Cryptographic Smooth Neighbors

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Surrey Centre for Cyber Security, University of Surrey, UK

Talk for the GRACE seminar at École Polytechnique

Meet-in-the-Middle

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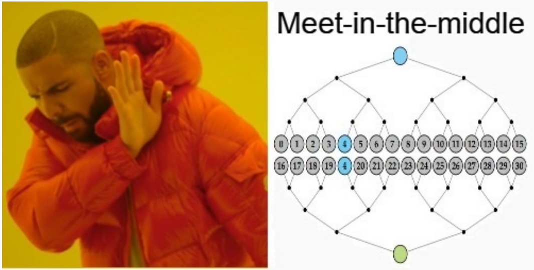
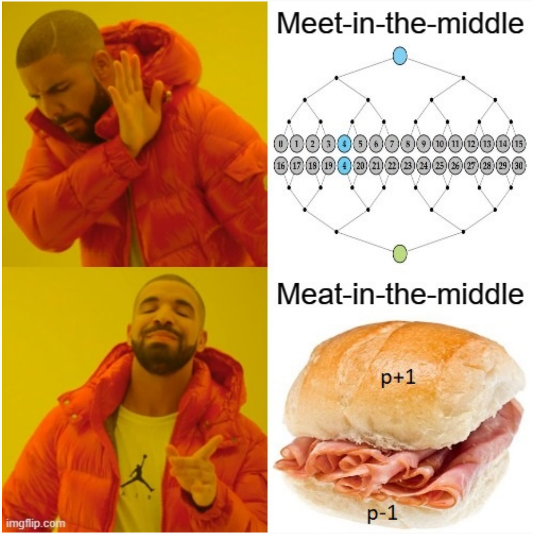


Meat-in-the-Middle



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The current state-of-the-art in SQISign uses the following prime

254-bit prime $p =$ `0x348757EADF5C9530B7311A63633F03DB535805FA6E9E48B1FFFFFFFFFFFFFFFF`:

$$p + 1 = 2^{65} \cdot 5^2 \cdot 7 \cdot 11 \cdot 19 \cdot 29^2 \cdot 37^2 \cdot 47 \cdot 197 \cdot 263 \cdot 281 \cdot 461 \cdot 521 \cdot 3923 \cdot 62731 \cdot 96362257 \cdot 3924006112952623, \text{ and}$$

$$p - 1 = 2 \cdot 3^{65} \cdot 13 \cdot 17 \cdot 43 \cdot 79 \cdot 157 \cdot 239 \cdot 271 \cdot 283 \cdot 307 \cdot 563 \cdot 599 \cdot 607 \cdot 619 \cdot 743 \cdot 827 \cdot 941 \cdot 2357 \cdot 10069$$

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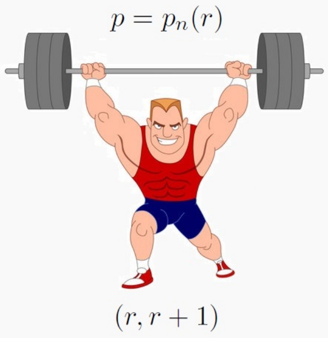

Outline

Finding Twin-Smooth Integers

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Surprisingly, for a fixed B there are finitely many B-smooth twins

Current landscape for finding smooth twins

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Constructive Methods

Pell equation
CHM

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Probabalistic methods

Naïve method
XGCD/CRT
$p_n(x) = 2x^n - 1$
Ideal PTE solutions

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For a B-smooth twin $(r, r+1)$, let $x = 2r+1$ so that $x^2 - 1$ is B-smooth and write $x^2 - 1 = Dy^2$ where D, y are B-smooth and D is squarefree. Then we can see that (x, y) is a solution to the Pell conic

$$X^2 - DY^2 = 1$$

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Lehmer ran this algorithm for $B = 41$, Luca and Najman (2011) ran it with $B = 100$ and most recently Costello (2019) ran it with $B = 113$

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The smallest smoothness bound of a $\approx$ 256-bit twin for which their sum is a prime is $B = 2^{15}$

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Repeat the above but for $S^{(1)}$ instead of $S^{(0)}$. Eventually we must have $S^{(d+1)} = S^{(d)}$ for some d and the algorithm terminates when this happens

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The original authors ran CHM with $B = 100$ and found all 100-smooth twins with the exception of 37 solutions. They subsequently ran it with $B = 200$ which took 2 weeks for them to compute

