Large Time Step scheme for Hyperbolic Systems

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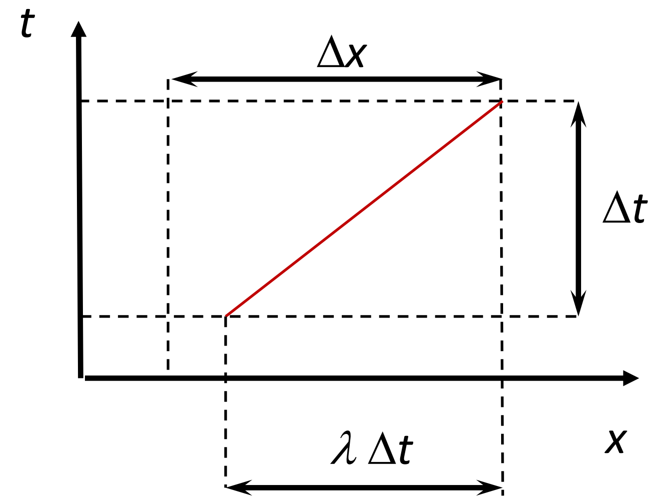
Content

1.1 Problem position

$$\partial_t \mathbf{u} + \partial_x \mathbf{f} = 0$$

$$\mathrm{u}_i^{n+1} = \mathrm{u}_i^n + \frac{\Delta t}{\Delta x_i}\left(\mathbf{f}_{i-1/2}^{n+1/2} - \mathbf{f}_{i+1/2}^{n+1/2}\right)$$



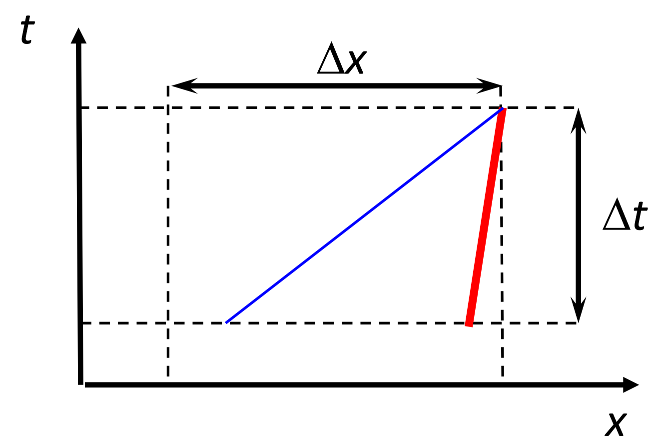
For stability

$$\nu = |\lambda| \frac{\Delta t}{\Delta x_i} \leq 1$$

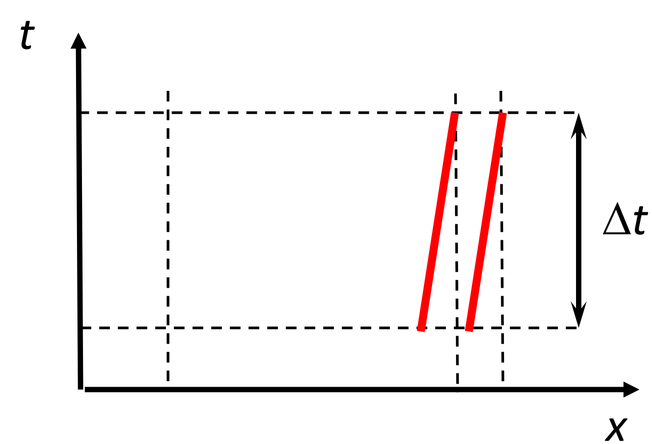
For accuracy : keep ν close to 1

1.1 Problem position

Achieving $\nu > 1$ or $\nu >> 1$ is desirable in at least 2 situations



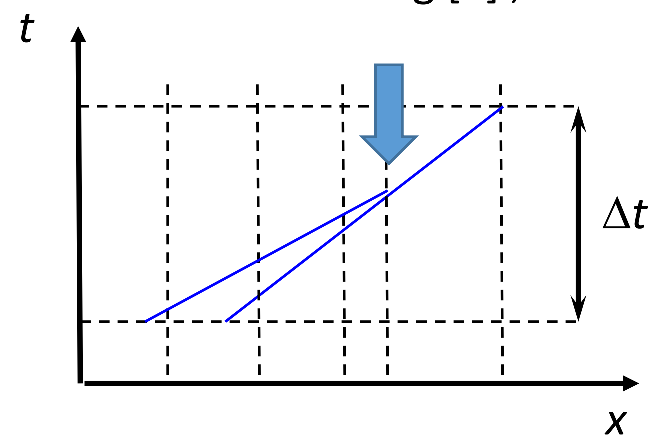
- Preserving the slower waves is more important than preserving the fast ones (passive transport)



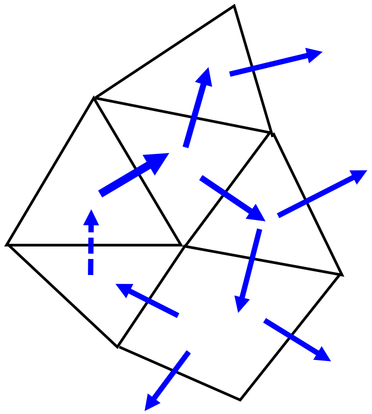
- A few narrow cells induce a locally much larger ν than elsewhere (classical in industrial meshes)

