \omega^2 u_b = f & \text{in} & \Omega_b, \\ A\nabla u_b \cdot \nu = 0 & \text{on} & \partial\Omega_b \cap \Omega, \\ u_\infty(x,y) = \sum_{k\geq 0} a_k^\infty e^{i\beta_k(x-a)} \varphi_k^+(y) & \text{in} & \Omega_\infty, \\ \forall k \in \mathbb{N},\ a_k^+ = a_k^\infty & \text{on} & \Sigma_a, \\ A\nabla u_b \cdot \nu = A\nabla u_\infty \cdot \nu & \text{on} & \Sigma_b. \end{array}\right.$$

(TC)



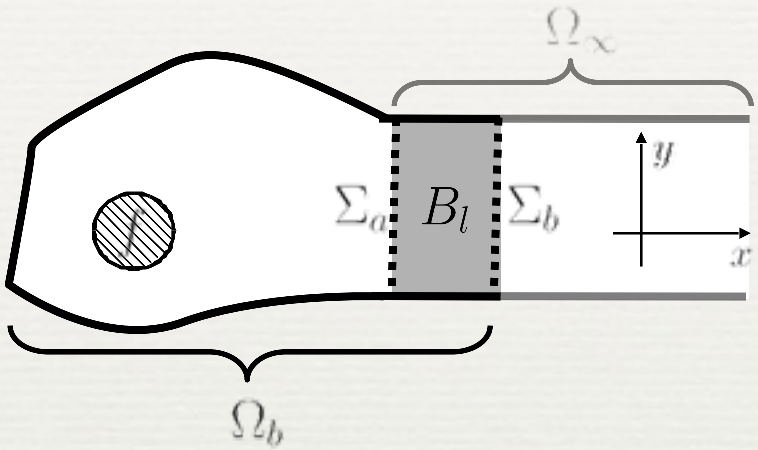
Formulation in a bounded domain

$$\left|\begin{array}{lll} \operatorname{div}(A\nabla u_b) + \omega^2 u_b = f & \text{in} & \Omega_b, \\ A\nabla u_b \cdot \nu = 0 & \text{on} & \partial\Omega_b, \cap\Omega, \\ A\nabla u_b \cdot \nu = T^l_{T\to N}(u_b) & \text{on} & \Sigma_b \end{array}\right.$$

TtN formulation

Equivalent multi-domain formulation:

$$
\left|
\begin{array}{lll}
\operatorname{div}(A\nabla \mathrm{u}_b) + \omega^2 \mathrm{u}_b = f & \text{in} & \Omega_b, \\
A\nabla \mathrm{u}_b \cdot \nu = 0 & \text{on} & \partial\Omega_b \cap \Omega, \\
\mathrm{u}_\infty(x,y) = \sum_{k\geq 0} a_k^\infty e^{i\beta_k(x-a)} \varphi_k^+(y) & \text{in} & \Omega_\infty, \\
\forall k \in \mathbb{N},\ a_k^+ = a_k^\infty & \text{on} & \Sigma_a, \\
A\nabla \mathrm{u}_b \cdot \nu = A\nabla \mathrm{u}_\infty \cdot \nu & \text{on} & \Sigma_b.
\end{array}
\right.
\quad \left.\begin{array}{l} \\ \\ \end{array}\right\} \textbf{(TC)}
$$



Formulation in a bounded domain

$$
\left|
\begin{array}{lll}
\operatorname{div}(A\nabla \mathrm{u}_b) + \omega^2 \mathrm{u}_b = f & \text{in} & \Omega_b, \\
A\nabla \mathrm{u}_b \cdot \nu = 0 & \text{on} & \partial\Omega_b, \cap\Omega, \\
A\nabla \mathrm{u}_b \cdot \nu = T^l_{T\to N}(\mathrm{u}_b) & \text{on} & \Sigma_b
\end{array}
\right.
$$

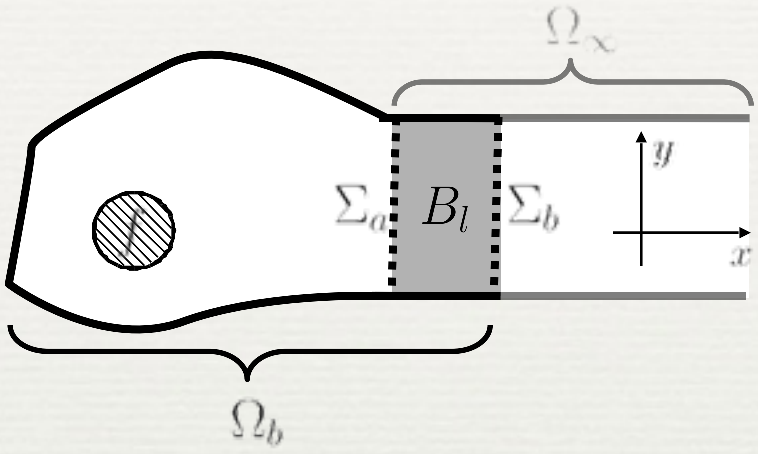
Remark:

$\forall l \geq l_{min},\quad T^l_{T\to N}(\cdot)$ is compact.

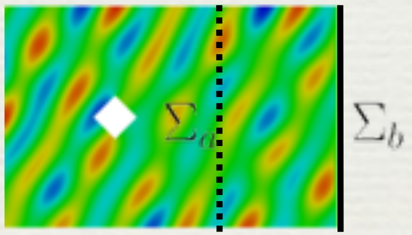
TtN formulation: illustration

Formulation in a bounded domain

$$\left|\begin{array}{lll} \operatorname{div}(A\nabla \mathbf{u}_b) + \omega^2 \mathbf{u}_b = f & \text{in} & \Omega_b, \\ A\nabla \mathbf{u}_b \cdot \nu = 0 & \text{on} & \partial\Omega_b, \cap \Omega, \\ A\nabla \mathbf{u}_b \cdot \nu = T^l_{T\to N}(\mathbf{u}_b) & \text{on} & \Sigma_b \end{array}\right.$$



