Multi-level elastic full waveform inversion in isotropic media via quantitative Lipschitz stability estimates

Florian Faucher[1] Maarten V. de Hoop[1] Henri Calandra[2]
Hélène Barucq[3]

Purdue University - Total - Inria Pau

November 17, 2014

[1]Department of Mathematics, Purdue University 150 N. University Street, West Lafayette IN 47907, USA.
[2]Total Research Exploration & Production, Houston, TX 77002, USA.
[3]Inria Magique 3d, Université de Pau, France.

Overview

1. Introduction to FWI
   - Seismic inverse problem
   - Iterative minimization
2. Elastic Full Waveform Inversion
   - Time-harmonic elastic wave equation
   - Convergence and stability results
   - Stability estimates and multi-level approach
3. Numerical Results
   - 2D Marmousi
   - 2D Pluto
   - 3D Model

Plan

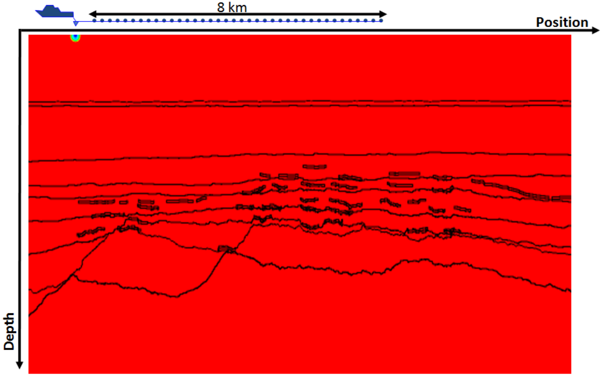
1. Introduction to FWI
    - Seismic inverse problem
    - Iterative minimization

Seismic acquisition

We want to reconstruct the unknown models m with data $F(m^\dagger)$ that are boundary observations

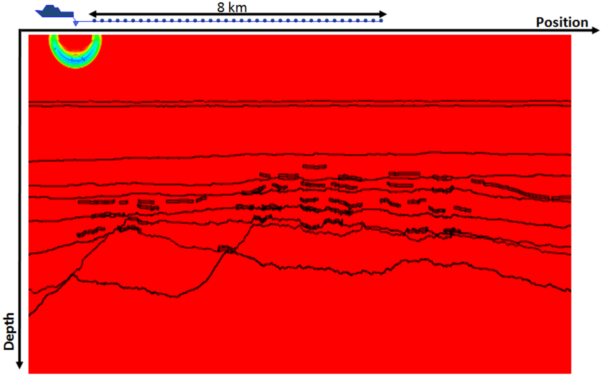
Seismic acquisition

We want to reconstruct the unknown models m with data $F(m^\dagger)$ that are boundary observations



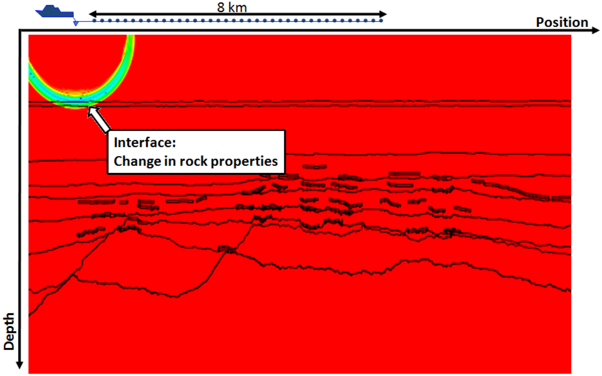
Seismic acquisition

We want to reconstruct the unknown models m with data $F(m^{\dagger})$ that are boundary observations



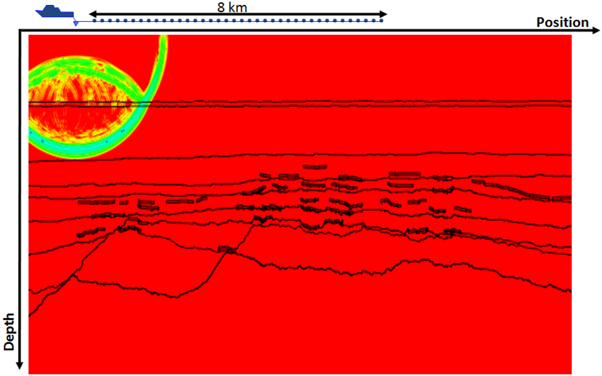
Seismic acquisition

We want to reconstruct the unknown models m with data $F(m^{\dagger})$ that are boundary observations



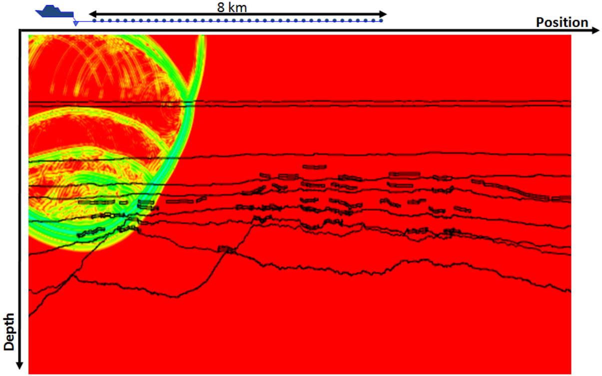
Seismic acquisition

We want to reconstruct the unknown models m with data $F(m^\dagger)$ that are boundary observations



