Modular polynomials and isogenies

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I. Introduction

Motivations for computing isogenies in ANT/crypto:

- original one (1989ff): Schoof-Elkies-Atkin (SEA);
- later (circa 2000): Kohel, Galbraith, Fouquet/FM (volcanoes);
- more recently (2006ff): Galbraith/Hess/Smart; Smart; Jao/Miller/Venkatesan; Teske; Couveignes, Rostovtsev/Stolbunov.
- post-quantum cryptography (2011ff): Defeo/Jao, etc.

Bibliography:

- Silverman; Lang's *Elliptic functions*.
- green book (Blake/Seroussi/Smart). Don't forget to read the original papers, when available. . .
- Gathen & Gerhard, etc.

Elliptic curves and isogenies

$$E : y^2 = x^3 + Ax + B \text{ over } \mathbf{K}, \mathrm{char}(\mathbf{K}) \not\in \{2,3\}.$$

Def. (torsion points) For $n \in \mathbb{N}$, $E[n] = \{P \in E(\overline{\mathbb{K}}), [n]P = O_E\}$.

Division polynomials:

$$[n](x,y) = \left(\frac{\varphi_n(x,y)}{\psi_n(x,y)^2}, \frac{\omega_n(x,y)}{\psi_n(x,y)^3}\right)$$

$$\varphi_n = x\psi_n^2 - \psi_{n+1}\psi_{n-1}$$

$$4y\omega_n = \psi_{n+2}\psi_{n-1}^2 - \psi_{n-2}\psi_{n+1}^2$$

In $\mathbf{K}[x,y]/(y^2 - (x^3 + Ax + B))$, one has:

$$\psi_{2m+1}(x,y) = f_{2m+1}(x), \quad \psi_{2m} = 2yf_{2m}(x)$$

for some $f_m(x) \in \mathbf{K}[A,B,x]$.

Isogenies

Def. $\phi : E \to E^*$, $\phi(O_E) = O_{E^*}$; induces a morphism of groups.

First examples

1. Separable:

$$[k](x,y) = \left(\frac{\varphi_k}{\psi_k^2}, \frac{\omega_k}{\psi_k^3}\right)$$

2. Complex multiplication: $[i](x,y) = (-x, iy)$ on $E : y^2 = x^3 - x$.

3. Inseparable: $\varphi(x,y) = (x^p, y^p)$, $\mathbf{K} = \mathbb{F}_p$.

In the sequel:

- only separable isogenies;
- finite fields of large characteristic (see comments at the end).

Finding isogenies

Thm. If F is a finite subgroup of $E(\overline{\mathbf{K}})$, there exists ϕ and E^* s.t.

$$\phi : E \rightarrow E^* = E/F, \quad \ker(\phi) = F.$$

Facts:

- An equation of E^* can be computed using Vélu's formulas;
- the kernel polynomial (== denominator of ϕ) is $\mathscr{K}_F = X^d - \sigma_1 X^{d-1} + \cdots$ is a factor of $f_\ell(X)$ (in case ℓ odd and $d = (\ell-1)/2$).

Thm. All isogenous curves of degree ℓ to a given E are characterized by $\Phi_\ell(j(E^*), j(E)) = 0$, where Φ_ℓ is the traditional modular equation.

But: having $j(E^*)$ is not enough to find an equation for E^* (quadratic twists), nor the explicit isogeny.

Basic algorithm

```
Function FindAllIsogenies($E$, $\ell$):
    Input   : $E/\mathbb{F}_q = [A,B]$ an elliptic curve, $\ell$ an odd prime
    Output: $\{(\sigma, A^*, B^*)\}$ parameters of curves $E^*$ that are
            $\ell$-isogenous to $E$ if any
    1. $\mathscr{L} \leftarrow$ roots of $\Phi_\ell(X, j(E)) = 0$ over $\mathbf{K}$
    2. $\mathscr{R} \leftarrow \emptyset$
    3. for $j^* \in \mathscr{L}$ do
        $\mathscr{R} \leftarrow \mathscr{R} \cup \{(\sigma, A^*, B^*)\}$, the parameters of $E^*$
    4. return $\mathscr{R}$.
```

Rem. $\#\mathscr{L} \in \{0, 2, 1, \ell+1\}$; more is known on the splitting of $\Phi_\ell(X, j(E))$ over $\mathbf{K}$, for instance using the action of the Frobenius over $E[\ell]$.