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Subsequently we ran it fully for $B = 547$ – the largest twin found was the following 122-bit twin

$$\begin{aligned} r &= 5^4 \cdot 7 \cdot 13^2 \cdot 17^2 \cdot 19 \cdot 29 \cdot 41 \cdot 109 \cdot 163 \cdot 173 \cdot 239 \cdot 241^2 \\ &\quad \cdot 271 \cdot 283 \cdot 499 \cdot 509, \text{ and} \\ r + 1 &= 2^8 \cdot 3^2 \cdot 31^2 \cdot 43^2 \cdot 47^2 \cdot 83^2 \cdot 103^2 \cdot 311^2 \cdot 479^2 \cdot 523^2. \end{aligned}$$

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We also introduced some other optimisations that made it possible for us to run larger values of B

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Optimisations

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Variant	Parameter	Runtime	Speedup	#twins	#twins from largest 100
Full CHM	-	4705s	1	2300724	100
`global-`k	$k = 2.0$	364s	13	2289000	86
	$k = 1.5$	226s	21	2282741	82
	$k = 1.05$	27s	174	2206656	65
`constant-range`	$R = 10000$	82s	57	2273197	93
	$R = 5000$	35s	134	2247121	87
	$R = 1000$	16s	294	2074530	75

Table 1: Performance results for different variants of our CHM implementation for smoothness bound $B = 300$. Speedup factors refer to the full CHM variant.

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Full CHM	-	4705s	1	2300724	100
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	$k = 1.5$	226s	21	2282741	82
	$k = 1.05$	27s	174	2206656	65
`constant-range`	$R = 10000$	82s	57	2273197	93
	$R = 5000$	35s	134	2247121	87
	$R = 1000$	16s	294	2074530	75

Table 1: Performance results for different variants of our CHM implementation for smoothness bound $B = 300$. Speedup factors refer to the full CHM variant.

`global-`k: Fix some $1 < k \leq 2$ and only check (r, s) with $r < s < k \cdot r$

`constant-range`: Fix a range R and only check (r, s) for the R successors s of r in each iteration

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CHM was run with $B = 1300$ using the `constant-range` optimisation with a range $R = 5000$, specifically targeting twins $(r, r+1)$ with $r > 2^{115}$ - the largest twin found was the following 145-bit twins

$$r = 2^5 \cdot 5 \cdot 7 \cdot 11^2 \cdot 13 \cdot 23 \cdot 53 \cdot 71 \cdot 109 \cdot 127 \cdot 131 \cdot 193 \cdot 251 \\ \cdot 283 \cdot 307 \cdot 359 \cdot 367 \cdot 461 \cdot 613 \cdot 653 \cdot 1277, \text{ and} \\ r + 1 = 3^2 \cdot 29^2 \cdot 31^2 \cdot 43^2 \cdot 59^2 \cdot 61^2 \cdot 73^2 \cdot 79^2 \cdot 89^2 \cdot 167^2 \cdot 401^2 \cdot 419^2.$$

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Unfortunately, choosing B large enough and running this to give you cryptographic sized twins is infeasible due to time and memory limitations

Parameter Setup for SQISign

SQISign requirements

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Cryptographic prime p (of $\approx 256, 384, 512$-bits), such that

$$p^2 - 1 = 2^f \cdot T \cdot R,$$

where f is a "relatively" large exponent, T is an odd smooth cofactor of size $\approx p^{5/4+\epsilon}$ and R can have rough factors

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If B is the smoothness bound of T, the quantity $\sqrt{B}/f$ is a rough cost metric for the signing algorithm in SQISign

XGCD/CRT method for finding SQISign parameters

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They forced

$$\begin{aligned} p \pm 1 &= 0 \mod 2^{\alpha}, \\ p \mp 1 &= 0 \mod 3^{\beta}, \\ p \pm 1 &= 0 \mod q \quad \text{for small primes } q, \\ p \mp 1 &= 0 \mod q' \quad \text{for other small primes } q' \end{aligned}$$

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With this technique, they found SQISign friendly primes whose smooth cofactor T is 2^{12}-smooth

Exhaustive search using $p_4(x)$

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Their idea is the following:

1. Replace[3] $x \mapsto 2^{15} \cdot x$ in the polynomial p_4
2. Sieve the interval $x \in [2^{47}, 2^{49}]$ to identify 2^{11}-smooth integers
3. Compute the 2^{11}-smooth odd cofactor, T, of
$$x^4(2^{15}x-1)(2^{15}x+1)(2^{30}x^2+1)$$
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They found 15 primes of this type