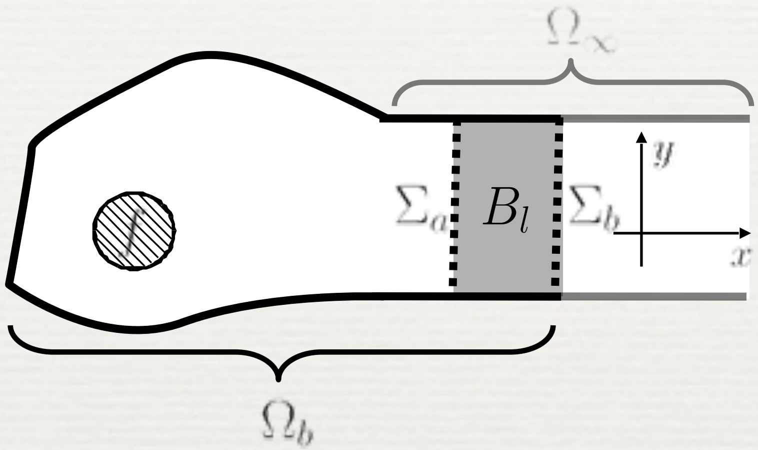
Real part of the solution:



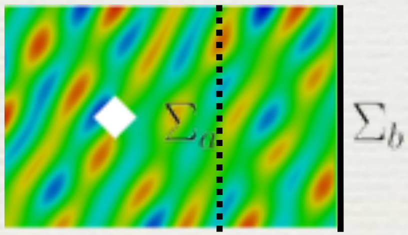
TtN formulation: illustration

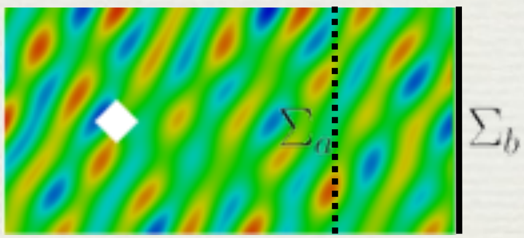
Formulation in a bounded domain

$$\left|\begin{array}{lr} \operatorname{div}(A\nabla \mathbf{u}_b) + \omega^2 \mathbf{u}_b = f \quad \text{in} & \Omega_b, \\ A\nabla \mathbf{u}_b \cdot \nu = 0 \quad \text{on} & \partial\Omega_b, \cap\Omega, \\ A\nabla \mathbf{u}_b \cdot \nu = T^l_{T\to N}(\mathbf{u}_b) \quad \text{on} & \Sigma_b \end{array}\right.$$



Real part of the solution:

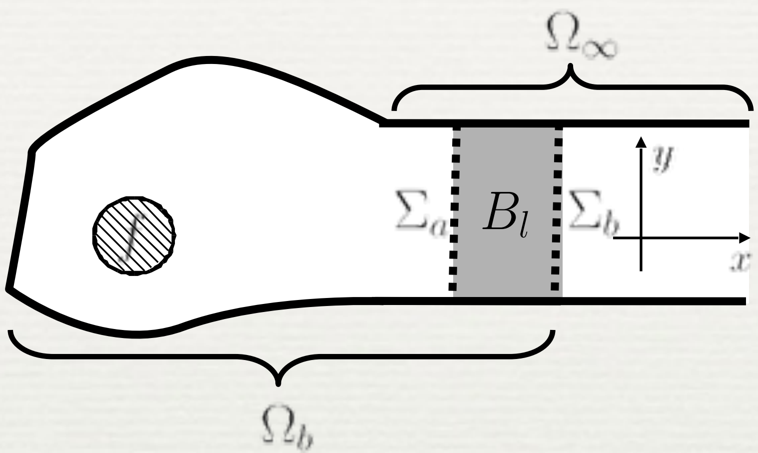




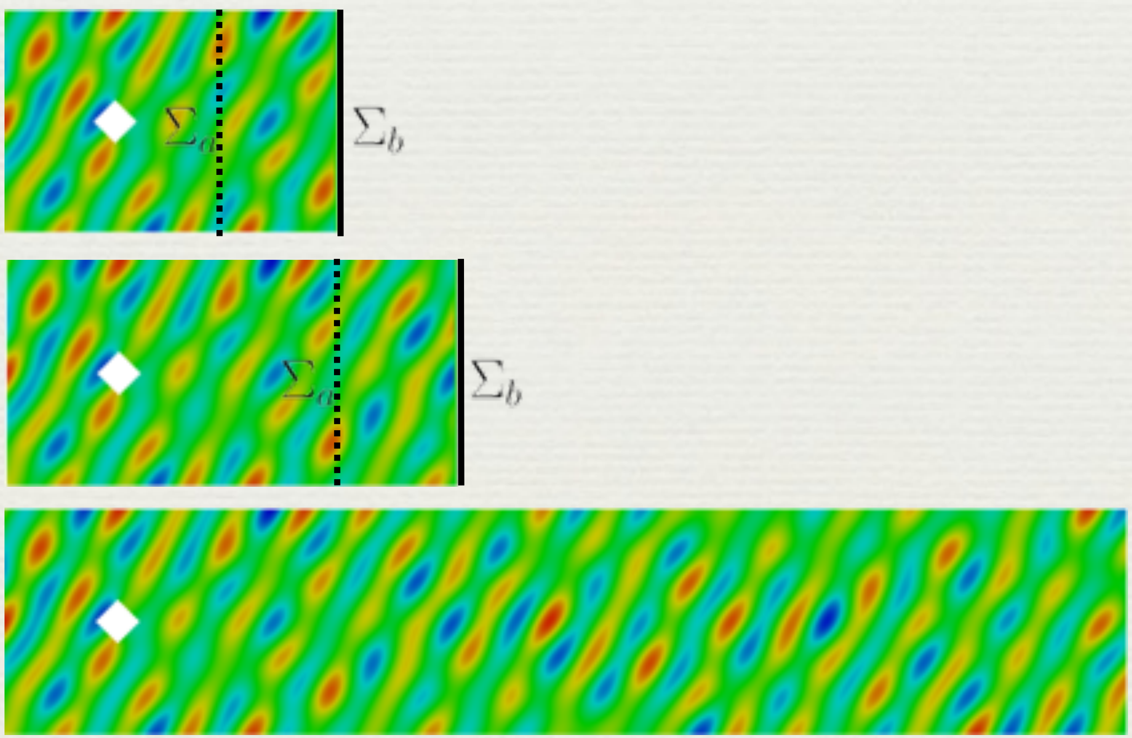
TtN formulation: illustration

Formulation in a bounded domain

$$
\left|\begin{array}{lr}
\operatorname{div}(A\nabla \mathbf{u}_b) + \omega^2 \mathbf{u}_b = f & \text{in} \quad \Omega_b, \\
A\nabla \mathbf{u}_b \cdot \nu = 0 & \text{on} \quad \partial\Omega_b, \cap\Omega, \\
A\nabla \mathbf{u}_b \cdot \nu = T^l_{T\to N}(\mathbf{u}_b) & \text{on} \quad \Sigma_b
\end{array}\right.
$$