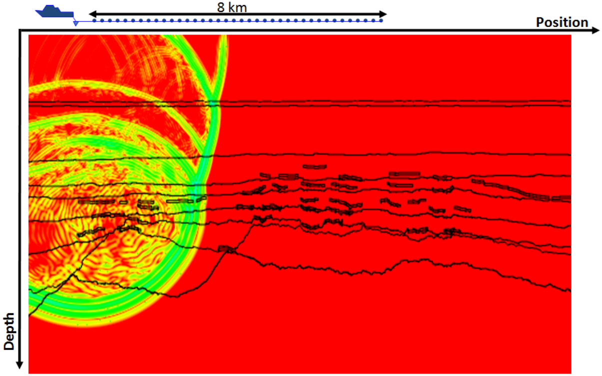
Seismic acquisition

We want to reconstruct the unknown models m with data $F(m^\dagger)$ that are boundary observations



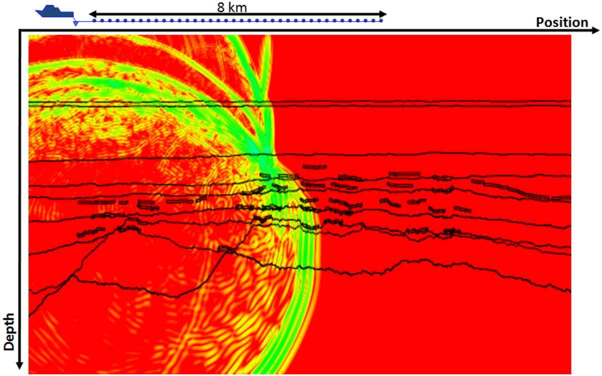
Seismic acquisition

We want to reconstruct the unknown models m with data $F(m^\dagger)$ that are boundary observations



Seismic acquisition

We want to reconstruct the unknown models m with data $F(m^\dagger)$ that are boundary observations



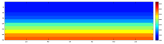
Full Waveform Inversion

Implement a flexible toolbox for modelling and inversion in Total architecture (DIP)

- (possibly complex) frequency-domain
- large computation (parallelism)
- acoustic and elastic problems
- minimization techniques

Full Waveform Inversion

starting model



↓

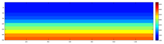
observation

wave simulation

$$\left(-\frac{\omega^2}{c^2} - \Delta\right)\hat{u} = \hat{g}$$

Full Waveform Inversion

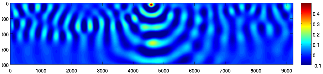
starting model



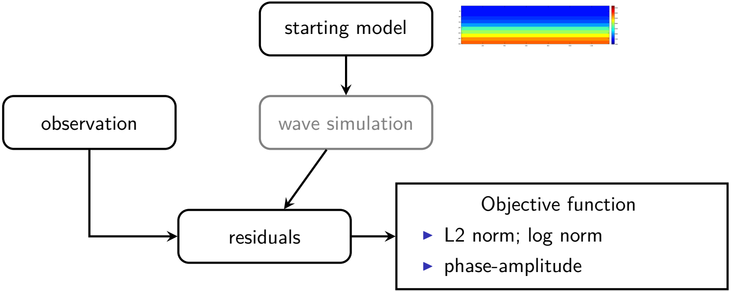
observation

wave simulation

- structured grid
- acoustic; elastic; TTI; VTI
- Hsolver

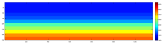


Full Waveform Inversion



Full Waveform Inversion

starting model



observation

wave simulation

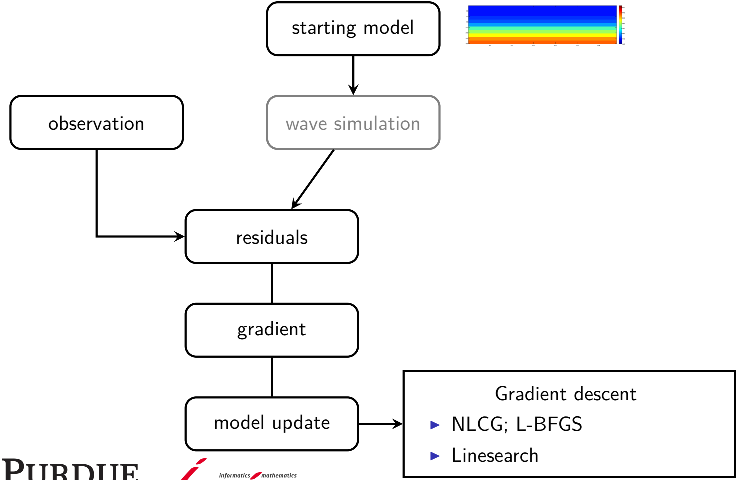
residuals

gradient

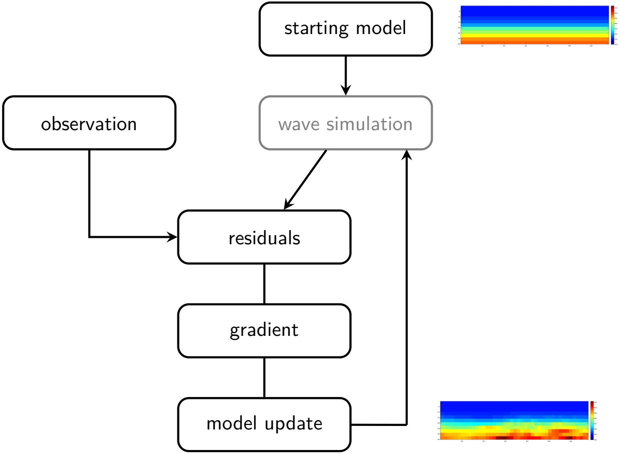
First-order adjoint method

- acoustic; elastic
- parametrization

Full Waveform Inversion



Full Waveform Inversion



Plan

2. Elastic Full Waveform Inversion
    - Time-harmonic elastic wave equation
    - Convergence and stability results
    - Stability estimates and multi-level approach