1.2 Most common approaches

Allow the waves to cross more than 1 cell within Δt : wave tracking [1] , time line interpolation [2]



- Easy for 1D linear laws
- Wave interactions when the laws are non-linear



- Tricky to implement in 2D

[1] LeVeque (1982, 1988), Harten (1986), Murillo & al. (2006), Morales & Murillo (2012)

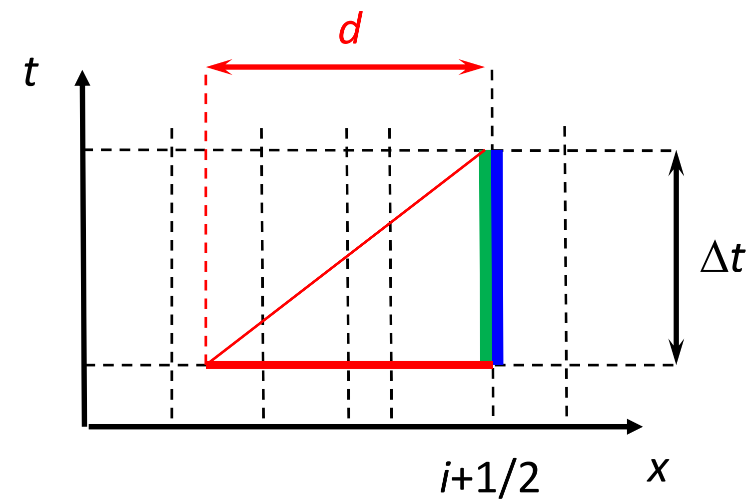
[2] Collins & al (1986), Guinot (2000)

2.1 The « naive » approach

$$\mathrm{u}_i^{n+1} = \mathrm{u}_i^n + \frac{\Delta t}{\Delta x_i}\left(\mathbf{f}_{i-1/2}^{n+1/2} - \mathbf{f}_{i+1/2}^{n+1/2}\right)$$

Only an *averaged* flux is needed

$$\mathbf{f}_{i+1/2}^{n+1/2} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \mathbf{f}(x_{i+1/2}, t)\mathrm{d}t$$



Approximation #1 : linearise wrt **u**

$$\mathbf{f}_{i+1/2}^{n+1/2} \approx \mathbf{f}\left(\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \mathbf{u}(x_{i+1/2}, t)\mathrm{d}t\right)$$

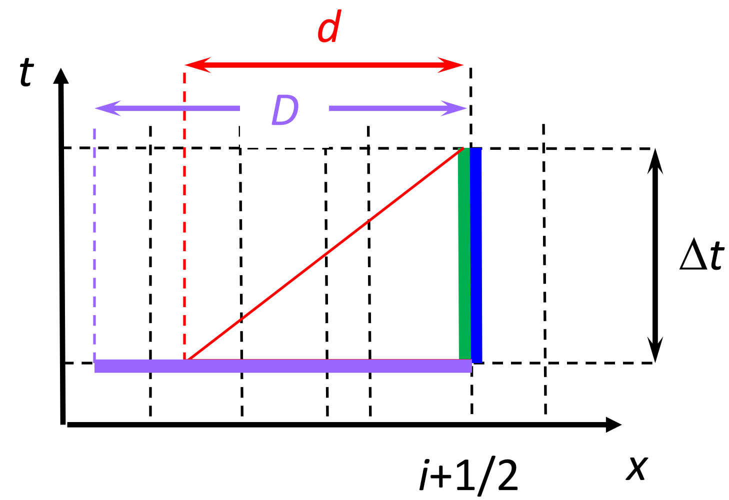
Approximation #2 : convert time average to space average (characteristic form)

$$\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \mathbf{u}(x_{i+1/2}, t)\mathrm{d}t \approx \frac{1}{d}\int_{x_{i+1/2}-d}^{x_{i+1/2}} \mathbf{u}(x, t_n)\mathrm{d}x$$

2.1 The « naive » approach

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{\Delta x_i}\left(\mathbf{f}_{i-1/2}^{n+1/2} - \mathbf{f}_{i+1/2}^{n+1/2}\right)$$

$$\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \mathbf{u}(x_{i+1/2}, t)\mathrm{d}t \approx \frac{1}{d}\int_{x_{i+1/2}-d}^{x_{i+1/2}} \mathbf{u}(x, t_n)\mathrm{d}x$$



BUT : computing *d* is not always straightforward (nonlinear laws)

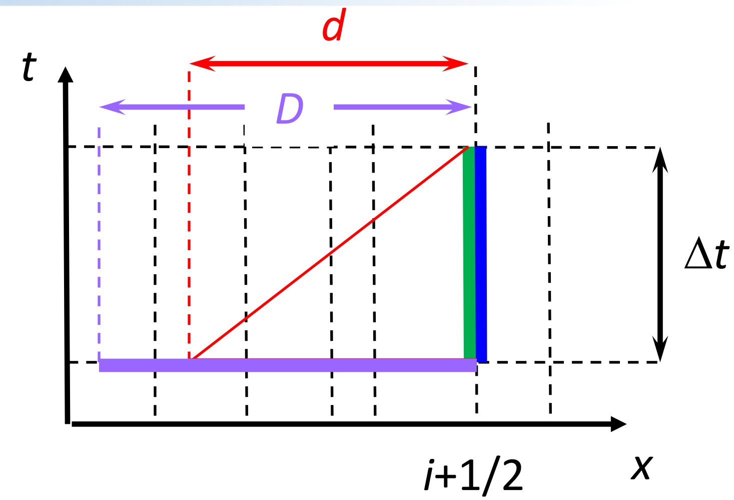
Approximation #3 (our contribution 1):
Do not use the exact *d* but a pre-defined *D*