Isogeny algorithms

Key ingredients:

- modular equations:
  - choose nice equations;
  - compute equations over $\mathbb{Z}[X]$ + instantiation over $\mathbf{K}$:
    • series over $\mathbb{Z}$ (or $\mathbb{Z}/p\mathbb{Z}$): (. . . , CCR, Atkin, . . .);
    • evaluation/interpolation: with floating points (Dupont/Enge); with curves modulo p (Charles + Lauter).
  - Compute $\Phi_\ell(X,j(E))$ directly using isogeny volcanoes (Sutherland *et al.*).
- compute ℓ-isogenies:
  - compute isgenous curve: magical (ugly) formulas by Atkin; alternatively: CCR.
  - compute isogeny: depends on q and p, BMSS, Lercier/Sirvent, etc.

A glimpse at Elkies's work

INPUT: E and E^* related via an ℓ-isogeny with trace σ.
OUTPUT: $I(x) = N(x)/D(x) = N(x)/f_\lambda(x)^2$.

$$E : y^2 = x^3 + Ax + B, \quad E^* : y^2 = x^3 + A^* x + B^*,$$

can be parametrized as $(x, y) = (\wp(z), \wp'(z)/2)$, where the function $\wp$ can be expanded as:

$$\wp(z) = \frac{1}{z^2} + \sum_{i \geq 1} c_i z^{2i},$$

with

$$c_1 = -\frac{A}{5}, c_2 = -\frac{B}{7}, \quad \text{for } k \geq 3, c_k = \frac{3}{(k-2)(2k+3)} \sum_{i=1}^{k-2} c_i c_{k-1-i}.$$

(see BMSS paper for fast expansion method)

Elkies's method

$$\frac{N(x)}{D(x)} = \wp^* \circ \wp^{-1}(x) = x + \sum_{i \geq 1} \frac{h_i}{x^i}$$

First: compute

$$h_k = \frac{3}{(k-2)(2k+3)} \sum_{i=1}^{k-2} h_i h_{k-1-i} - \frac{2k-3}{2k+3} A h_{k-2} - \frac{2(k-3)}{2k+3} B h_{k-3}$$

for all $k \geq 3$ with $h_1 = (A - A^*)/5$ and $h_2 = (B - B^*)/7$.

$\Rightarrow O(\ell^2)$ operations in $\mathbf{K}$.

Second: get σ_i's using:

$$h_i = (2i+1)\sigma_{i+1} + (2i-1)A\sigma_{i-1} + (2i-2)B\sigma_{i-2}, \quad \text{for all } i \geq 1,$$

Third: recover $D(x)$ using Newton's formulas in $O(\ell^2)$ operations, or perhaps in $O(\mathrm{M}(\ell))$ with Schönhage's algorithm.

Total complexity: $O(\ell^2)$.

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II. Classical theory and computations

Eisenstein series: $\delta_r(n) = \sum_{d|n} d^r$

$$E_2(q) = 1 - 24 \sum_{n=1}^{\infty} \delta_1(n) q^n,$$

$$E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \delta_3(n) q^n,$$

$$E_6(q) = 1 - 504 \sum_{n=1}^{\infty} \delta_5(n) q^n,$$

Fact: E_4 and E_6 are modular forms of weight 4 and 6 respectively, E_2 is almost modular.

$$\Delta(q) = \frac{E_4^3 - E_6^2}{1718} = q \prod_{n \geq 1} (1 - q^n)^{24} = \eta(q)^{24}$$

$$j(q) = \frac{E_4^3}{\Delta} = \frac{1}{q} + 744 + \sum_{n \geq 1} c_n\, q^n, c_n \in \mathbb{N}.$$