Isotropic Elastic time-harmonic wave propagation

$$-\rho\omega^2\hat{u} - \nabla\left(\lambda\nabla\cdot\hat{u}\right) - \nabla\cdot\left(\mu\left[\nabla\hat{u} + (\nabla\hat{u})^T\right]\right) = \hat{g},$$

Multi-parameter inversion: (λ, μ, ρ) to reconstruct

- $\lambda(x)$ and $\mu(x)$ are the Lamé parameters; $\rho(x)$ the density
- ω is the (possibly complex) frequency
- $g(x)$ is the source (near surface)
- $u(x)$ represent the wavefields
- P and S-wavespeed: $C_p = \sqrt{\dfrac{\lambda + 2\mu}{\rho}}$; $C_s = \sqrt{\dfrac{\mu}{\rho}}$

Convergence and stability results

For piecewise constant m, we have the Lipschitz-type stability [Beretta, de Hoop, Francini & Zhai, 2014]

$$\|m - m^{\dagger}\| \leq \mathcal{C} \, \|F(m) - F(m^{\dagger})\|$$

- m: models Lamé parameters,density
- $m^{\dagger}$: 'true' models
- $F(m) \sim$ forward operator restricted at the receivers
- $\mathcal{C}$: stability constant

Convergence and stability results

For piecewise constant m, we have the Lipschitz-type stability [Beretta, de Hoop, Francini & Zhai, 2014]

$$\|m - m^{\dagger}\| \leq \mathcal{C} \, \|F(m) - F(m^{\dagger})\|$$

- Lamé parameters and density must be decoupled

Convergence and stability results

For piecewise constant m, we have the Lipschitz-type stability [Beretta, de Hoop, Francini & Zhai, 2014]

$$\|m - m^{\dagger}\| \leq \mathcal{C}\,\|F(m) - F(m^{\dagger})\|$$

- Lamé parameters and density must be decoupled
- stability linked with the lower bound of the Fréchet derivative

$$\mathcal{C} \sim 2/l_0$$

$$l_0 \leq \min_{m;h} \|D_m F(m)[h]\|$$

Computational stability estimates

Use Gauss-Newton Hessian to estimate the stability

$$H^{GN}(.,.) = (R\partial_m u)^* \, R\partial_m u(.,.) = D_m F(.)^* D_m F(.)$$

Stability estimate: $2/\sqrt{\text{smallest sv}(H^{GN})}$

Computational stability estimates

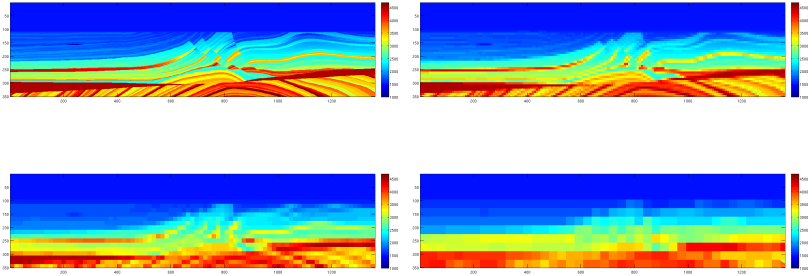
Use Gauss-Newton Hessian to estimate the stability

$$H^{GN}(.,.) = (R\partial_m u)^* \, R\partial_m u(.,.) = D_m F(.)^* D_m F(.)$$

Stability estimate: $2/\sqrt{\text{smallest sv}(H^{GN})}$

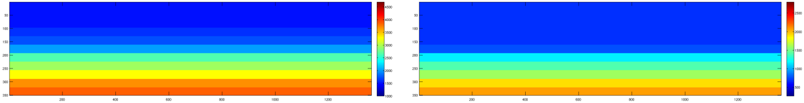
- Computation: 2nd order adjoint state; GN Hessian vector-product [Fichtner and Trampert,2010] [Métivier et al,2013]
- Stability estimates for different partitioning and parametrization

Gauss-Newton Hessian hierarchical compression



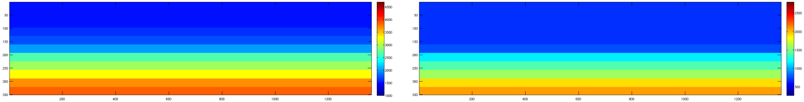
full scale (1361x351 domains) Haar compression level 3 (170x43) 4 (85x21) and 5 (42x10)

Gauss-Newton Hessian hierarchical compression

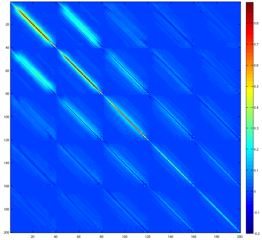


starting P-wavespeed (left) and S-wavespeed (right), Haar level 5 (42x10)

Gauss-Newton Hessian hierarchical compression



starting P-wavespeed (left) and S-wavespeed (right), Haar level 5 (42x10)



2Hz GN Hessian projection for parameter λ

Parametrization

Stability estimates for different parametrization (1/., log(.) and .)
1361x351 domains

Parametrization

Stability estimates for different parametrization ($1/.$, $\log(.)$ and $.$)
1361x351 domains

2Hz	λ			μ		
parametrization	$1/.$	$\log(.)$	$.$	$1/.$	$\log(.)$	$.$
	Stability estimates					
Haar level 5 (42x10)	0.3	330	33300	0.06	10	1000

Multi-level approach

Smallest singular value for different partitioning and frequency.
1361x351 domains

Multi-level approach

Smallest singular value for different partitioning and frequency.
1361x351 domains

2Hz	$1/\lambda$	$1/\mu$
Haar level 5 (42x10)	0.3	0.06
Haar level 4 (85x21)	330	10

Multi-level approach

Smallest singular value for different partitioning and frequency.
1361x351 domains