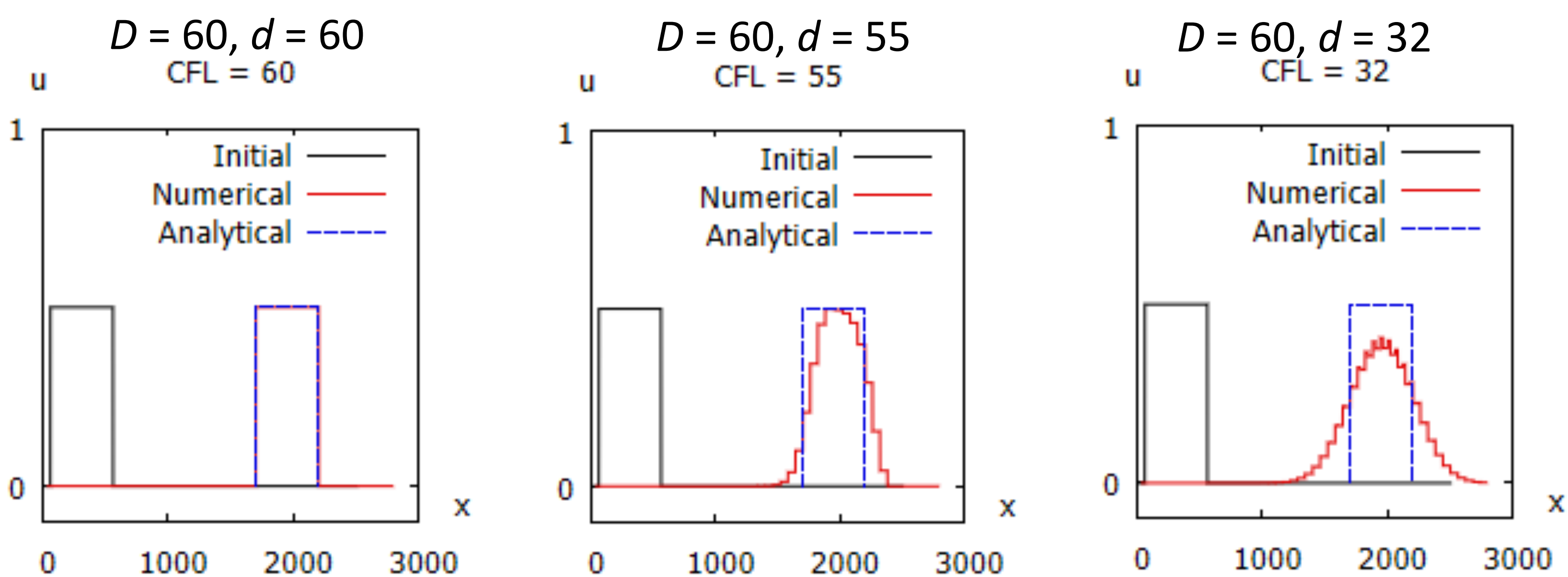
$$\frac{1}{d}\int_{x_{i+1/2}-d}^{x_{i+1/2}} \mathbf{u}(x, t_n)\mathrm{d}x \approx \frac{1}{D}\int_{x_{i+1/2}-D}^{x_{i+1/2}} \mathbf{u}(x, t_n)\mathrm{d}x$$

2.1 The « naive » approach

What happens if $D \neq d$?

Linear advection of a hat function





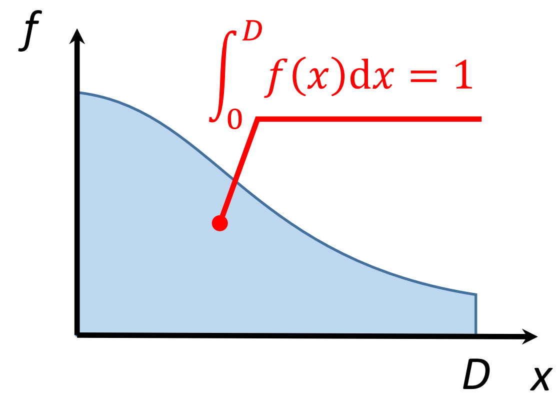
2.2 Convolution in 1D

The method performs poorly when $D \neq d$ even for a small (relative) difference between the two

$$\frac{1}{d}\int_{x_{i+1/2}-d}^{x_{i+1/2}} \mathbf{u}(x,t_n)\mathrm{d}x \approx \frac{1}{D}\int_{x_{i+1/2}-D}^{x_{i+1/2}} \mathbf{u}(x,t_n)\mathrm{d}x$$

Cure

Approximation #4 (our contribution 2): give less weight to those cells that are farther away from the interface (weighted averaging/convolution)