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Comparison of their primes

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254-bit prime p:

$$p+1 = 2^{65}\cdot 5^2\cdot 7\cdot 11\cdot 19\cdot 29^2\cdot 37^2$$
$$\cdot 47\cdot 197\cdot 263\cdot 281\cdot 461$$
$$\cdot 521\cdot 3923\cdot R,\ \text{and}$$
$$p-1 = 2\cdot 3^{65}\cdot 13\cdot 17\cdot 43\cdot 79\cdot 157$$
$$\cdot 239\cdot 271\cdot 283\cdot 307\cdot 563$$
$$\cdot 599\cdot 607\cdot 619\cdot 743\cdot 827$$
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256-bit prime $p = p_4(r) = 2r^4 - 1$ with $r = 2^{15}\cdot 4110994464096 99$:

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$$p-1 = 2\cdot 5^2\cdot 13\cdot 17\cdot 29\cdot 37\cdot 41$$
$$\cdot 103\cdot 109\cdot 149\cdot 191\cdot 269$$
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In practice, the first of these performs slightly better despite having a larger signing cost metric - owing in large part to the large power of 3 but also the amount of small smoothness[4] is also larger

[4] A cofactor that is, say, 100-smooth

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389-bit prime $p = p_{12}(r) = 2r^{12} - 1$ with $r = 5114946480$:

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This could be used in theory but we find better more applicable primes

Our Method

General framework

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For the polynomials $p_n(x) = 2x^n - 1$, we have

$$
\begin{aligned}
4x^n(x-1) &\mid p_n^2(x) - 1 \quad \text{for all } n, \text{ and} \\
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If so then compute the other smooth factors of $p^2 - 1$ and check to see if the combined cofactor is larger than $p^{5/4+\epsilon}$

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One can estimate the probability of this happening using a result by Banks and Shaparlinski (2006)

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We give a worst case probability

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$\mathbf{n = 6}$: For a smooth twin $(r, r \pm 1)$, let $p = 2r^6 - 1$. Here we have

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The amount of guaranteed smoothness from the twin alone is $p^{7/6}$

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Here we exploit the multiple factors, $(r \mp 1)(r^2 - r + 1)(r^2 + r + 1)$, to give a better chance of finding enough smoothness for SQISign parameters

Requirements and smoothness probabilities in each case

	n	$\log(r)$	**Probability of B-smooth** $(r, r \pm 1)$	**Probability of $p^2 - 1$ $\log T'$-bits B-smooth given $(r, r \pm 1)$ twin smooth**	**Extra Smoothness Needed**
NIST-I	2	≈ 127.5	$2^{-58.5}$	1	0
$B = 2^9$	3	≈ 85.0	$2^{-32.1}$	$2^{-12.1}$	42
$\log p = 256$	4	≈ 63.75	$2^{-20.5}$	$\approx 2^{-22.4}$	63.25
$\log T' = 384$	6	≈ 42.5	$2^{-10.4}$	$\approx 2^{-32.2}$	84.5
NIST-III	2	≈ 191.5	$2^{-55.7}$	1	0
$B = 2^{14}$	3	≈ 127.67	$2^{-30.5}$	$2^{-11.7}$	63.33
$\log p = 384$	4	≈ 95.75	$2^{-19.4}$	$\approx 2^{-15.7}$	95.25
$\log T' = 576$	6	≈ 63.83	$2^{-9.7}$	$\approx 2^{-19.2}$	127.17
NIST-V	2	≈ 255.5	$2^{-63.7}$	1	0
$B = 2^{17}$	3	≈ 170.33	$2^{-35.2}$	$2^{-13.5}$	84.67
$\log p = 512$	4	≈ 127.75	$2^{-22.6}$	$\approx 2^{-18.2}$	127.25
$\log T' = 768$	6	≈ 85.17	$2^{-11.5}$	$\approx 2^{-22.5}$	169.83

Table 2: Assuming that $(r, r \pm 1)$ are twin smooth integers and p has $\log p$ bits, calculates the probability of having a B-smooth divisor $T' \mid p^2 - 1$ of size $\approx p^{3/2}$.