2Hz	$1/\lambda$	$1/\mu$
Haar level 5 (42x10)	0.3	0.06
Haar level 4 (85x21)	330	10

5Hz	$1/\lambda$	$1/\mu$
Haar level 5 (42x10)	0.04	0.02
Haar level 4 (85x21)	0.2	0.04
Haar level 3 (170x43)	330	10

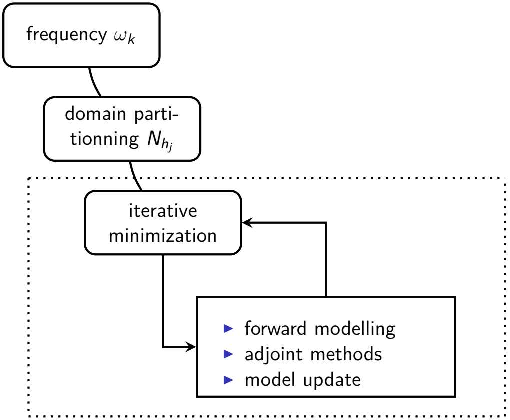
Multi-level approach

Smallest singular value for different partitioning and frequency.
1361x351 domains

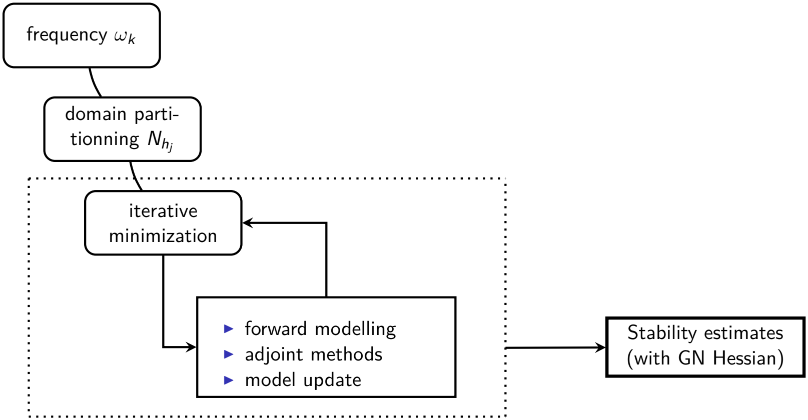
5Hz	$1/\lambda$	$1/\mu$
Haar level 5 (42x10)	0.04	0.02
Haar level 4 (85x21)	0.2	0.04
Haar level 3 (170x43)	330	10

8Hz	$1/\lambda$	$1/\mu$
Haar level 5 (42x10)	0.04	0.01
Haar level 4 (85x21)	0.08	0.03
Haar level 3 (170x43)	6	0.09

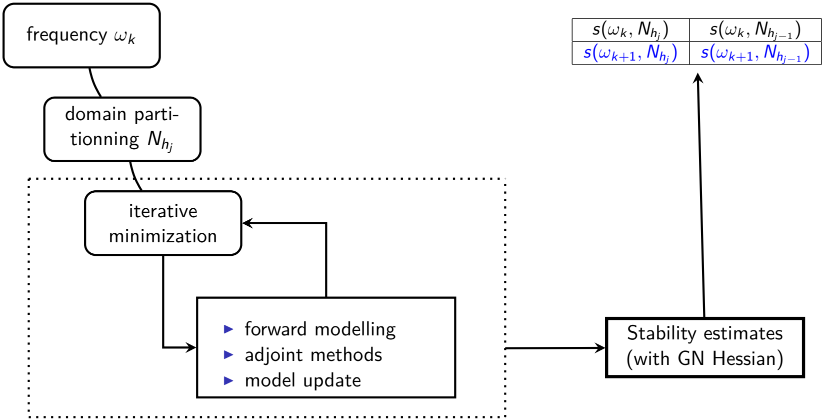
Multi-level, multi-frequency elastic inversion



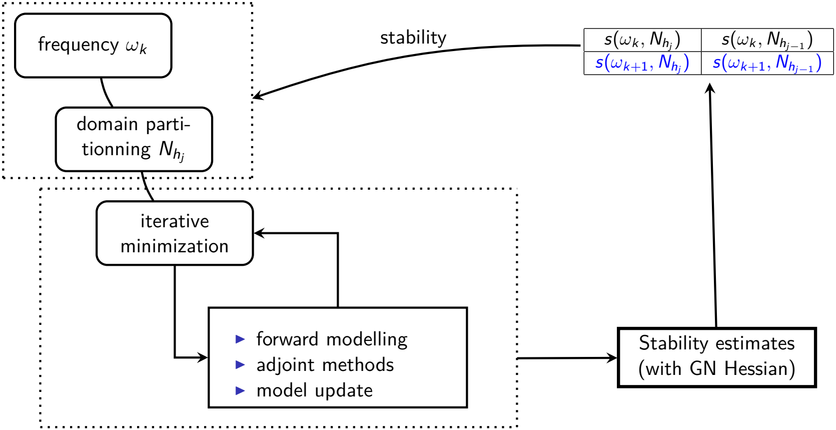
Multi-level, multi-frequency elastic inversion