Real part of the solution:

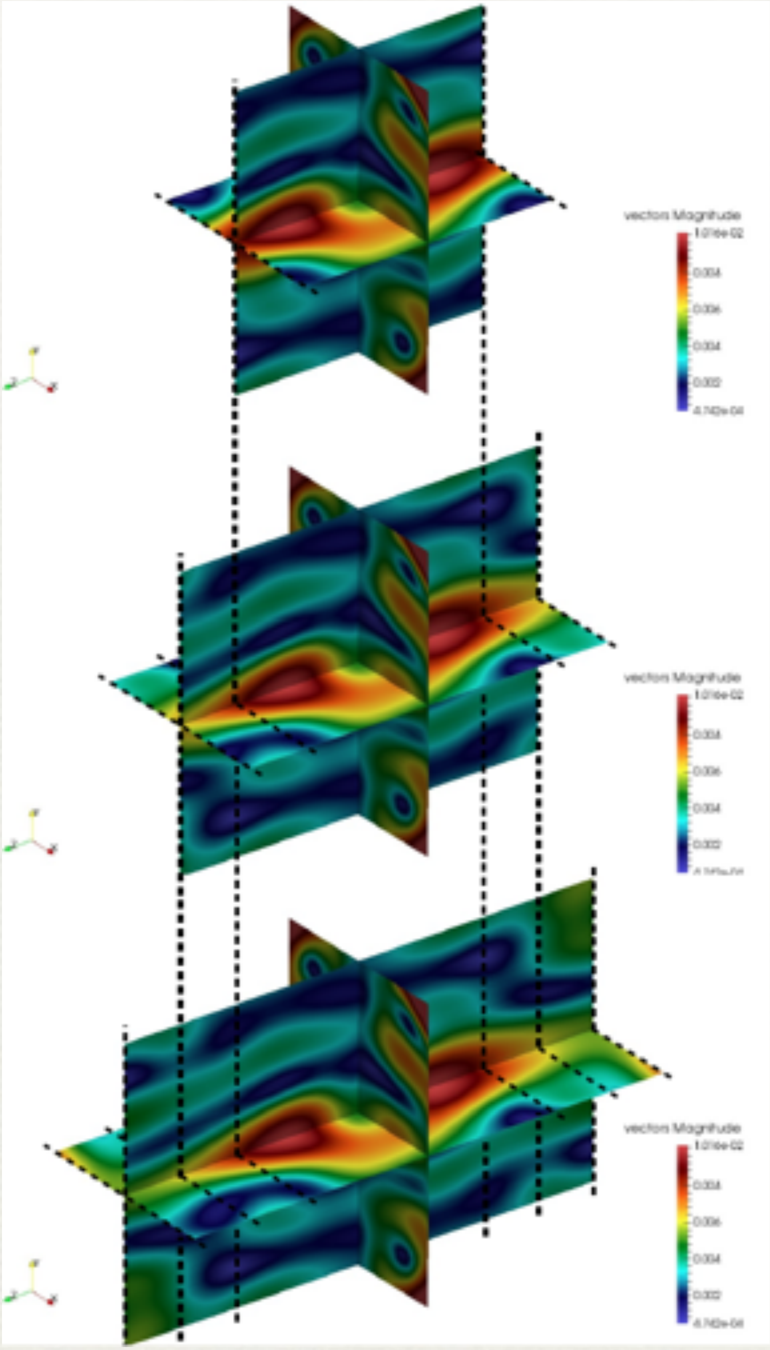


TtN formulation for the elastic waveguide

And it works also in 3D

Straight waveguide with a rectangular cross-section, made of an anisotropic material:

$$\mathbf{C} = \begin{bmatrix} 32.4 & 20.2 & 11.9 & 7.1 & -9.1 & -15.7 \\ & 24.8 & 10.7 & 5.7 & -6.2 & -12.5 \\ & & 15.4 & 2.4 & -3.4 & -4.2 \\ & & & 6.6 & -4.1 & -6.1 \\ & & & & 8.3 & 7.9 \\ & & & & & 17.4 \end{bmatrix} \times 10^{10}\,\mathrm{Pa}$$

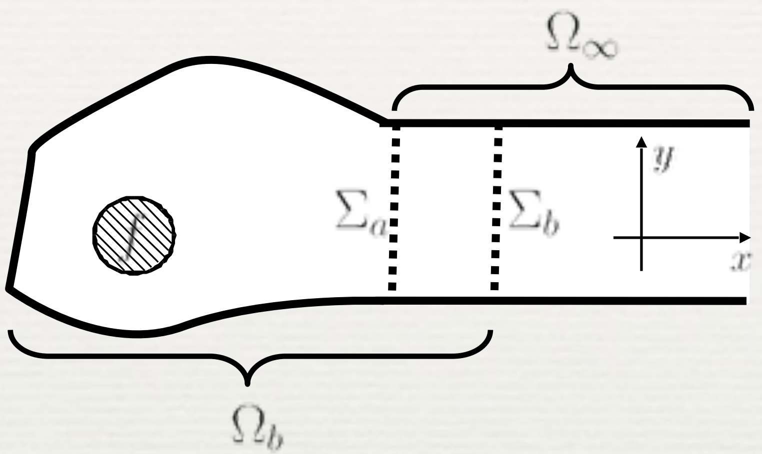


The result is independent of the position of the artificial boundaries.

In what follows...