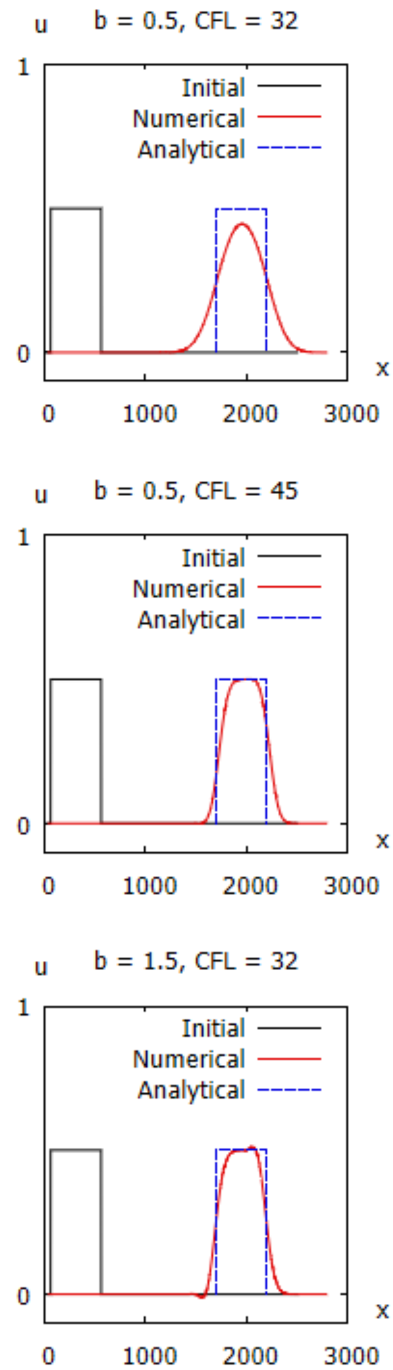


$$\frac{1}{d}\int_{x_{i+1/2}-d}^{x_{i+1/2}} \mathbf{u}(x,t_n)\,\mathrm{d}x \approx \int_{x_{i+\frac{1}{2}}-D}^{x_{i+\frac{1}{2}}} f(x_{i+1/2}-x)\,\mathbf{u}(x,t_n)\,\mathrm{d}x = f * \mathbf{u}$$

2.2 Convolution in 1D

Results for a power law convolution kernel (D = 60)

$$f(x) = a\left(\frac{x}{D}\right)^{b} \qquad a = \frac{b+1}{D}$$

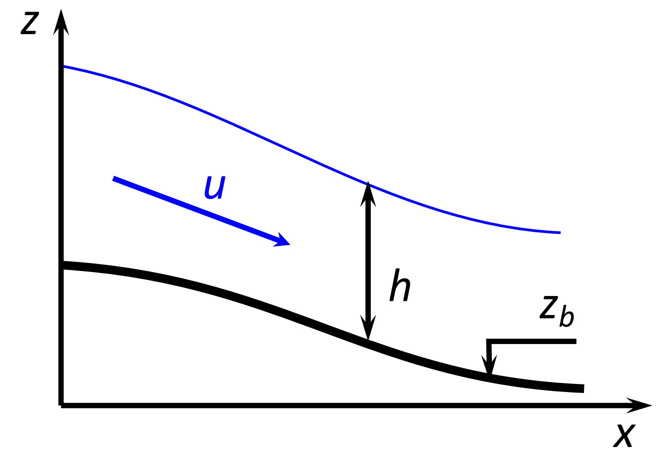


2.3 Shallow water equations & bottom source term

$$\partial_t \mathbf{u} + \partial_x \mathbf{f} = \mathbf{s} \qquad \mathbf{u} = \begin{bmatrix} h \\ hu \end{bmatrix}$$

$$\partial_t \mathbf{u} + \mathbf{A}\partial_x \mathbf{u} = \mathbf{s} \qquad \mathbf{f} = \begin{bmatrix} hu \\ hu^2 + gh^2/2 \end{bmatrix}$$

$$\mathbf{s} = \begin{bmatrix} 0 \\ (S_0 - S_f)gh \end{bmatrix}$$



EVR approach : reconstructing *z* alone is sufficient

$$\mathbf{u} = \frac{h}{2}\mathbf{k}_1 + \frac{h}{2}\mathbf{k}_2 = \frac{h}{2}\begin{bmatrix} 1 \\ u - c \end{bmatrix} + \frac{h}{2}\begin{bmatrix} 1 \\ u + c \end{bmatrix}$$

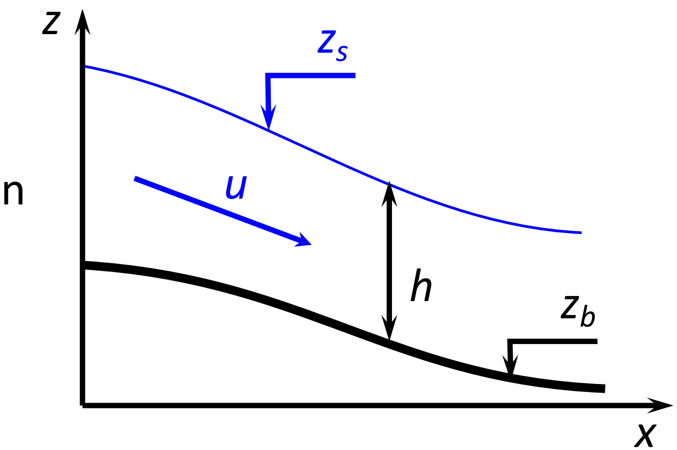
$$\overline{\mathbf{u}} \approx \frac{\overline{h}}{2}\overline{\mathbf{k}_1} + \frac{\overline{h}}{2}\overline{\mathbf{k}_2}$$