Multi-level, multi-frequency elastic inversion



Multi-level, multi-frequency elastic inversion

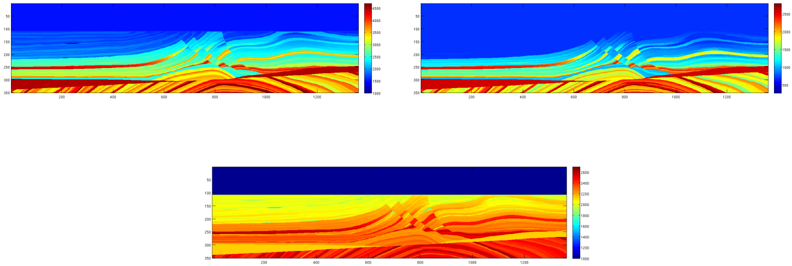


Plan

3. Numerical Results
   - 2D Marmousi
   - 2D Pluto
   - 3D Model

Marmousi models

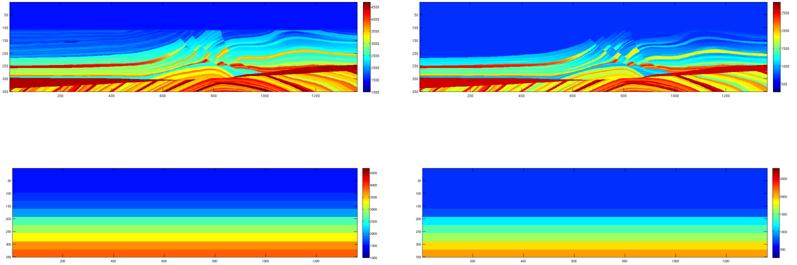
13.6km x 3.5km fully elastic



P-wavespeed, S-wavespeed and Density true models

Marmousi multi-level reconstruction

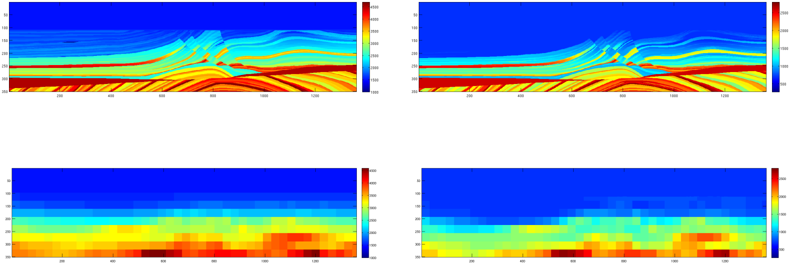
13.6km x 3.5km grid size 10m, frequency list from 1 to 10Hz, unknown density



P-wavespeed and S-wavespeed, true and starting models

Marmousi multi-level reconstruction

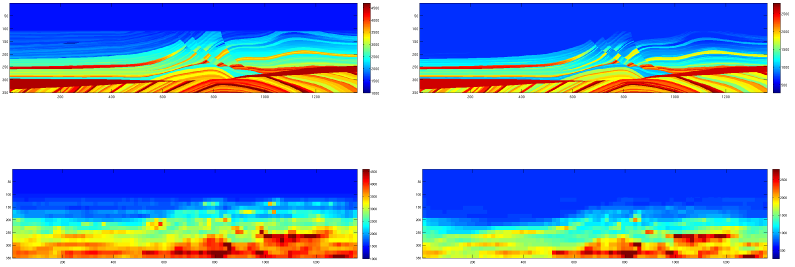
13.6km x 3.5km grid size 10m, frequency list from 1 to 10Hz, unknown density



P-wavespeed and S-wavespeed, true and 1Hz reconstruction (Haar level 5 - 42x10)

Marmousi multi-level reconstruction

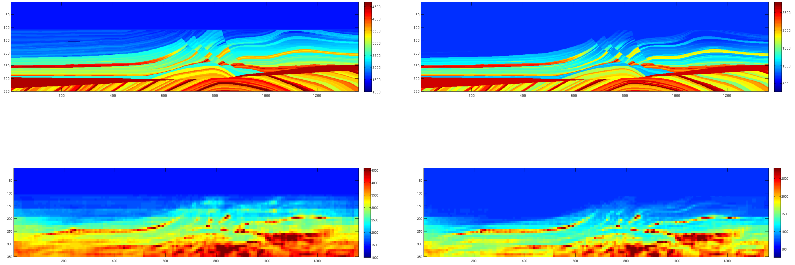
13.6km x 3.5km grid size 10m, frequency list from 1 to 10Hz, unknown density



P-wavespeed and S-wavespeed, true and 2Hz reconstruction (Haar level 4 - 85x21)

Marmousi multi-level reconstruction

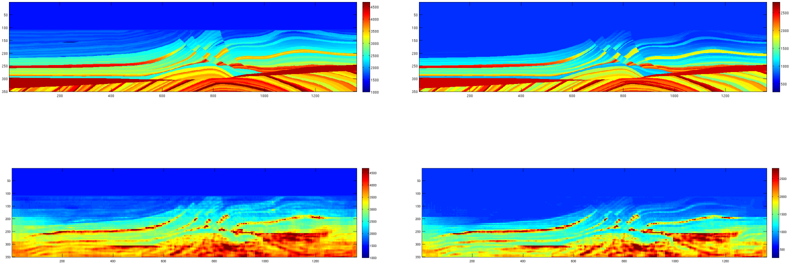
13.6km x 3.5km grid size 10m, frequency list from 1 to 10Hz, unknown density



P-wavespeed and S-wavespeed, true and 4Hz reconstruction (Haar level 3 - 170x43)

Marmousi multi-level reconstruction

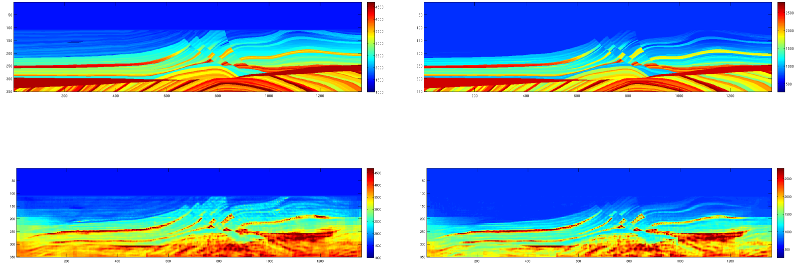
13.6km x 3.5km grid size 10m, frequency list from 1 to 10Hz, unknown density