Identities involving Eisenstein's series

When $F(q) = \sum_{n \geq n_0} a_n q^n$, we introduce the operator

$$F'(q) = \frac{1}{2i\pi} \frac{dF}{d\tau} = q \frac{dF}{dq} = \sum_{n \geq n_0} n a_n q^n.$$

$$\Delta = \frac{E_4^3 - E_6^2}{1728}, \quad \frac{\Delta'}{\Delta} = E_2, \quad j = \frac{E_4^3}{\Delta}, \quad j - 1728 = \frac{E_6^2}{\Delta}, \qquad (1)$$

$$\frac{j'}{j} = -\frac{E_6}{E_4}, \quad \frac{j'}{j - 1728} = -\frac{E_4^2}{E_6}, \quad j' = -\frac{E_4^2 E_6}{\Delta}, \qquad (2)$$

$$3E_4{}' = E_2 E_4 - E_6, \quad 2E_6{}' = E_2 E_6 - E_4^2, \quad 12E_2{}' = E_2^2 - E_4. \qquad (3)$$

(The last line is due to Ramanujan.)

A) Lattices, etc.

Def. $\mathscr{L} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $\mathscr{L}' = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$ are isomorphic iff there exists P in $SL_2(\mathbb{Z})$ s.t.

$$\left(\begin{array}{c}\omega_1'\\ \omega_2'\end{array}\right) = P\left(\begin{array}{c}\omega_1\\ \omega_2\end{array}\right).$$

Thm. $\mathscr{L}$ and $\mathscr{L}'$ are isomorphic iff $j(\mathscr{L}) = j(\mathscr{L}')$.

Def. $\mathscr{L}$ and $\mathscr{M}$ are isogenous iff $\exists \alpha \in \mathbb{C}, \alpha\mathscr{L} \subset \mathscr{M}$.

Most interesting case: $\mathscr{M}$ is a sublattice of $\mathscr{L}$ s.t. $\mathscr{L}/\mathscr{M}$ is cyclic of finite index. In other words:

$$\mathscr{M} = (a\omega_1 + b\omega_2)\mathbb{Z} + (c\omega_1 + d\omega_2)\mathbb{Z}$$

and $ad - bc = m$ with $\gcd(a,b,c,d) = 1$.

Fundamental theorem (modular polynomial):

Thm. $\exists \alpha \in \mathbb{C}$ s.t. $\alpha\mathscr{M} \subset \mathscr{L}$ iff $\exists m$ s.t. $\Phi_m(j(\mathscr{M}), j(\mathscr{L})) = 0$ s.t. with $\tau = \omega_2/\omega_1$ (imag. part > 0), $q = \exp(2i\pi\tau)$:

$$\Phi_m(X, \tau) = \prod_{A \in \mathscr{S}_m} (X - j(A\tau)) = \sum_{k=0}^{\mu_0(m)} C_k(\tau) X^k,$$

$$\mathscr{S}_m = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, ad = m, \gcd(a, b, d) = 1, a > 0, d > b \geq 0 \right\}$$

of cardinality $\mu_0(m) = m \prod_{p|m} (1 + 1/p)$.

When $m = \ell$ is prime:

$$\mathscr{S}_\ell = \left\{ \begin{pmatrix} 1 & b \\ 0 & \ell \end{pmatrix}, 0 \leq b < \ell \right\} \cup \left\{ \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

of cardinality $\ell + 1$.

Modular polynomials

Thm.

- $\Phi_m(X,Y) \in \mathbb{Z}[X,Y]$;
- $\Phi_m(Y,X) = \Phi_m(X,Y)$;
- if m is squarefree, then the coefficient of highest degree of $\Phi_m(X,X)$ is ± 1.

Prop. ("Cyclotomic" properties)
(a) If $(m_1,m_2) = 1$, then

$$\Phi_{m_1 m_2}(X,J) = \mathrm{Resultant}_Z(\Phi_{m_1}(X,Z), \Phi_{m_2}(Z,J)).$$

(b) If $m = \ell^e$ with $e > 1$, then

$$\Phi_{\ell^e}(X,J) = \mathrm{Resultant}_Z(\Phi_\ell(X,Z), \Phi_{\ell^{e-1}}(Z,J))/\Phi_{\ell^{e-2}}(Z,J)^\ell.$$

Thm. (Kronecker) If ℓ is prime, then

$$\Phi_\ell(X,Y) \equiv (X^\ell - Y)(Y^\ell - X) \bmod \ell.$$

Height

Thm. (P. Cohen)
$H(\Phi_m) = 6\mu_0(m)(\log m - 2\sum_{p|m}(\log p)/p + O(1))$.