2.3 Shallow water equations & bottom source term

We proved that static conditions are preserved provided that

- Averaging is done for the free surface elevation z_s instead of the depth h

$$\partial_t \mathbf{u}' + \partial_x \mathbf{f} = \mathbf{s} \qquad \mathbf{u}' = \begin{bmatrix} z_s \\ hu \end{bmatrix}$$



- The bottom elevation z_b is taken from the local values over the cells immediately next to the interface

2.4 Stability

For the linear advection equation

$$u_i^{n+1} = u_i^n + \nu\left(\tilde{u}_{i-1/2,L}^n - \tilde{u}_{i+1/2,L}^n\right)$$

$$\nu = c\frac{\Delta t}{\Delta x}$$

$$\tilde{u}_{i+1/2,L}^n = w_L * u(x,t_n) \approx \sum_{k=1}^{K} f_k u_{i-k+1}^n \Delta x$$

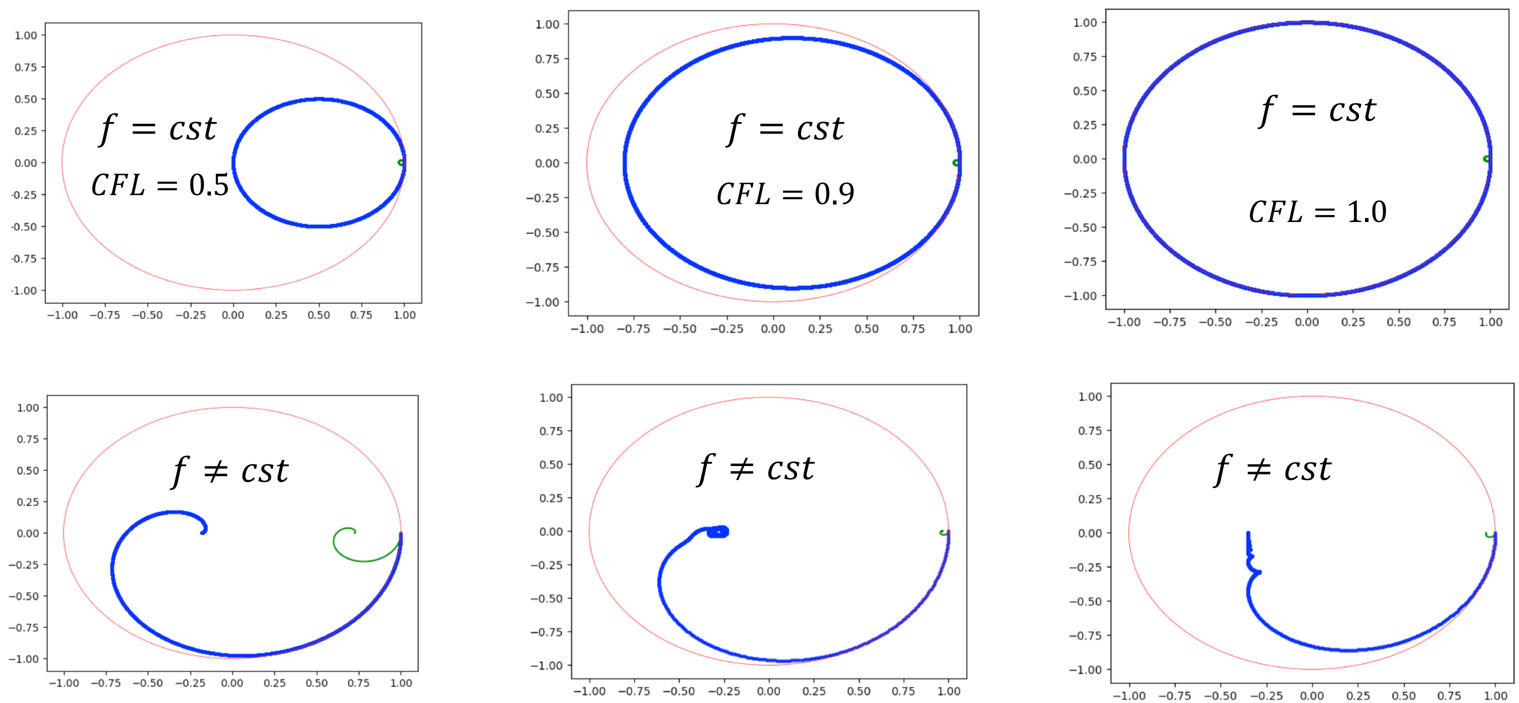
$$\sum_{k=1}^{K} f_k \Delta x = 1$$

Von Neumann analysis: $u_i^n = u_0^0 \exp(\mu t + j\sigma i\Delta x)$

Need to prove that this amplification factor remains « in the unity disk »

$$A_K \equiv \frac{u_i^{n+1}}{u_i^n} = 1 + \nu\left(1 - e^{j\sigma\Delta x}\right)\sum_{k=1}^{K} f_k e^{-jk\sigma\Delta x}\Delta x$$