Practical SQISign Results

NIST-I parameters

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We used $n = 2, 3, 4$ to find a collection of 256-bit SQISign friendly primes

NIST-I parameters

We used $n = 2, 3, 4$ to find a collection of 256-bit SQISign friendly primes

243-bit prime $p = 2r^2 - 1$ with $r{=}20910230141429718023578160841527132 16$:

$$p + 1 = 2^{49} \cdot 3^4 \cdot 7^2 \cdot 11^2 \cdot 31^2 \cdot 41^2 \cdot 47^2 \cdot 67^2 \cdot 151^2 \cdot 173^2 \cdot 193^2 \cdot 223^2 \cdot 307^2 \cdot 317^2 \cdot 463^2 \cdot 887^2, \text{ and}$$

$$p - 1 = 2 \cdot 5 \cdot 13^2 \cdot 19 \cdot 29 \cdot 53 \cdot 61 \cdot 113 \cdot 211 \cdot 311 \cdot 337 \cdot 479 \cdot 599 \cdot 691 \cdot 739 \cdot 773 \cdot 811 \cdot 1277 \cdot 99100616784027099637811188822240347$$

NIST-I parameters

We used $n = 2, 3, 4$ to find a collection of 256-bit SQISign friendly primes

243-bit prime $p = 2r^2 - 1$ with $r{=}209102301414297180235781608415271321$6:

$$
\begin{aligned}
p + 1 &= 2^{49} \cdot 3^4 \cdot 7^2 \cdot 11^2 \cdot 31^2 \cdot 41^2 \cdot 47^2 \cdot 67^2 \cdot 151^2 \cdot 173^2 \cdot 193^2 \cdot 223^2 \\
&\quad \cdot 307^2 \cdot 317^2 \cdot 463^2 \cdot 887^2, \text{ and} \\
p - 1 &= 2 \cdot 5 \cdot 13^2 \cdot 19 \cdot 29 \cdot 53 \cdot 61 \cdot 113 \cdot 211 \cdot 311 \cdot 337 \cdot 479 \cdot 599 \cdot 691 \\
&\quad \cdot 739 \cdot 773 \cdot 811 \cdot 1277 \cdot 9910061678402709963781118882240347
\end{aligned}
$$

255-bit prime $p = 2r^3 - 1$ with $r = 26606682403634464748953600$:

$$
\begin{aligned}
p + 1 &= 2^{40} \cdot 5^6 \cdot 11^3 \cdot 47^3 \cdot 67^6 \cdot 101^3 \cdot 113^3 \cdot 137^3 \cdot 277^3 \cdot 307^3 \cdot 421^3, \text{and} \\
p - 1 &= 2 \cdot 3^2 \cdot 19^3 \cdot 37 \cdot 59 \cdot 61 \cdot 97 \cdot 181^2 \cdot 197 \cdot 223 \cdot 271 \cdot 281 \cdot 311 \\
&\quad \cdot 397 \cdot 547 \cdot R
\end{aligned}
$$

NIST-I parameters

We used $n = 2, 3, 4$ to find a collection of 256-bit SQISign friendly primes

NIST-I parameters

We used $n = 2, 3, 4$ to find a collection of 256-bit SQISign friendly primes

253-bit prime $p = 2r^4 - 1$ with $r = 8077251317941145600$:

$$p + 1 = 2^{49} \cdot 5^8 \cdot 13^4 \cdot 41^4 \cdot 71^4 \cdot 113^4 \cdot 181^4 \cdot 223^4 \cdot 457^4, \text{ and}$$

$$p - 1 = 2 \cdot 3^2 \cdot 7^5 \cdot 17 \cdot 31 \cdot 53 \cdot 61 \cdot 73 \cdot 83 \cdot 127 \cdot 149 \cdot 233 \cdot 293 \cdot 313 \cdot 347 \cdot 397 \cdot 467 \cdot 479 \cdot R$$

NIST-I parameters

We used $n = 2, 3, 4$ to find a collection of 256-bit SQISign friendly primes

253-bit prime $p = 2r^4 - 1$ with $r = 8077251317941145600$:

$$
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p + 1 &= 2^{49} \cdot 5^8 \cdot 13^4 \cdot 41^4 \cdot 71^4 \cdot 113^4 \cdot 181^4 \cdot 223^4 \cdot 457^4, \text{ and} \\
p - 1 &= 2 \cdot 3^2 \cdot 7^5 \cdot 17 \cdot 31 \cdot 53 \cdot 61 \cdot 73 \cdot 83 \cdot 127 \cdot 149 \cdot 233 \cdot 293 \cdot 313 \\
&\quad \cdot 347 \cdot 397 \cdot 467 \cdot 479 \cdot R
\end{aligned}
$$