P-wavespeed and S-wavespeed, true and 6Hz reconstruction (Haar level 2 - 340x87)

Marmousi multi-level reconstruction

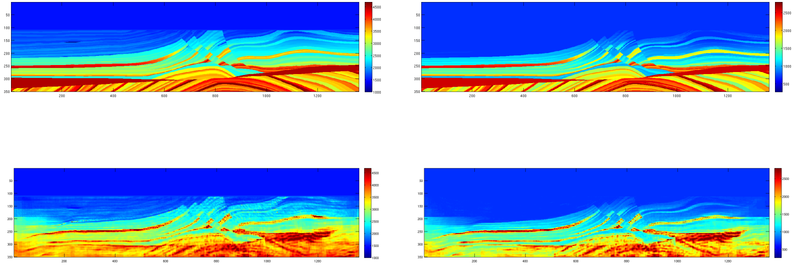
13.6km x 3.5km grid size 10m, frequency list from 1 to 10Hz, unknown density



P-wavespeed and S-wavespeed, true and 8Hz reconstruction (Haar level 1 - 680x175)

Marmousi multi-level reconstruction

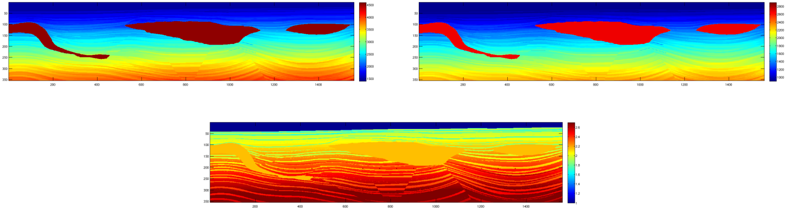
13.6km x 3.5km grid size 10m, frequency list from 1 to 10Hz, unknown density



P-wavespeed and S-wavespeed, true and 10Hz reconstruction - 1361x351

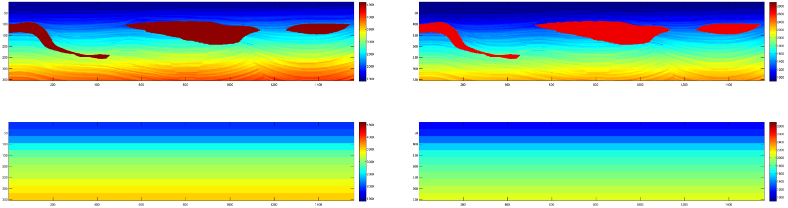
Pluto models

31.16km by 7.08km grid size 20m



True Pluto models for P-wavespeed (upper left) S-wavespeed (upper right) and Density (bottom)

Pluto multi-level reconstruction



P-wavespeed and S-wavespeed, true and starting models

Pluto multi-level reconstruction

complex frequency distant from spectrum

Pluto multi-level reconstruction

complex frequency distant from spectrum

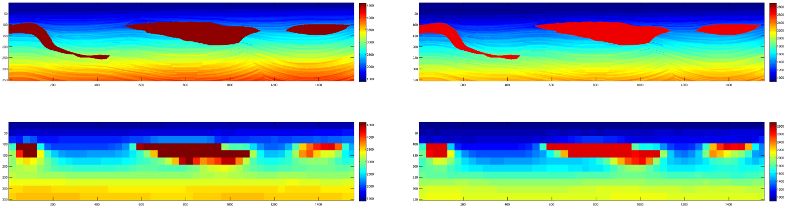
stability estimates

	$1/\lambda$	$1/\mu$
0.1Hz		
Haar level 5 (42x10)	20	0.2
$(0.1\text{Hz} + i)$		
Haar level 5 (42x10)	1	0.1

- improve radius of convergence needed for the starting iterations
- small frequency not available from field data

Pluto multi-level reconstruction

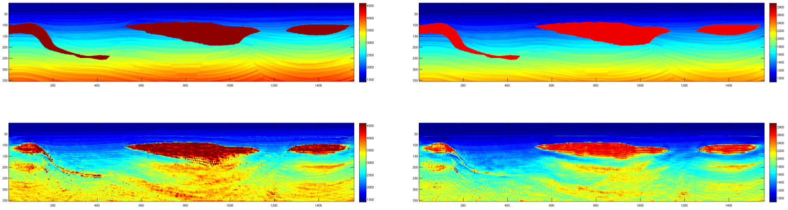
Complex frequency set, $\{(0.1\text{Hz}+i);(0.5\text{Hz}+i);(1\text{Hz}+i)\}$. Unknown density



P-wavespeed and S-wavespeed, Complex frequency reconstruction - Haar level 5, 49x12 domains

Pluto multi-level reconstruction

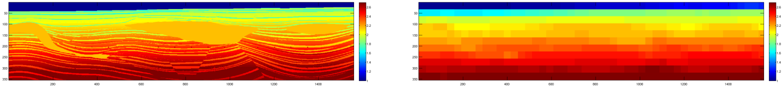
Frequency set $\{4, 5, 6, 7, 8\}$ Hz. Unknown density



P-wavespeed and S-wavespeed, 8Hz reconstruction - 1559x355 domains

Pluto density multi-level reconstruction

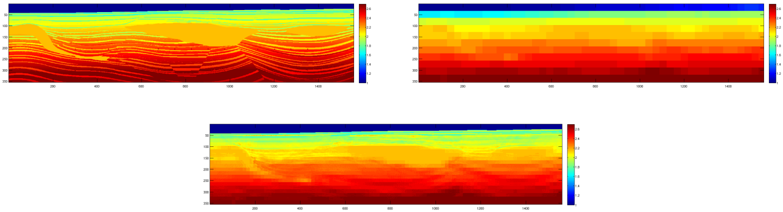
31.16km by 7.08km grid size 20m. We assume the knowledge of the Lamé parameters λ and μ frequency from 4 to 8Hz.