Find $\mathrm{u} \in H^1_{\mathrm{loc}}(\Omega)$ the outgoing solution of

$$\left|\begin{array}{ll} \operatorname{div}(A\nabla \mathrm{u}) + \omega^2 \mathrm{u} = f & \text{in} \quad \Omega, \\ A\nabla \mathrm{u} \cdot \nu = 0 & \text{on} \quad \partial\Omega, \end{array}\right.$$



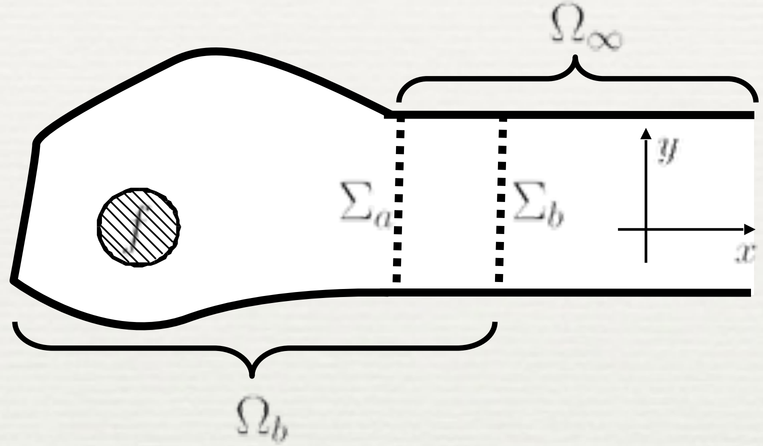
Outline:

- The modal representation in Ω_∞
- The classical « Dirichlet - Neumann » multi-domain formulation
  Isotropic case $A = Id$
- New multi-domain formulation « Transparent - Neumann »
  Anisotropic case $A \neq Id$
- $\boxed{\text{Iterative resolution} \quad \text{Isotropic case} \quad A = Id}$

Two approaches in the isotropic case

Formulation in a bounded domain

$$\left|\begin{array}{lll} \Delta \mathbf{u}_b + \omega^2 \mathbf{u}_b = f & \text{in} & \Omega_b, \\ \partial_\nu \mathbf{u}_b = 0 & \text{on} & \partial\Omega_b, \cap \Omega, \\ \partial_\nu \mathbf{u}_b = T^l_{\dots\to N}(\mathbf{u}_b) & \text{on} & \Sigma_b \end{array}\right.$$



where:

- $T^l_{\dots\to N} = T^l_{D\to N}$

  for the Dirichlet - Neumann formulation

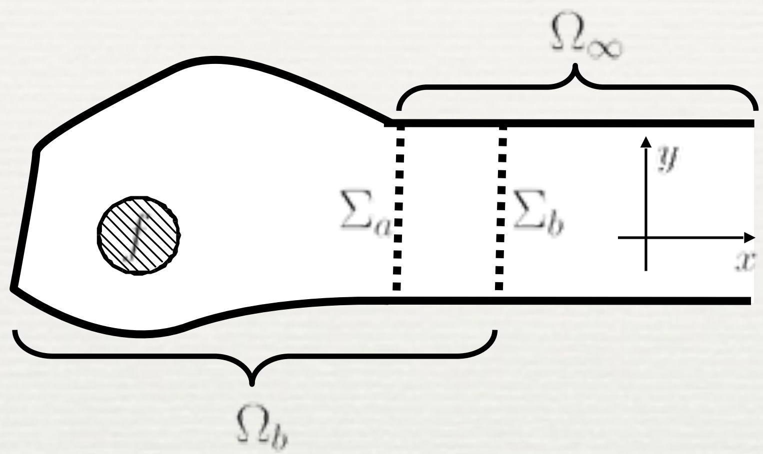
- $T^l_{\dots\to N} = T^l_{T\to N}$

  for the Transparent - Neumann formulation

Two approaches in the isotropic case

Formulation in a bounded domain

$$\left|\begin{array}{lll}\Delta \mathrm{u}_b + \omega^2 \mathrm{u}_b = f & \text{in} & \Omega_b, \\ \partial_\nu \mathrm{u}_b = 0 & \text{on} & \partial\Omega_b, \cap \Omega, \\ \partial_\nu \mathrm{u}_b = T^l_{\dots\to N}(\mathrm{u}_b) & \text{on} & \Sigma_b\end{array}\right.$$



where:

- $T^l_{\dots\to N} = T^l_{D\to N}$

  for the Dirichlet - Neumann formulation

- $T^l_{\dots\to N} = T^l_{T\to N}$

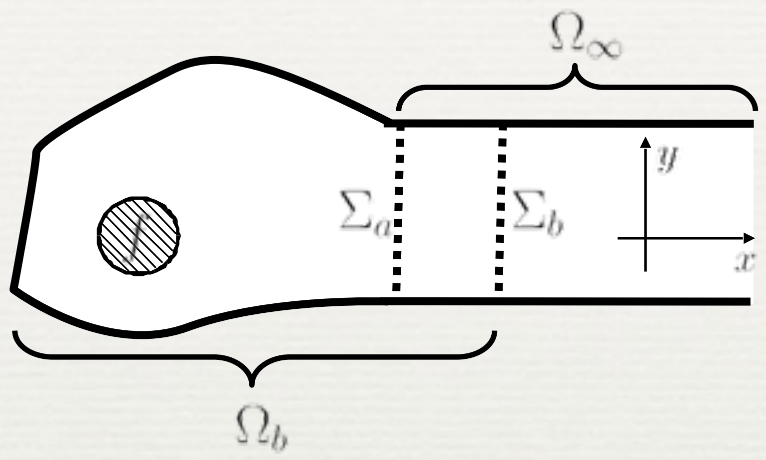
  for the Transparent - Neumann formulation

→ Non local condition

Schwarz algorithm

Iterative algorithm

$$\left|\begin{array}{lll} \Delta \mathbf{u}_b^{(n)} + \omega^2 \mathbf{u}_b^{(n)} = f & \text{in} & \Omega_b, \\ \partial_\nu \mathbf{u}_b^{(n)} = 0 & \text{on} & \partial\Omega_b, \cap\Omega, \\ \partial_\nu \mathbf{u}_b^{(n)} = T^l_{\dots\to N}(\mathbf{u}_b^{(n-1)}) & \text{on} & \Sigma_b \end{array}\right.$$



where:

- $T^l_{\dots\to N} = T^l_{D\to N}$

  for the Dirichlet - Neumann formulation

- $T^l_{\dots\to N} = T^l_{T\to N}$

  for the Transparent - Neumann formulation

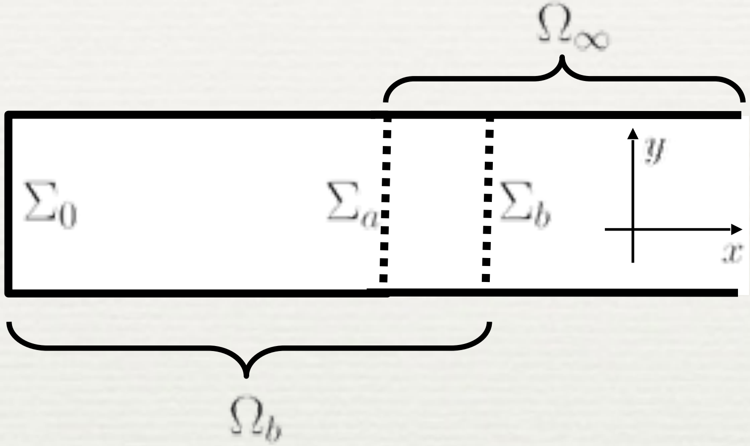
$\longrightarrow$ Local condition

Convergence of the algorithm ?

Convergence analysis in a straight guide

Iterative algorithm

$$\left|\begin{array}{lll}\Delta \mathbf{u}_b^{(n)} + \omega^2 \mathbf{u}_b^{(n)} = 0 & \text{in} & [0,b]\times[0,1], \\ \partial_\nu \mathbf{u}_b^{(n)} = 0 & \text{on} & [0,b]\times\{0,1\}\cup\Sigma_0, \\ \partial_\nu \mathbf{u}_b^{(n)} = T^l_{\dots\to N}(\mathbf{u}_b^{(n-1)}) & \text{on} & \Sigma_b \end{array}\right.$$



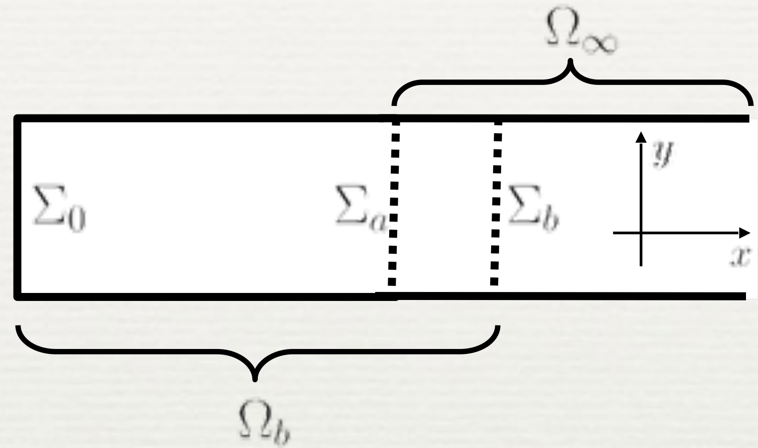
Thanks to the separable geometry and the condition on Σ_0, it comes:

$$\mathbf{u}_b^{(n)} = \sum_{k\geq 0} a_k^{(n)} \left(e^{i\beta_k x} + e^{-i\beta_k x}\right) \varphi_k(y) \qquad \text{in} \quad \Omega_b$$

Convergence analysis in a straight guide

Iterative algorithm

$$\left|\begin{array}{llr} \Delta \mathbf{u}_b^{(n)} + \omega^2 \mathbf{u}_b^{(n)} = 0 & \text{in} & [0,b]\times[0,1], \\ \partial_\nu \mathbf{u}_b^{(n)} = 0 & \text{on} & [0,b]\times\{0,1\}\cup\Sigma_0, \\ \partial_\nu \mathbf{u}_b^{(n)} = T^l_{\dots\to N}(\mathbf{u}_b^{(n-1)}) & \text{on} & \Sigma_b \end{array}\right.$$



Thanks to the separable geometry and the condition on Σ_0, it comes:

$$\mathbf{u}_b^{(n)} = \sum_{k\geq 0} a_k^{(n)} \left(e^{i\beta_k x} + e^{-i\beta_k x}\right) \varphi_k(y) \qquad \text{in} \quad \Omega_b$$

Dirichlet - Neumann

$$a_k^{(n)} = a_k^{(n-1)} \frac{\left(e^{i\beta_k a} + e^{-i\beta_k a}\right) e^{i\beta_k l}}{e^{i\beta_k b} - e^{-i\beta_k b}}$$

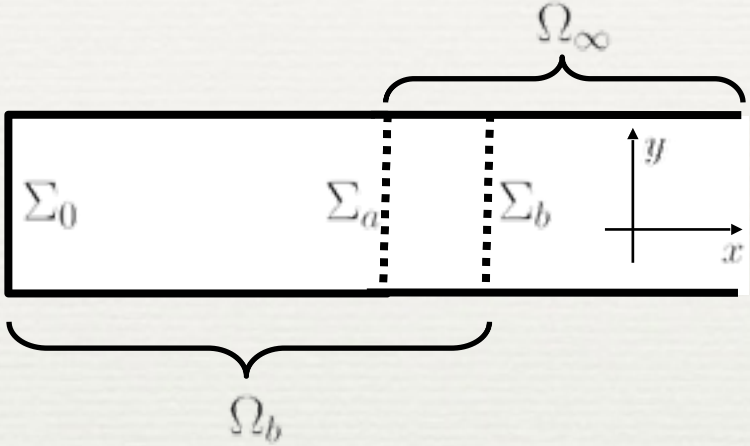
Transparent - Neumann

$$a_k^{(n)} = a_k^{(n-1)} \frac{e^{i\beta_k a} e^{i\beta_k l}}{e^{i\beta_k b} - e^{-i\beta_k b}}$$