ℓ	101	211	503	1009	2003
$H(\Phi_\ell)$	3985	9256	24736	53820	115125
PCohen	2768	6743	18736	41832	91320

Thm. (Bröker & Sutherland)

$$H(\Phi_\ell) \leq 6\ell\log\ell + 16\ell + 14\sqrt{\ell}\log\ell.$$

$\Rightarrow$ Φ_ℓ has $O(\ell^2)$ coefficients of size $\ell\log\ell$, or a $\tilde{O}(\ell^3)$-bit object.

Ex.

$$\begin{aligned}\Phi_2(X,Y) &= X^3 + X^2\left(-Y^2 + 1488\,Y - 162000\right)\\ &+X\left(1488\,Y^2 + 40773375\,Y + 8748000000\right)\\ &+Y^3 - 162000\,Y^2 + 8748000000\,Y - 157464000000000.\end{aligned}$$

B) Computing modular polynomials over $\mathbb{Z}[X]$

Remember that

$$j(q) = \frac{1}{q} + 744 + \sum_{n\geq 1} c_n q^n, \quad c_n \in \mathbb{Z}^+$$

Then $\Phi_\ell(X,Y)$ is such that $\Phi_\ell(j(q), j(q^\ell))$ vanishes identically.

Naive method: indeterminate coefficients (over $\mathbb{Q}$ or small p's); at least $\tilde{O}((\ell^2)^\omega)$ operations over $\mathbb{Q}$.

Ex.

$$\Phi_2(X,Y) = X^3 + X^2\left(-Y^2 + 1488\,Y - 162000\right)$$
$$+X\left(1488\,Y^2 + 40773375\,Y + 8748000000\right)$$
$$+Y^3 - 162000\,Y^2 + 8748000000\,Y - 157464000000000.$$

a) Series computations

Enneper (1890) use q-expansion of j and $j(q^\ell)$ with $O(\ell^2)$ terms; Atkin used this modulo FFT-friendly primes + CRT (embarassingly parallel). $\tilde{O}(\ell^3 \mathsf{M}(p))$

1. Compute power sums for $1 \leq r \leq \ell$:

$$S_r(q) = j(\ell\tau)^r + \sum_{k=0}^{\ell-1} j\left(\frac{\tau+k}{\ell}\right)^r = S_{r,0}(q) + S_{r,1}(w)$$

with $w = q^{1/\ell}$; $S_{r,1}$ *a priori* in $\mathbb{Q}(\zeta_\ell)$, but in fact over $\mathbb{Q}$, hence $S_{r,1}(w) = S_{r,1}(q)$;

2. a) Go back to $\Phi(X, J)$ using Newton formulas.

b) recognize $C_r(q) = C_r(J)$.

Numerical example

$\ell = 3$, $p = 101$: $j = q^{-1} + 37 + 35\,q + 51\,q^2 + 45\,q^3 + 34\,q^4 + O(q^5)$.

Over $\mathbb{F}_p[\zeta] = \mathbb{F}_p[X]/(X^2+X+1)$:

$$j_0 = \frac{1}{w} + 37 + 35w + 51w^2 + 45w^3 + 34w^4 + 55w^5 + w^6 + 78w^7 + 77w^8 + 28w^9$$

$$j_1 = (100\zeta + 100)\frac{1}{w} + 37 + 35\zeta w + (50\zeta + 50)w^2 + 45w^3 + 34\zeta w^4 + (46\zeta +$$

$$j_2 = \zeta\frac{1}{w} + 37 + (66\zeta + 66)w + 51\zeta w^2 + 45w^3 + (67\zeta + 67)w^4 + 55\zeta w^5 + w^6 +$$

r	$S_{r,1}$
1	$10 + 34w^3 + 3w^6 + 84w^9 + O(w^{10})$,
2	$75 + 98w^3 + 35w^6 + O(w^9)$,
3	$3w^{-}3 + 89 + 3w^3 + w^6 + O(w^8)$,
4	$40w^{-}3 + 2 + w^3 + 13w^6 + O(w^7)$.