2.4 Stability



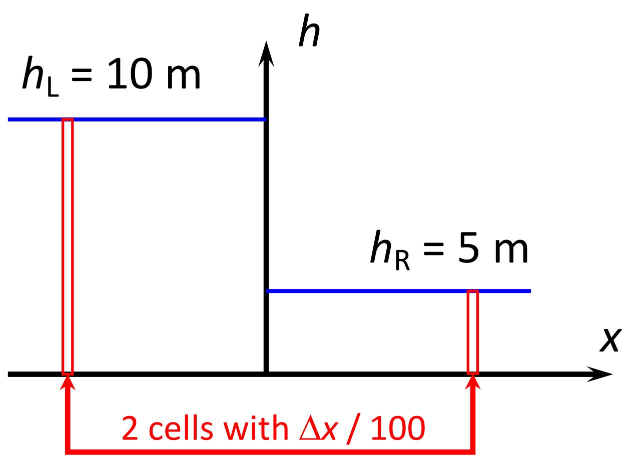
Theorem 1. *For every $D > 0$ and every decreasing kernel function f on $[0, D]$,*

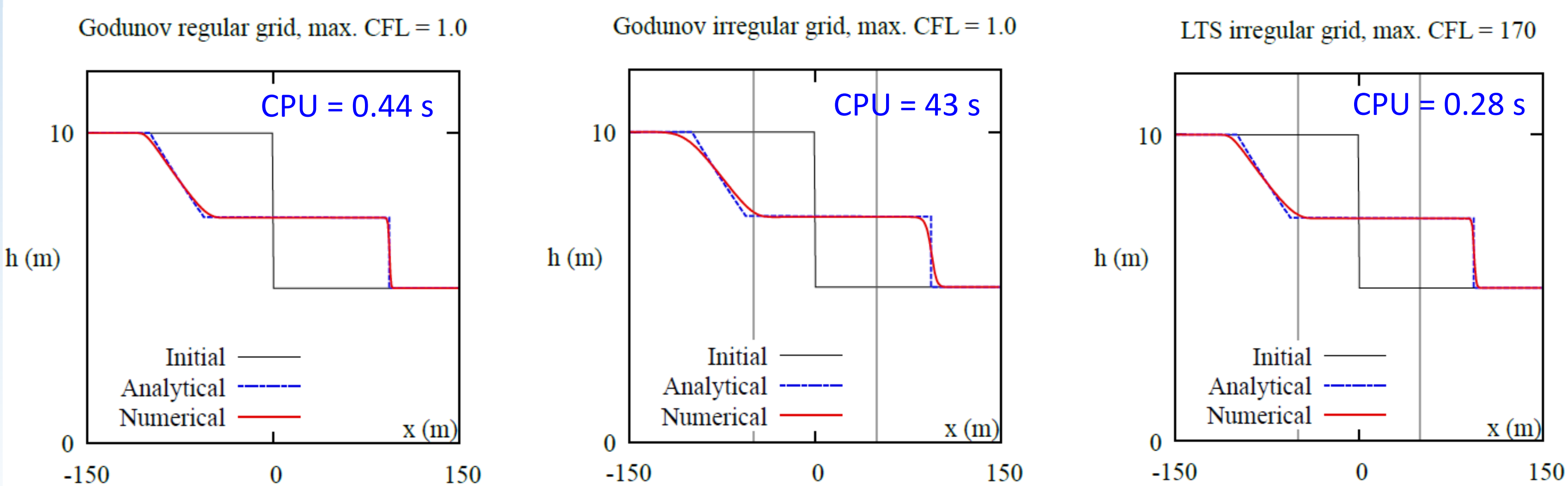
$$\exists \nu > 0 : \forall \sigma \geq 0, |A_\infty(\sigma)| \leq 1$$

The same result *should* hold for $A_K(\sigma)$…

Application to the shallow water equations

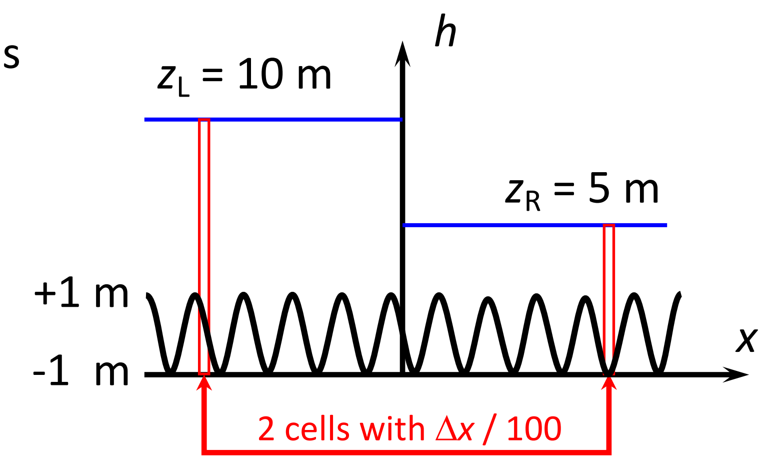
- Dambreak problem, flat bottom
  (∃ analytical solution)