Convergence analysis in a straight guide

Iterative algorithm

$$\left|\begin{array}{lll} \Delta \mathbf{u}_b^{(n)} + \omega^2 \mathbf{u}_b^{(n)} = 0 & \text{in} & [0,b]\times[0,1], \\ \partial_\nu \mathbf{u}_b^{(n)} = 0 & \text{on} & [0,b]\times\{0,1\}\cup\Sigma_0, \\ \partial_\nu \mathbf{u}_b^{(n)} = T^l_{\dots\to N}(\mathbf{u}_b^{(n-1)}) & \text{on} & \Sigma_b \end{array}\right.$$



Thanks to the separable geometry and the condition on Σ_0, it comes:

$$\mathbf{u}_b^{(n)} = \sum_{k\geq 0} a_k^{(n)} \left(e^{i\beta_k x} + e^{-i\beta_k x}\right) \varphi_k(y) \qquad \text{in} \quad \Omega_b$$

Dirichlet - Neumann

$$a_k^{(n)} = a_k^{(n-1)} \underbrace{\frac{\left(e^{i\beta_k a} + e^{-i\beta_k a}\right) e^{i\beta_k l}}{e^{i\beta_k b} - e^{-i\beta_k b}}}_{\lambda_k}$$

Transparent - Neumann

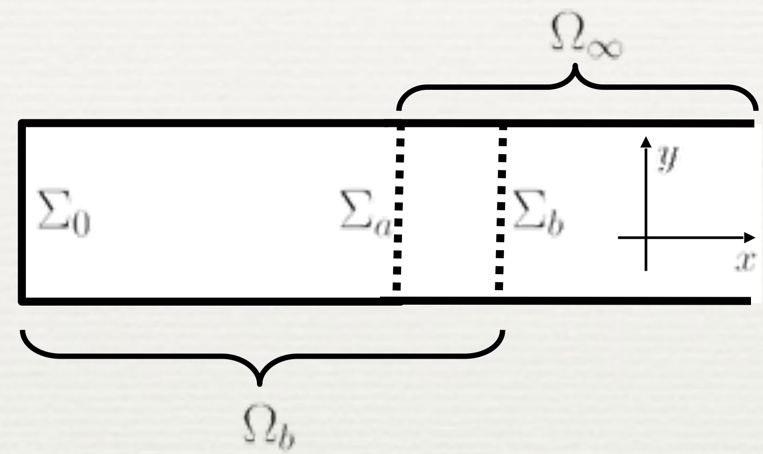
$$a_k^{(n)} = a_k^{(n-1)} \underbrace{\frac{e^{i\beta_k a} e^{i\beta_k l}}{e^{i\beta_k b} - e^{-i\beta_k b}}}_{\lambda_k}$$

The convergence is ensured if and only if $\sup |\lambda_k| < 1$

Convergence analysis in a straight guide

Iterative algorithm

$$\left|\begin{array}{lll} \Delta \mathbf{u}_b^{(n)} + \omega^2 \mathbf{u}_b^{(n)} = 0 & \text{in} & [0,b]\times[0,1], \\ \partial_\nu \mathbf{u}_b^{(n)} = 0 & \text{on} & [0,b]\times\{0,1\}\cup\Sigma_0, \\ \partial_\nu \mathbf{u}_b^{(n)} = T^l_{\dots\to N}(\mathbf{u}_b^{(n-1)}) & \text{on} & \Sigma_b \end{array}\right.$$



Thanks to the separable geometry and the condition on Σ_0, it comes:

$$\mathbf{u}_b^{(n)} = \sum_{k\geq 0} a_k^{(n)} \left(e^{i\beta_k x} + e^{-i\beta_k x}\right) \varphi_k(y) \qquad \text{in} \quad \Omega_b$$

Dirichlet - Neumann

$$a_k^{(n)} = a_k^{(n-1)} \underbrace{\frac{\left(e^{i\beta_k a} + e^{-i\beta_k a}\right) e^{i\beta_k l}}{e^{i\beta_k b} - e^{-i\beta_k b}}}_{\lambda_k}$$

The convergence cannot be ensured !

Transparent - Neumann

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Convergence analysis in a straight guide

Iterative algorithm

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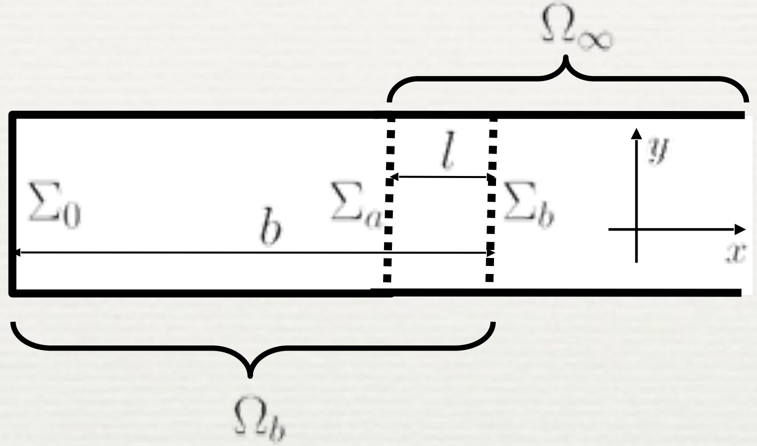

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$$\lambda_k \underset{k\to\infty}{\sim} e^{-2|\beta_k| l}$$

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Transparent - Neumann

$$a_k^{(n)} = a_k^{(n-1)} \underbrace{\frac{e^{i\beta_k a} e^{i\beta_k l}}{e^{i\beta_k b} - e^{-i\beta_k b}}}$$

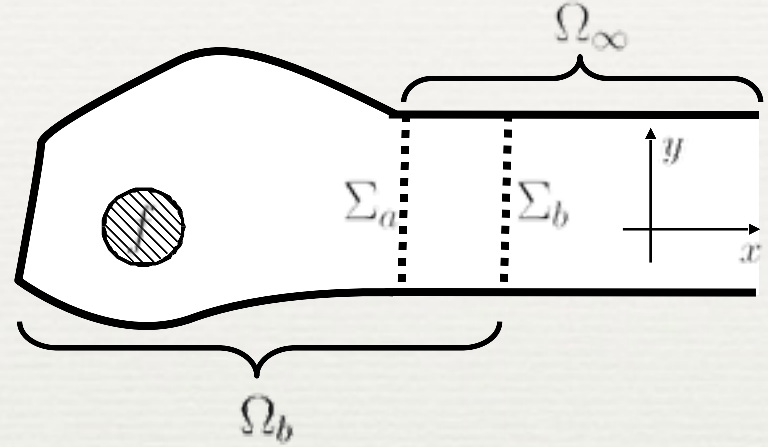
$$\lambda_k \underset{k\to\infty}{\sim} e^{-2|\beta_k| b}$$

The convergence cannot be ensured !