True and starting density models

Pluto density multi-level reconstruction

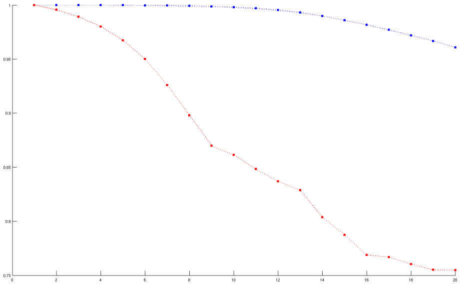
31.16km by 7.08km grid size 20m. We assume the knowledge of the Lamé parameters λ and μ frequency from 4 to 8Hz.



True and starting density models (top). 8Hz multi-level reconstruction (bottom)

Pluto density multi-level reconstruction

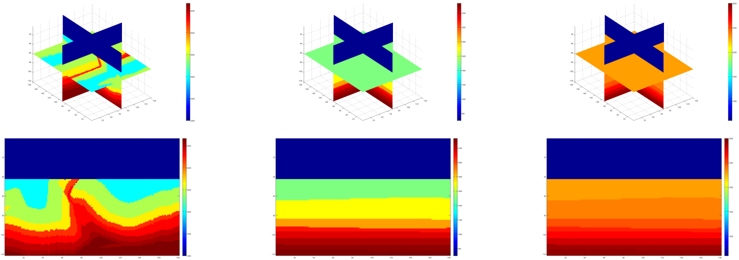
31.16km by 7.08km grid size 20m. We assume the knowledge of the Lamé parameters λ and μ frequency from 4 to 8Hz.



scaled residuals comparison between the Lamé iterations (red) and density iterations (blue)

3D models

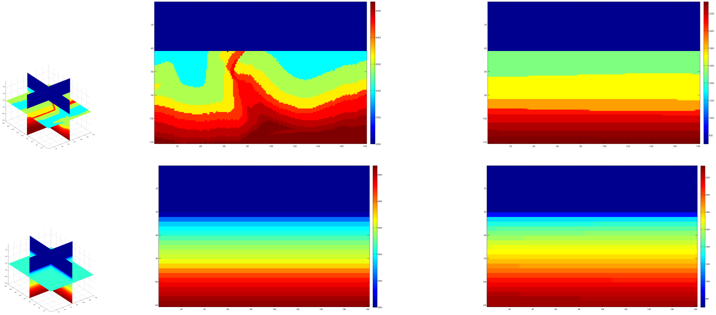
1.8km x 1.4km x 1.2km



P-wavespeed, S-wavespeed and Density models: 3D and vertical sections

3D multi-level reconstruction

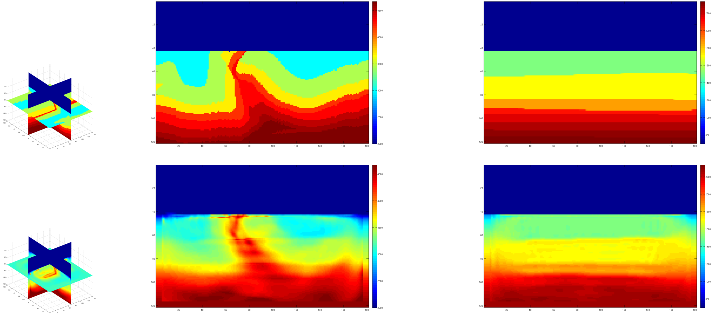
1.8km x 1.4km x 1.2km frequency from 2 to 14Hz



P-wavespeed and S-wavespeed true and starting Vertical sections

3D multi-level reconstruction

1.8km x 1.4km x 1.2km frequency from 2 to 14Hz



P-wavespeed and S-wavespeed true and 14Hz reconstruction Vertical sections

Conclusion

- Complex frequency domain toolbox
- Hierarchical multi-level approach
- Quantitative estimate of the stability constant
- Isotropic elastic reconstruction (Lamé parameters and density)

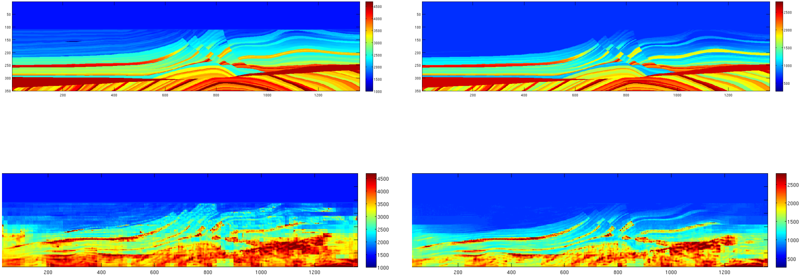
Future Work

- Field data
- Stochastic approach for low frequency
- Unstructured mesh adaptation and deformation

Thank you

Marmousi multi-level reconstruction

When we add up to 20% noise in the data



P-wavespeed and S-wavespeed, true and 10Hz reconstruction - 1361x351

Ajoint state

First order Adjoint state

$$D_m\epsilon\left\{m\right\}\delta m = \int -\omega^2 u.u_r\delta\rho \\ + (\nabla.u)(\nabla.u_r)\delta\lambda \\ + (\nabla u_r) : (\nabla u + (\nabla u)^T)\delta\mu dX$$

2^{nd} order ajoint state: Hessian formulation

using Hessian vector product [Fichtner and Trampert,2010] [Métivier *et al*,2013] [Shi *et al*, 2014] (higher order). Using vector product $H.v$ in 2 more linear system solves