Numerical example (cont'd)

$$S_1 = S_{0,1} + S_{1,1} = q^{-3} + 47 + 34q + 3q^2 + 18q^3 + 51q^6 + 45q^9 + O(q^{12}),$$

$$S_2 = S_{0,2} + S_{1,2} = q^{-6} + 74q^{-3} + 100 + 98q + 35q^2 + 47q^3 + 58q^4 + 39q^6 + O($$

$$S_3 = S_{0,3} + S_{1,3} = q^{-9} + 10q^{-6} + 71q^{-3} + 3q^{-1} + 85 + 3q + +q^2 + 29q^3 + O(q$$

$$S_4 = S_{0,4} + S_{1,4} = q^{-12} + 47q^{-9} + 72q^{-6} + 95q^{-3} + 40q^{-1} + 70 + q + 13q^2 + O$$

Newton's formulas give:

$$c_0 = q^{-4} + 57q^{-3} + 37q^{-2} + 58q^{-1} + 92 + 7q + 82q^2 + O(q^3),$$

$$c_1 = 38q^{-3} + 12q^{-2} + 65q^{-1} + 44 + 44q + 68q^2 + 93q^3 + 57q^4 + 29q^5 + O(q^6$$

$$c_2 = 10q^{-3} + 34q^{-2} + 3q^{-1} + 12 + 75q + 45q^2 + 16q^3 + 40q^4 + 34q^5 + 31q^6 +$$

$$c_3 = 100q^{-3} + 54 + 67q + 98q^2 + 83q^3 + 60q^4 + 50q^6 + 56q^9 + O(q^{12}).$$

Example (cont'd)

$$\Phi_3(X,Y) = X^4 + C_3(Y)X^3 + C_2(Y)X^2 + C_1(Y)X + C_0(X), \quad \deg(C_i) \leq 3$$

$$c_0 = q^{-4} + 57q^{-3} + 37q^{-2} + 58q^{-1} + 92 + 7q + 82q^2 + O(q^3),$$

$$C_0 = J^4 + 10J^3 + 67J^2 + 52J,$$

$$c_1 = 100q^{-3} + 54 + 67q + 98q^2 + 83q^3 + 60q^4 + 50q^6 + 56q^9 + O(q^{12}),$$

$$C_1 = 38J^3 + 36J^2 + 56J + 52;$$

$$c_2 = 10q^{-3} + 34q^{-2} + 3q^{-1} + 12 + 75q + 45q^2 + 16q^3 + 40q^4 + 34q^5 + 31q^6 +$$

$$C_2 = 10J^3 + 35J^2 + 36J + 67,$$

$$C_3 = 100J^3 + 10J^2 + 38J + 10.$$

b) Evaluation/interpolation (Enge; Dupont)

$$\Phi_\ell(X,J) = X^{\ell+1} + \sum_{u=0}^{\ell} C_u(J)X^u, \quad C_u(J) \in \mathbb{Z}[J], \deg(C_u(J)) \leq \ell+1.$$

All computations are done using precision $H = O(\ell \log \ell)$.

```
Function ComputePhi($\ell$, $j$, $(j_r)$, $\deg_X$):
    Input   : $\ell$ an odd prime; $f$ a function, $j_r$ conjugates
    Output: $\Phi_\ell(X,J)$ with degree $\deg_X$ in $X$
    for $\deg_X + 1$ values of $z_i$ do
        compute $j_r(z_i)$ to precision $H$ and build
        $\prod_{r=1}^{\ell+1}(X - j_r(z_i)) = X^{\ell+1} + \sum_{u=0}^{\ell} C_u(j(z_i))X^u$;
          $O(\mathsf{M}(\ell)\log\ell)$ ops
    for $u \leftarrow 0$ to $\ell$ do
        interpolate $C_u$ from $(j(z_i), C_u(j(z_i)))$ for $1 \leq i \leq \deg_X + 1$
    return $\Phi_\ell(X,J)$
```

All 1.2 + 2 has cost $O(\ell\mathsf{M}(\ell)(\log\ell)\mathsf{M}(H)) = \tilde{O}(\ell^3)$.

C) Isogeny volcanoes

Bröker, Lauter, Sutherland (2010): Under the Generalized Riemann Hypothesis (GRH), expected running time of $O(\ell^3 (\log \ell)^3 \log\log \ell)$, and compute $\Phi_\ell(X, j(E)) \bmod p$ using $O(\ell^2 (\log \ell)^2 + \ell^2 \log p)$ space.