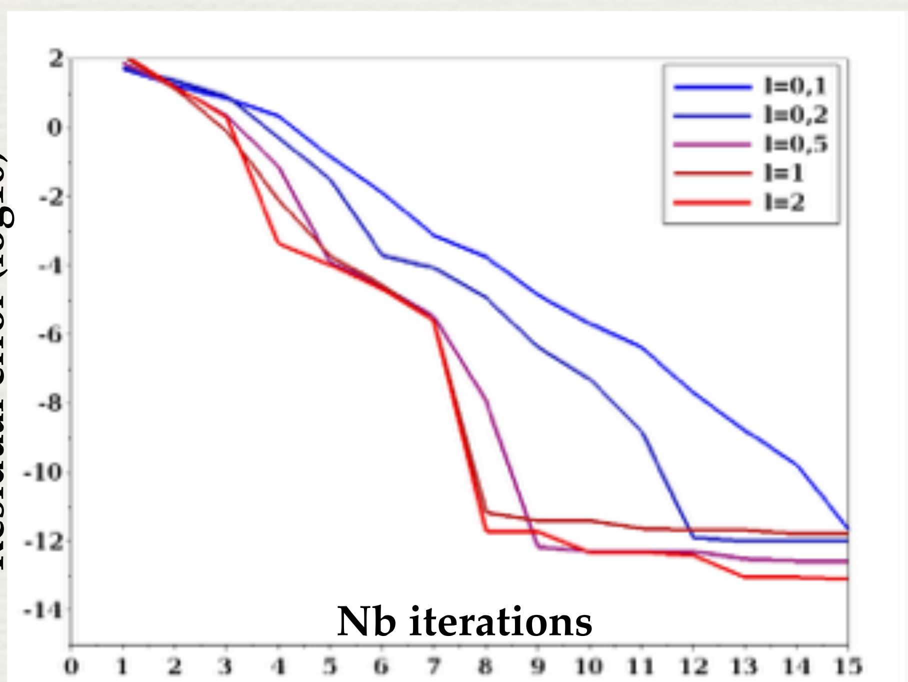
GMRES convergence: illustration

Formulation in a bounded domain

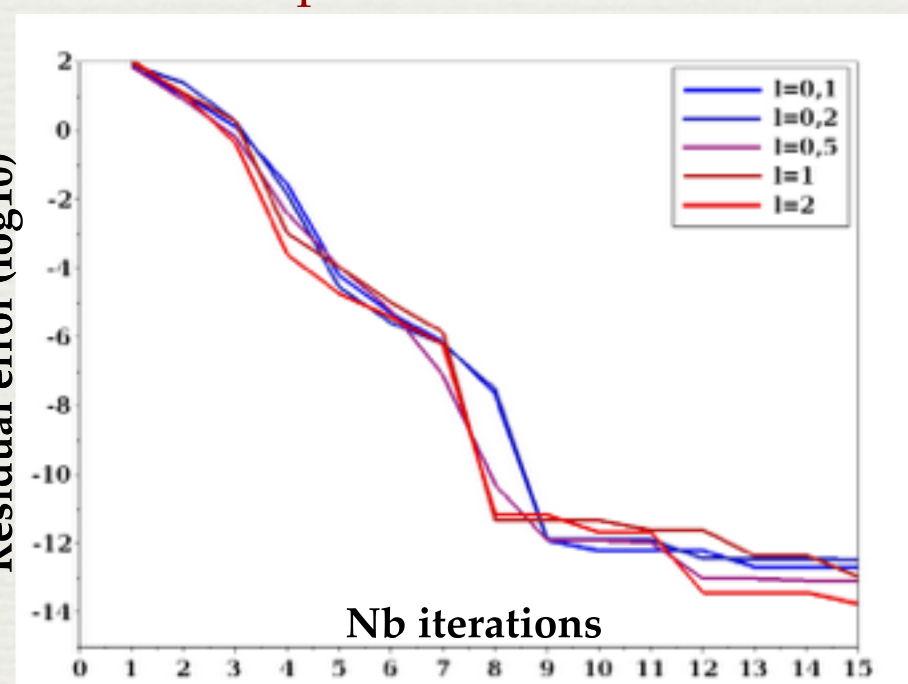
$$\left|\begin{array}{lll} \Delta \mathrm{u}_b + \omega^2 \mathrm{u}_b = f & \text{in} & \Omega_b, \\ \partial_\nu \mathrm{u}_b = 0 & \text{on} & \partial\Omega_b, \cap \Omega, \\ \partial_\nu \mathrm{u}_b = T^l_{\dots\to N}(\mathrm{u}_b) & \text{on} & \Sigma_b \end{array}\right.$$