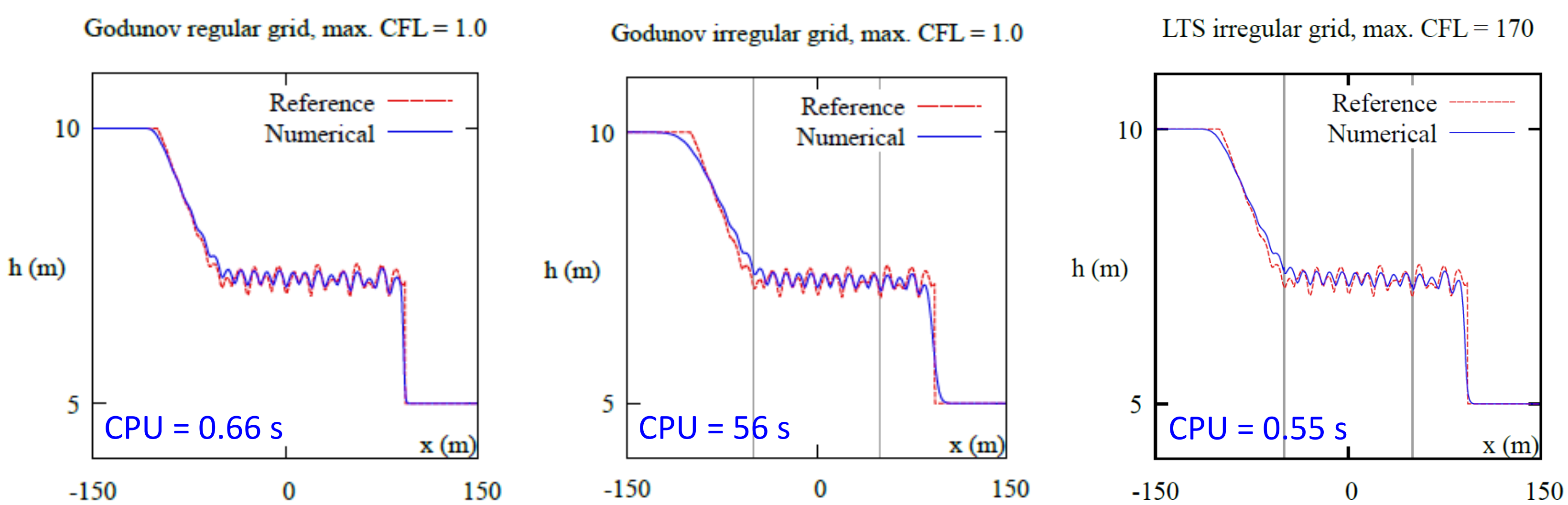




Application to the shallow water equations

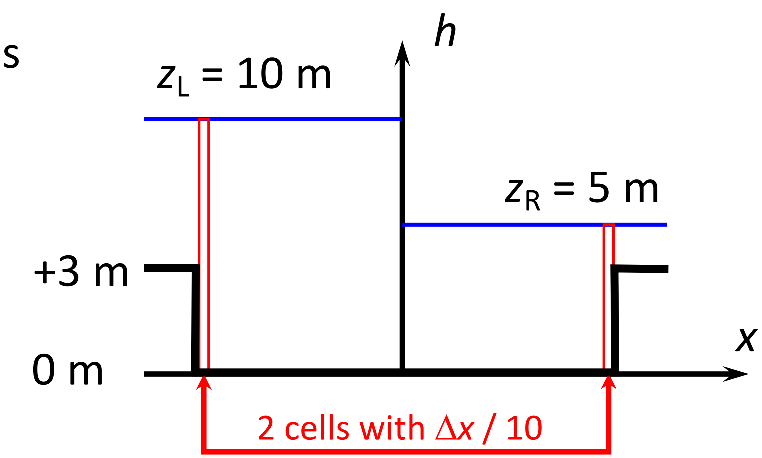
- Dambreak problem, sinusoidal bottom (∄ analytical solution)