Full Hessian

$$\text{a)} \quad S(m)^{\dagger}\alpha = -\sum_{j=1}^{M}(\partial_{m_j}S(m).v_j)u$$

$$\text{b)} \quad S(m)^{\dagger}\beta = -R^{\dagger}R\alpha - \sum_{j=1}^{M}(\partial_{m_j}S(m)v_j)^{\dagger}u_r$$

$$\begin{aligned} H(m)v = \mathrm{Re}\{ & < (\partial_m S)(m)(.)u(m), \beta(m) > + \\ & < (\partial_m S)(m)(.)\alpha(m), u_r(m) > + \\ & < \partial_m \{(\partial_m S)(m)vu(m)\}, u_r(m) > \} \end{aligned}$$

2^{nd} order ajoint state: Hessian formulation

using Hessian vector product [Fichtner and Trampert,2010] [Métivier *et al*,2013] [Shi *et al*, 2014] (higher order). Using vector product $H.v$ in 2 more linear system solves

Gauss Newton Hessian

$$\text{a)} \quad S(m)^{\dagger}\alpha = -\sum_{j=1}^{M}(\partial_{m_j}S(m).v_j)u$$

$$\text{b)} \quad S(m)^{\dagger}\beta = -R^{\dagger}R\alpha$$

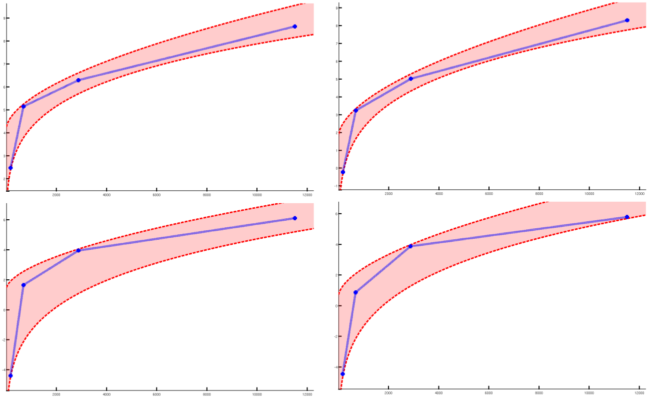
$$H(m)v = \text{Re}\{<(\partial_m S)(m)(.)u(m), \beta(m)>\}$$

stability estimate general behaviour - Acoustic

$$\frac{1}{4} e^{K_1 N^{1/5}} \leq \mathcal{C} \leq \frac{1}{\omega^2} e^{\left(K(1+\omega^2 B_2) N^{4/7}\right)}$$

stability estimate general behaviour - Acoustic

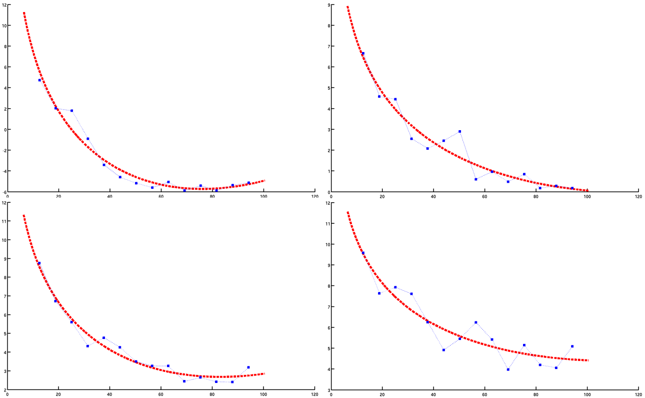
$$\frac{1}{4} e^{K_1 N^{1/5}} \leq \mathcal{C} \leq \frac{1}{\omega^2} e^{\left(K(1+\omega^2 B_2)N^{4/7}\right)}$$



evolution of the smallest singular value depending on the number of domain (blue) and best approximation of the bounds (red) - logarithmic scale

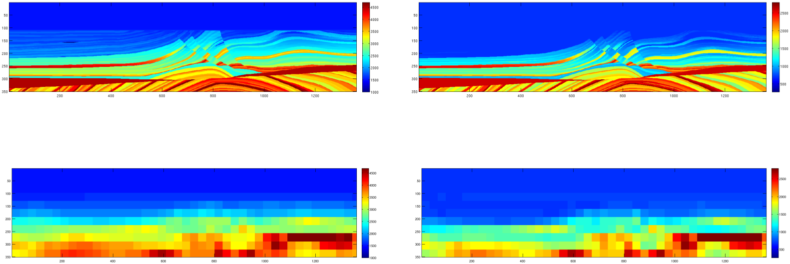
stability estimate general behaviour - Acoustic

$$\frac{1}{4}e^{K_1 N^{1/5}} \leq \mathcal{C} \leq \frac{1}{\omega^2}e^{\left(K(1+\omega^2 B_2)N^{4/7}\right)}$$



evolution of the smallest singular value depending on the frequency (blue) and best approximation for frequency dependency (red) - logarithmic scale

Marmousi elastic - complex frequency



P-wavespeed and S-wavespeed, true and complex frequency reconstruction (Haar level 5 - 43x11).
It can be complicated to retrieve low frequency from field data. Complex frequency instead

(local) optimization, seismic inverse problem

some references on Full Waveform Inversion (FWI)

- Bamberger, Chavent & Lailly (1977, 1979); Chavent (1983)
- Tarantola & Valette (1982)
- Pratt, Shin & Hicks (1998); Pratt (1999); Sirgue & Pratt (2004)
- Operto, Virieux, Dessa & Pascal (2006); Virieux & Operto (2009)
- Métivier, Brossier, Virieux & Operto (2013)