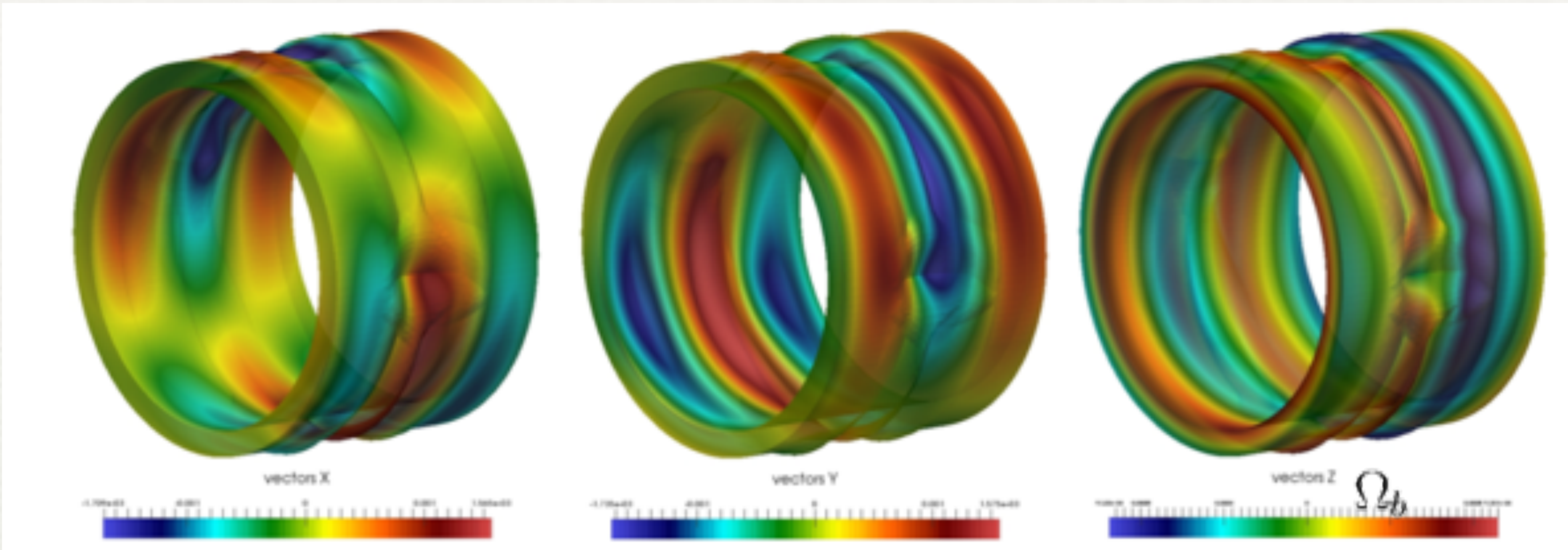
Dirichlet - Neumann



Transparent - Neumann



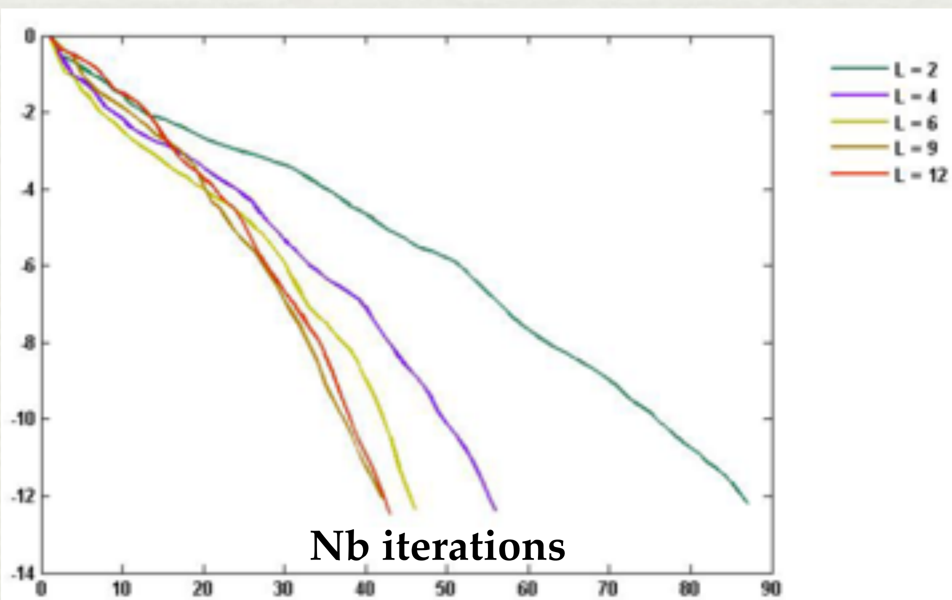
GMRES convergence: illustration in 3D

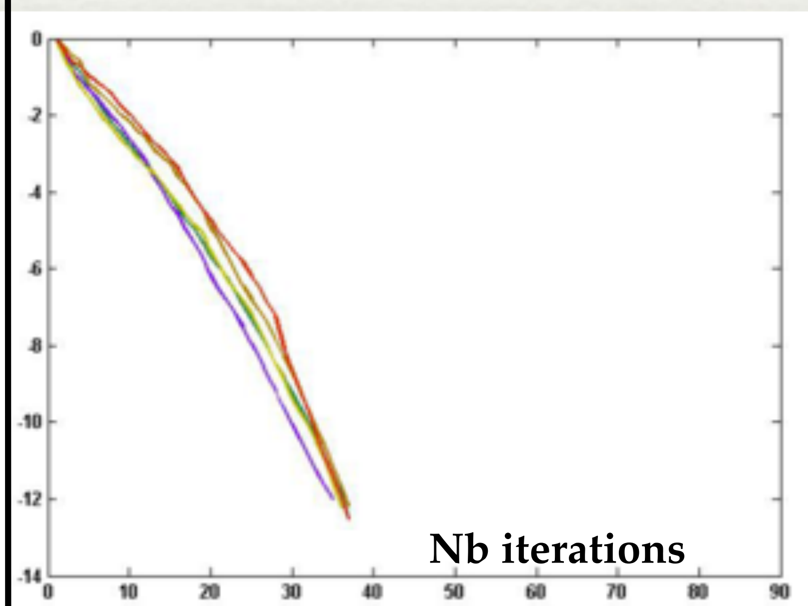


Y - Robin

Transparent - Robin

Residual error (log10)





Conclusion

Advantages of the Transparent – Neumann formulation:

- Works for anisotropic elastic waveguides

- No « box frequencies »

- Benefit for an iterative resolution whatever the size of the overlap is !

Vahan Baronian, Anne-Sophie Bonnet-Ben Dhia, Sonia Fliss and Antoine Tonnoir
Iterative methods for scattering problems in isotropic and anisotropic elastic waveguides, to appear in Wave Motion

Questions ?

Advantages of the Transparent – Neumann formulation:

- Works for anisotropic elastic waveguides
- No « box frequencies »
- Benefit for an iterative resolution whatever the size of the overlap is !

Thank you for your attention !

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