Remarks:

- This prime is out of scope for De Feo, Leroux and Wesolowski to find since they “maximised” the power of two in $p + 1$
- No conclusions should be made about how these primes compare to the state-of-the-art without an implementation

NIST-III parameters

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We used $n = 3, 4, 6$ to find a collection of 384-bit SQISign friendly primes

NIST-III parameters

We used $n = 3, 4, 6$ to find a collection of 384-bit SQISign friendly primes

375-bit prime $p = 2r^4 - 1$ with $r = 1232621228336746350727292518 4$:

$$p + 1 = 2^{77} \cdot 11^4 \cdot 29^4 \cdot 59^4 \cdot 67^4 \cdot 149^4 \cdot 331^4 \cdot 443^4 \cdot 593^4 \cdot 1091^4 \cdot 1319^4, \text{ and}$$

$$p - 1 = 2 \cdot 3 \cdot 5 \cdot 13 \cdot 17 \cdot 31 \cdot 37 \cdot 53 \cdot 83 \cdot 109 \cdot 131 \cdot 241 \cdot 269 \cdot 277 \cdot 283 \cdot 353 \cdot 419 \cdot 499 \cdot 661 \cdot 877 \cdot 1877 \cdot 3709 \cdot 9613 \cdot 44017 \cdot 55967 \cdot R$$

NIST-III parameters

We used $n = 3, 4, 6$ to find a collection of 384-bit SQISign friendly primes

375-bit prime $p = 2r^4 - 1$ with $r = 123262122833674635072729251 84$:

$$p + 1 = 2^{77} \cdot 11^4 \cdot 29^4 \cdot 59^4 \cdot 67^4 \cdot 149^4 \cdot 331^4 \cdot 443^4 \cdot 593^4 \cdot 1091^4 \cdot 1319^4, \text{ and}$$

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382-bit prime $p = 2r^6 - 1$ with $r = 1189664338866214502 4$:

$$p + 1 = 2^{79} \cdot 3^6 \cdot 23^{12} \cdot 107^6 \cdot 127^6 \cdot 307^6 \cdot 401^6 \cdot 547^6, \text{ and}$$

$$p - 1 = 2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17 \cdot 19 \cdot 47 \cdot 71 \cdot 79 \cdot 109 \cdot 149 \cdot 229 \cdot 269 \cdot 283 \cdot 349 \cdot 449 \cdot 463 \cdot 1019 \cdot 1033 \cdot 1657 \cdot 2179 \cdot 2293 \cdot 4099 \cdot 5119 \cdot 10243 \cdot R$$

NIST-V parameters

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We used $n = 4, 6$ to find a collection of 512-bit SQISign friendly primes

NIST-V parameters

We used $n = 4, 6$ to find a collection of 512-bit SQISign friendly primes

499-bit prime $p = 2r^6 - 1$ with $r = 9469787780580604464332800$:

$$p + 1 = 2^{109} \cdot 5^{12} \cdot 7^{12} \cdot 13^6 \cdot 61^6 \cdot 179^6 \cdot 281^6 \cdot 379^6 \cdot 1367^6 \cdot 1427^6, \text{ and}$$

$$\begin{aligned} p - 1 = 2 \cdot 3^3 \cdot 19 \cdot 23^3 \cdot 31 \cdot 43^2 \cdot 73 \cdot 139 \cdot 337 \cdot 461 \cdot 641 \cdot 971 \cdot 1069 \\ \cdot 1097 \cdot 5843 \cdot 12841 \cdot 23671 \cdot 39667 \cdot 51193 \cdot 75223 \cdot 459317 \\ \cdot 703981 \cdot R \end{aligned}$$

NIST-V parameters

We used $n = 4, 6$ to find a collection of 512-bit SQISign friendly primes

499-bit prime $p = 2r^6 - 1$ with $r = 946978778058060446433280 0$:

$$p + 1 = 2^{109} \cdot 5^{12} \cdot 7^{12} \cdot 13^6 \cdot 61^6 \cdot 179^6 \cdot 281^6 \cdot 379^6 \cdot 1367^6 \cdot 1427^6, \text{ and}$$
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508-bit prime $p = 2r^6 - 1$ with $r = 26697973900446483680608256$:

$$p + 1 = 2^{85} \cdot 17^{12} \cdot 37^6 \cdot 59^6 \cdot 97^6 \cdot 233^6 \cdot 311^{12} \cdot 911^6 \cdot 1297^6, \text{ and}$$
$$\begin{aligned} p - 1 = 2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11^2 \cdot 23^2 \cdot 29 \cdot 127 \cdot 163 \cdot 173 \cdot 191 \cdot 193 \cdot 211 \cdot 277 \\ \cdot 347 \cdot 617 \cdot 661 \cdot 761 \cdot 1039 \cdot 4637 \cdot 5821 \cdot 15649 \cdot 19139 \\ \cdot 143443 \cdot 150151 \cdot R \end{aligned}$$