- Need class polynomials $H_D(X)$ (sometimes $H_{\ell^2 D}(X)$).
- Interpolate the values of all quantities modulo p.
- Extensible to other functions, partial differentials.

- Works also in Sutherland's algo for direct evaluation over $\mathbf{K}$ using explicit CRT.

D) Elkies's approach to the isogeny problem

Tate curve:

$$Y^2 = X^3 - \frac{E_4(q)}{48} X + \frac{E_6(q)}{864}$$

is ℓ-isogenous to

$$Y^2 = X^3 - \frac{E_4(q^\ell)}{48} X + \frac{E_6(q^\ell)}{864}$$

$\sigma_r =$ power sums of the roots of the kernel polynomial:

$$\sigma_1(q) = \frac{\ell}{2}(\ell E_2(q^\ell) - E_2(q)).$$

Use series identities to get formulas for $E_4(q^\ell)$ and $E_6(q^\ell)$, $\sigma_1(q)$ from known values.
Also:

$$A - A^* = 5(6\sigma_2 + 2A\sigma_0), \qquad B - B^* = 7(10\sigma_3 + 6A\sigma_1 + 4B\sigma_0),$$

$+$ induction relation for σ_k with $k > 3$.

Consequence: A^* and B^* belong to $\mathbb{Q}[\sigma_1, A, B]$.

Formulas for the kernel of the isogeny

Points in the kernel:

$$x(\zeta;q) = \frac{1}{12} - 2\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} + \sum_{n\in\mathbb{Z}} \frac{\zeta q^n}{(1-\zeta q^n)^2};$$

$$y(\zeta;q) = \frac{1}{2}\sum_{n\in\mathbb{Z}} \frac{\zeta q^n(1+\zeta q^n)}{(1-\zeta q^n)^3}$$

satisfy

$$y^2 = x^3 - \frac{E_4(q)}{48}x + \frac{E_6(q)}{864}.$$

$$\sigma_1 = \sum_{\zeta\in\mu_\ell,\zeta\neq 1} x(\zeta;q) = \frac{\ell}{12}(E_2(q) - \ell E_2(q^\ell)).$$

The case of $\Phi_\ell(X,Y)$

Thm. (Schoof95)
With $\tilde{j} = j(q^\ell)$:

1) $\Phi_\ell(j,\tilde{j}) = 0$.

2) $j'\partial_X + \ell\tilde{j}'\partial_Y = 0$ yields $\tilde{j}'$.

3)

$$\frac{j''}{j'} - \ell\frac{\tilde{j}''}{\tilde{j}'} = -\frac{{j'}^2\partial_{XX} + \cdots}{j'\partial_X}.$$

All these yield algebraic expressions for $E_4(q^\ell)$, $E_6(q^\ell)$, σ_1, involving all these derivatives.

Cost: $O(\ell^2)$ operations in $\mathbf{K}$.

Finding smaller equations and their formulas

- Traditional approach: $\Phi_\ell(X, Y)$. Formulas given by Schoof.
- Elkies 1992: *ad hoc* modular equations + formulas for each ℓ; cumbersome.
- Atkin: canonical with η-products, laundry method (conjecturally smallest); magical formulas using differentials of order 1 and 2.
- Müller (Enge): Hecke operators + somewhat *ad hoc* tables. Same Atkin formulas.
- Smallest models for $X_0(\ell)$ in two variables, not related to j; formulas?

Alternative: Charlap-Coley-Robbins (1991).

Enge

ℓ	r	H	$\deg(J)$	eval(s)	interp(s)	tot (d)	Mb gz
3011	5	7560	200				368
3079	97	9018	254	7790	640	23	547
3527	13	9894	268	799	1440	3	746
3517	97	10746	290	12400	1110	42	850
4003	13	11408	308	1130	2320	4	1127
5009	5	13349	334	880	3110	3	1819
6029	5	16418	402	1550	6370	7	3251
7001	5	19473	466	2440	11700	13	5182
8009	5	22515	534	3500	20000	22	7905
9029	5	25507	602	5030	33100	35	11460
10079	5	28825	672	7690	56300	61	16152

III. Charlap-Coley-Robbins

Ref.: `arxiv.org/abs/2303.00346`, `arxiv.org/abs/2302.05217` + Magma files on my webpage.