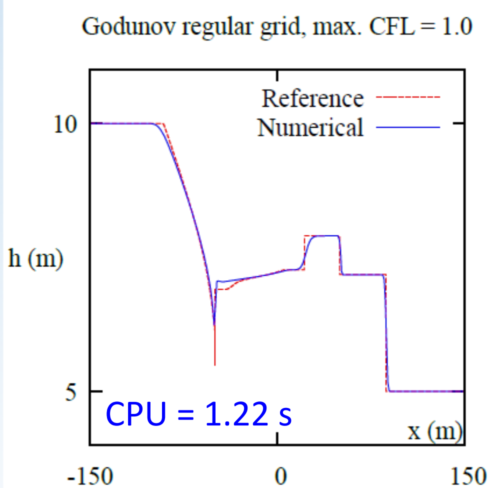


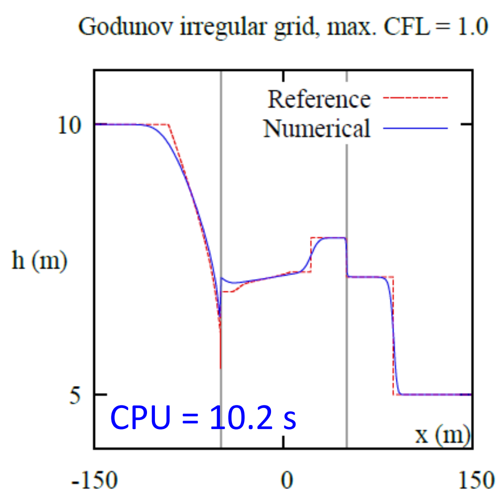


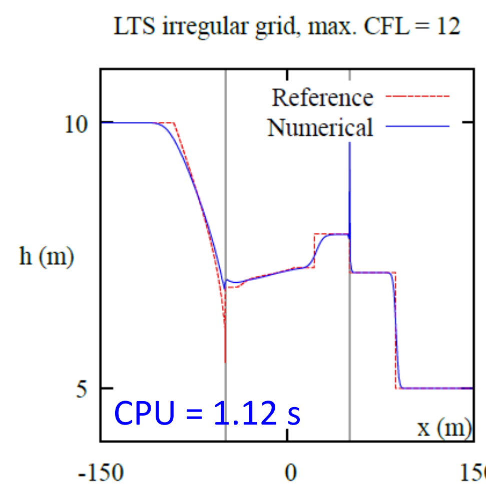
Application to the shallow water equations

- Dambreak problem, discontinuous bottom (∄ analytical solution)



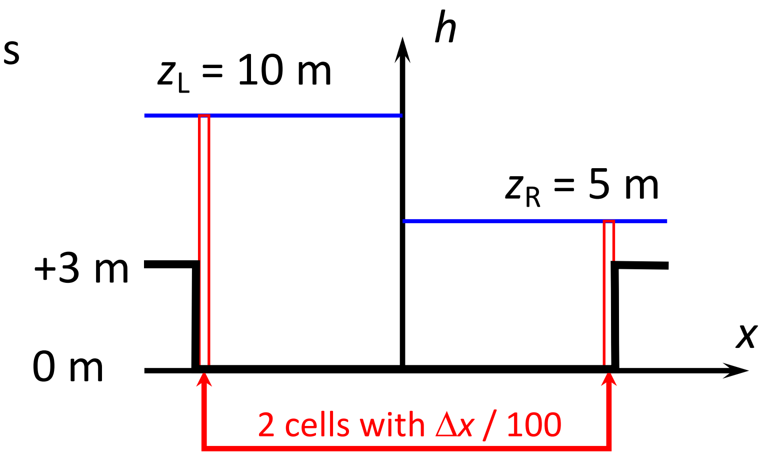


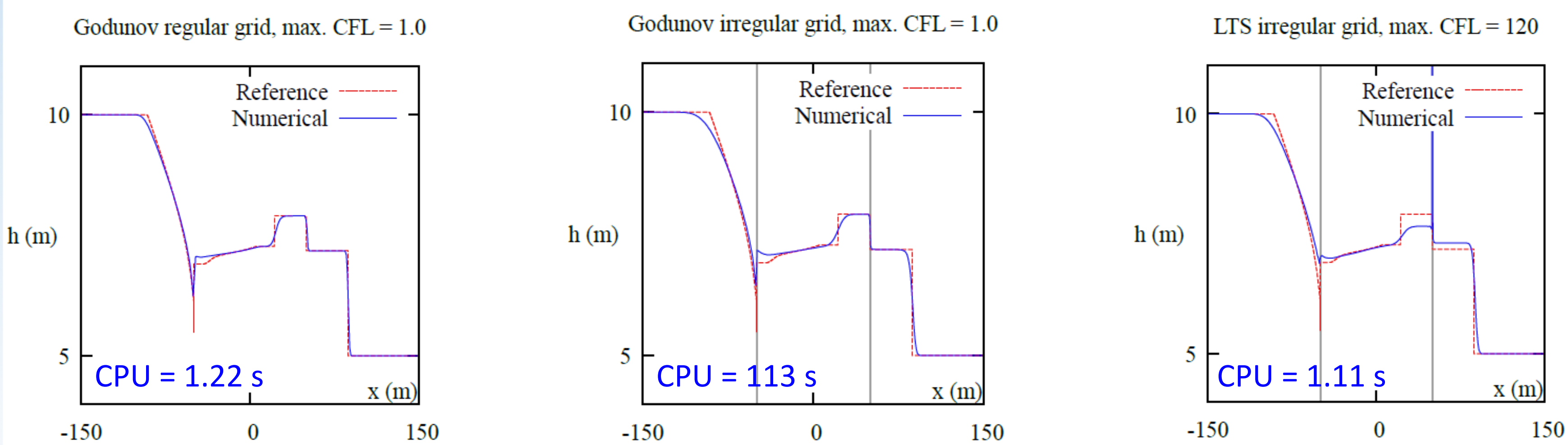




Application to the shallow water equations

- Dambreak problem, discontinuous bottom (∄ analytical solution)





So far

- No theoretical CFL restriction for homogeneous PDEs
- Smooth geometrical source terms (continuous topography) : no noticeable balancing issues/instability identified
- Stiff geometrical source terms (discontinuous topography): conservation issues as CFL increases (balancing errors are amplified), no instability identified
- Large *l* contrasts may require iterative estimate of *D*

Pending questions / ongoing developments

- Treatment of boundaries
- How to handle dry/disconnected regions?
- Going 2D