Table of primes

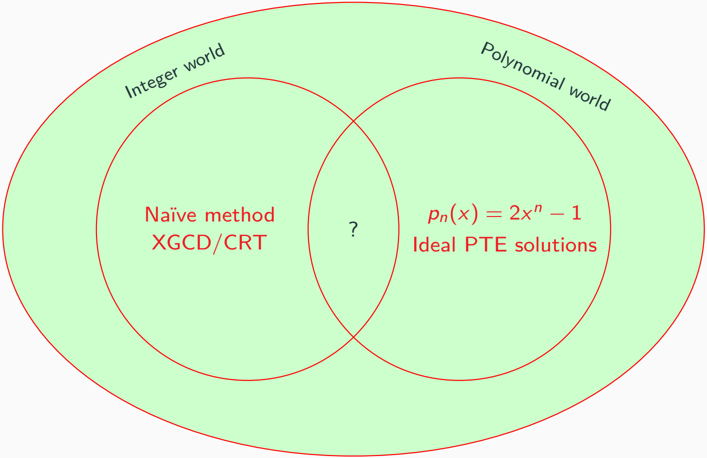
NIST security level	n	r	$\lceil \log_2(p) \rceil$	f	B	$\sqrt{B}/f$	$\log_p(T)$
NIST-I	2	1211460311716772790566574529001291776	241	49	1091	0.67	1.28
		2091023014142971802357816084152713216	243	49	887	0.61	1.28
	3	34742728167898672973 57824	246	43	547	0.54	1.29
		1022731837578822719958 9376	251	31	383	0.63	1.31
		2161173603326087887680 0000	254	31	421	0.66	1.28
		2046144912550037474885 6320	254	46	523	0.50	1.26
		2660668240363446474895 3600	255	40	547	0.58	1.28
	4	1466873880764125184	243	49	701	0.54	1.28
		8077251317941145600	253	49	479	0.45	1.30
		34848218231355211776*	261	77	2311	0.62	1.30
NIST-III	3	1374002035005713149550405343373848576	362	37	1277	0.97	1.25
	4	5139734876262390964070873088	370	45	11789	2.41	1.26
		12326212283367463507272925184	375	77	55967	3.07	1.31
		18080754980295452456023326720	377	61	95569	5.07	1.26
		27464400309146790228660255744	379	41	13127	2.79	1.29
	6	2628583629218279424	369	73	13219	1.58	1.27
		5417690118774595584	375	79	58153	3.05	1.27
		11896643388662145024	382	79	10243	1.28	1.30
NIST-V	4	11421678154858170943951287580127979 1104*	507	65	75941	4.24	1.26
		12379427438747429891274254381924258 7136*	508	41	15263	3.01	1.29
	6	9469787780580604464332800	499	109	703981	7.70	1.25
		12233468605740686007808000	502	73	376963	8.41	1.28
		26697973900446483680608256	508	85	150151	4.56	1.26
		31929740427944870006521856	510	91	550657	8.15	1.25
		41340248200900819056793600	512	67	224911	7.08	1.28

Table 3: A table of SQISign parameters $p = p_n(r)$ found using twin-smooth integers $(r, r \pm 1)$ at each security level. The f is the power of two dividing $(p^2 - 1)/2$ and B is the smoothness bound of the odd cofactor $T \approx p^{5/4+\epsilon}$. The r marked with an asterisk correspond to primes p not found using the CHM machinery.