A) Theory

$$\begin{tikzcd}
\mathbb{Q}(A,B)[X]/(f_\ell(X,A,B)) \arrow[d, dash, "(\ell-1)/2"] \\
\mathbb{Q}(A,B)[X]/(U_\ell(X,A,B)) \arrow[d, dash, "\ell+1"] \\
\mathbb{Q}(A,B)
\end{tikzcd}$$

Using traces

Classical: use the trace T_1 of an element in $\mathbb{Q}(A,B)[X]/(f_\ell(X,A,B))$.
Let $P=(x_1,y_1)\neq O_E$. Other points are $P_j=[j]P=(x_j,y_j)$ can be expressed using division polynomials.
For $0\leq k\leq \ell+1$

$$T_k=\sum_{j=1}^{d} x_j^k=\sum_{j=1}^{d}\left(x_1-\frac{\psi_{j-1}(x_1)\psi_{j+1}(x_1)}{\psi_j(x_1)^2}\right)^k$$

so that $T_1=x_1+\cdots+x_d$ and $T_0=d=(\ell-1)/2$. The minimal polynomial $U_\ell(X)=X^{\ell+1}+u_1X^\ell+\cdots+u_0$ of T_1 defines the lower extension.

Use Newton's identities to reconstruct the factor $\prod_{i=1}^{d}(X-x_i)=X^d-T_1X^{d-1}+\cdots$ over the intermediate extension.

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Use Newton's identities to reconstruct the factor $\prod_{i=1}^{d}(X-x_i)=X^d-T_1X^{d-1}+\cdots$ over the intermediate extension. $\leftarrow$ kernel polynomial!

CCR polynomials

Thm. There exists three polynomials $U_\ell(X,Y,Z)$, $V_\ell(X,Y,Z)$, $W_\ell(X,Y,Z)$ in $\mathbb{Z}[X,Y,Z,1/\ell]$ of degree $\ell+1$ in X such that $U_\ell(\sigma,A,B)=0$, $V_\ell(A^*,A,B)=0$, $W_\ell(B^*,A,B)=0$.

Thm. When $\ell>3$, U_ℓ, V_ℓ, W_ℓ live in $\mathbb{Z}[X,Y,Z]$.

Prop. Assigning respective weights 1, 2, 3 to X, Y, Z, the monomials in U_ℓ, V_ℓ and W_ℓ have generalized degree $\ell+1$.

Ex. $U_5(X,Y,Z)=X^6+20YX^4+160ZX^3-80Y^2X^2-128YZX-80Z^2$.

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The working loop

```
Function UseCCR($E$, $\ell$):
    Input  : $E/\mathbb{F}_q = [A,B]$ an elliptic curve, $\ell$ an odd prime
    Output: $(\sigma, A^*, B^*)$ parameters of a curve $E^*$ that is
            $\ell$-isogenous to $E$
    1. $\mathscr{L}_U \leftarrow$ roots of $U_\ell(X,A,B)$ over $\mathbb{F}_q$
    2. if $\mathscr{L}_U \neq \emptyset$ then
        2.0. Let $\sigma$ be an element of $\mathscr{L}_U$
        2.1. $\mathscr{L}_V \leftarrow$ roots of $V_\ell(X,A,B)$ over $\mathbb{F}_q$
        2.2. $\mathscr{L}_W \leftarrow$ roots of $W_\ell(X,A,B)$ over $\mathbb{F}_q$
        for $v \in \mathscr{L}_V$ do
            for $w \in \mathscr{L}_W$ do
                if $(\sigma, v, w)$ is an $\ell$-isogeny then
                    return $(\sigma, v, w)$.
```

Cost: 3 polynomial exponentiations + ≤ 4 isogeny tests.

B) Computing CCR polynomials

Prop. [multiplier] The function $\mathscr{F}_n = E_2(\tau) - nE_2(n\tau)$ is a modular form of weight 2 and trivial multiplier system for $\Gamma_0(n)$.