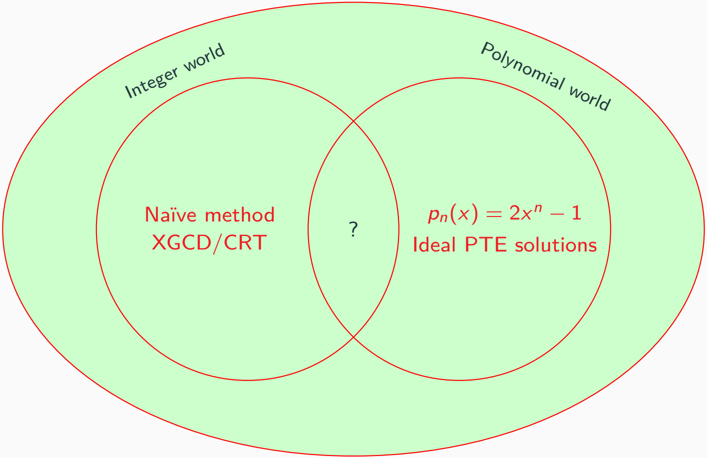
Smooth Twins from XGCD over Polynomial Rings

Probabilistic methods for finding smooth twins

Probabilistic methods for finding smooth twins



Probabilistic methods for finding smooth twins



Is there something that can bind these methods together?

Smooth Twins from XGCD over $\mathbb{Q}[x]$

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Choose two polynomials, $F, G \in \mathbb{Z}[x]$, that split completely into linear factors and the number of distinct roots of $F \cdot G$ is small[5]

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Use the XGCD algorithm over $\mathbb{Q}[x]$ to find two polynomials $S, T \in \mathbb{Q}[x]$ such that

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Then $(\hat{F}(r), \hat{G}(r))$ generates a smooth twin if and only if $S(r)T(r)$ is smooth

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Realising the generalisation

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- Computing the XGCD of $F(x) = x^n$ and $G(x) = x - 1$ results in the polynomials

  $$S(x) = 1, \text{ and } T(x) = -x^{n-1} - \cdots - x - 1$$

  Hence we get $\hat{F}(x) = x^n$ and $\hat{G}(x) = x^n - 1$

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We can reverse the latter remark - i.e. use XGCD over $\mathbb{Q}[x]$ as a tool to find ideal PTE solutions

In fact, we were able to find a completely new class of ideal size 4 PTE solutions that haven't appeared in the literature or any known database

New ideal PTE solutions of size 4

a	b	c	d	e	a	b	c	d	e	a	b	c	d	e
17	5	35	27	32	6620	1940	13289	10985	11664	22572	6660	47545	35152	44217
86	26	221	125	216	6830	2210	53261	9261	53240	22715	6755	50759	34295	48384
171	51	391	256	375	7398	2250	20125	10648	19773	23579	7619	176039	32000	175959
243	75	775	343	768	7749	2289	16459	12000	15379	26010	8070	88501	36501	87880
524	164	2009	729	2000	8021	2561	43931	10976	43875	26672	8720	314465	35937	314432
594	174	1189	1000	1029	8987	2915	76055	12167	76032	28170	8790	103429	39304	102885
605	185	1739	864	1715	10269	3129	28459	14739	28000	29358	8610	59245	48013	52728
965	305	4331	1331	4320	11556	3756	105481	15625	105456	31160	9320	72929	46305	70304
1463	455	5135	2048	5103	12015	3855	73759	16384	73695	31437	10185	255595	42592	255507
1602	510	8245	2197	8232	12386	3806	37541	17576	37125	31841	10421	396611	42875	396576
1790	530	3869	2744	3645	13076	3836	26441	21296	23625	33561	10461	121411	46875	120736
2471	791	14351	3375	14336	14472	4440	43105	20577	42592	33885	9945	68731	54880	61731
2628	780	5785	3993	5488	14573	4745	142715	19683	142688	34047	10335	90895	49152	89167
2889	909	12019	4000	11979	15930	4710	34069	24565	31944	35684	10604	79289	54000	75449
3608	1160	23345	4913	23328	17153	5525	116675	23328	116603	37638	12330	493885	50653	493848
3735	1095	7519	6144	6655	18074	5894	189029	24389	189000	39542	12410	158045	54872	157437
3962	1190	9605	5832	9317	19214	5954	64349	27000	63869	40871	13271	359471	55296	359375
4455	1335	10591	6591	10240	20195	5915	40391	34391	34560	41445	12465	101659	60835	98784
5027	1595	24215	6912	24167	22095	7215	245791	29791	245760	44099	14459	608039	59319	608000
5049	1629	36019	6859	36000	22473	6765	55555	32928	54043					

Table 4: List of all inequivalent and normalised sized 4 ideal PTE solutions of the form $[0, a, a, c] =_3 [b, b, d, e]$ with $0 < b < a < 50000$ and $c, d, e > 0$.

Strategy for finding SQISign primes

Strategy for finding SQISign primes

The idea is to replace the polynomials $p_n(x)$ with other polynomials $p_{i,j}(x)$ such that

$$x^i(x+1)^j \mid p_{i,j}^2(x) - 1, \quad \text{with } i,j \geq 2, i \neq j \text{ and } \deg(p_{i,j}) < i+j$$

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To do this, we compute the XGCD of $F_i(x) = x^i$ and $G_j(x) = (x+1)^j$, which gives us

$$S_{i,j}(x) = (-1)^i \sum_{k=0}^{j-1} \binom{i+k-1}{k} (x+1)^k$$

$$T_{i,j}(x) = \sum_{k=0}^{i-1} (-1)^k \binom{j+k-1}{k} x^k$$

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Then we set

$$\begin{aligned} p_{i,j}(x) &:= (-1)^i \left(x^i S_{i,j}(x) - (x+1)^j T_{i,j}(x) \right) \\ &= (-1)^i \left(2x^i S_{i,j}(x) - 1 \right) \\ &= (-1)^{i+1} \left(2(x+1)^j T_{i,j}(x) + 1 \right) \end{aligned}$$

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Note that $\deg(p_{i,j}) = i + j - 1 < i + j$ and, by the uniqueness of XGCD, no other polynomials exists whose degree is smaller than this one

Strategy for finding SQISign primes

Strategy for finding SQISign primes

For instance when $i, j \in \{2, 3\}$ with $i \neq j$, we have

$$p_{2,3}(x) = 6x^4 + 16x^3 + 12x^2 - 1$$
$$p_{3,2}(x) = 6x^4 + 8x^3 + 1$$

Strategy for finding SQISign primes

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$$p_{2,3}(x) = 6x^4 + 16x^3 + 12x^2 - 1$$
$$p_{3,2}(x) = 6x^4 + 8x^3 + 1$$

We can adopt the same strategy as before, namely take a smooth twin $(r, r+1)$ and compute the evaluation

$$p = p_{i,j}(r)$$

and see whether it is a suitable SQISign parameter

Strategy for finding SQISign primes

For instance when $i, j \in \{2, 3\}$ with $i \neq j$, we have

$$p_{2,3}(x) = 6x^4 + 16x^3 + 12x^2 - 1$$
$$p_{3,2}(x) = 6x^4 + 8x^3 + 1$$

We can adopt the same strategy as before, namely take a smooth twin $(r, r+1)$ and compute the evaluation

$$p = p_{i,j}(r)$$

and see whether it is a suitable SQISign parameter

We limit ourselves to small $i, j \geq 2$ since the polynomials $S_{i,j}$, $T_{i,j}$ are irreducible[6] for small $i, j \geq 2$

[6] Moreover, we conjecture that these polynomials are irreducible for all $i, j \geq 2$

Practical Results

255-bit prime $p = p_{3,2}(r)$ with $r = 5964933197580566528$:

$$p + 1 = 2 \cdot 3^5 \cdot 19 \cdot 31^2 \cdot 37^2 \cdot 67 \cdot 83^2 \cdot 89^2 \cdot 113^2 \cdot 157^4 \cdot 173^2 \cdot 233 \cdot 487^2 \cdot 641 \cdot R, \text{ and}$$

$$p - 1 = 2^{48} \cdot 11^3 \cdot 29^2 \cdot 47^3 \cdot 53^3 \cdot 79 \cdot 131^3 \cdot 331^3 \cdot 349^3 \cdot 439^3 \cdot 691 \cdot R'$$

382-bit prime $p = p_{3,2}(r)$ with $r = 24412952691406071260714369024$:

$$p + 1 = 2 \cdot 3^7 \cdot 7^{10} \cdot 19^6 \cdot 67^2 \cdot 131 \cdot 241^2 \cdot 313^2 \cdot 379^2 \cdot 641 \cdot 883^2 \cdot 1103^2 \cdot 1117^2 \cdot 2689 \cdot 11177 \cdot R, \text{ and}$$

$$p - 1 = 2^{66} \cdot 5 \cdot 13^3 \cdot 17^3 \cdot 23^3 \cdot 41^3 \cdot 59^3 \cdot 61^3 \cdot 83^6 \cdot 127 \cdot 389 \cdot 491^3 \cdot 787^3 \cdot 983 \cdot 1549^3 \cdot R'$$

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The general strategies deployed to find these primes can be applied in future applications

Merci pour votre attention
Questions?

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