Prop. The roots of $U_\ell(X, A(q), B(q))$ are $-\ell\mathscr{F}_\ell(q)/2$ and $\mathscr{F}_\ell(w\zeta_\ell^k)/2$ for $0 \leq k < \ell$, where $w^\ell = q$ and ζ_ℓ is a root of unity.

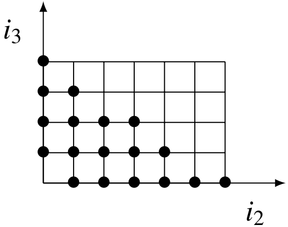
Prop. The height of U_ℓ is approximately $2(\ell+1)\log\ell$.

Setting

$$\begin{aligned} U_\ell(X,Y,Z) &= X^{\ell+1} + \sum_{i_1+2i_2+3i_3=\ell+1} u_{i_1,i_2,i_3} X^{i_1} Y^{i_2} Z^{i_3}, \\ &= X^{\ell+1} + \sum_{r=0}^{\ell} X^r \underbrace{\sum_{2i_2+3i_3=\ell+1-r} u_{r,i_2,i_3} Y^{i_2} Z^{i_3}}_{c_r(Y,Z)}. \end{aligned}$$

Rem. $\#\{i_2, i_3\} = O(\ell - r)$.

Grid for U is $\left((\ell+1)/2, \left\lfloor \frac{\ell+1}{3} \right\rfloor\right)$.



$\Rightarrow$ precompute the whole grid with the $A^{i_2} \cdot B^{i_3}$ (+FFT sharing).

Using series again

$$\sigma_1(q) = -\frac{\ell \mathscr{F}_\ell(q)}{2} = \frac{\ell(\ell-1)}{2} + 12\ell \sum_{n=1}^{\infty} \sigma_1'(n) q^n$$

where $\sigma_1'(n)$ is the sum of the divisors of n prime to ℓ.

$$\sigma_r(q) = \sum_{2i_2+3i_3=r} u_{r,i_2,i_3} A(q)^{i_2} B(q)^{i_3}.$$

→ linear system $\mathscr{S}_r$ (independent of ℓ) which is $O(r) \times O(r)$.

It can be solved with $O(r^\omega)$ operations over $\mathbb{Z}$,

for a total of $O(\sum_r r^\omega) = O(\ell^{\omega+1}) = O(\ell^4)$ generally.

Rem. all these systems may be solved in parallel.

Other algorithms

Evaluation/interpolation: all work for power sums.

Isogeny volcanoes: same.

Purely algebraic approaches

Triangular sets: Schost *et al.*; change of order algorithm.

Noro/Yasuda/Yokoyama (2020):
In particular (representation à la Hecke; aka RUR form):

$$A^* = \frac{N_{\ell,A}(X,A,B)}{U'_\ell(X)}, \quad B^* = \frac{N_{\ell,B}(X,A,B)}{U'_\ell(X)}$$

(Only here: $U'_\ell(X) = \frac{\partial U_\ell}{\partial X}$.)
$N_{\ell,A}$ (resp. $N_{\ell,B}$) are polynomials with integer coefficients and of generalized weight $2\ell+4$ (resp. $2\ell+6$). Computations by any evaluation/interpolation method.

Ex. (with a sign flip)

$$N_{5,A} = 630AX^5 - 9360BX^4 - 8240A^2X^3 + 24480BAX^2$$

$$+(1120A^3 - 28800B^2)X - 3200BA^2.$$

NoYaYo: algorithms

authors Use Gröbner basis: heavy, stopping at $\ell = 89$.

M. We know the expansions of A^*, A, B and U'_ℓ, we really solve

$$U'_\ell(q) \cdot A^*(q) = N_{\ell,A}(\sigma_1(q), A(q), B(q))$$

the way we want (series, floating points, isogeny volcanoes); much less costly, $\ell = 181$ in a couple of hours in Magma.

C) Finding the isogenous curve

Atkin:

We also discuss here the alternative modular equation suggested by (CCR). They use an equation of degree (q+1) in E2*,whose coefficients are forms of appropriate weights expressible in terms of E4 and E6 (or,by applying Wq, in terms of E4q and E6q). In the equivalent of cases 1 and 3 above,they find a value of E2* in GF(p).
The procedure with which they then continue is however intolerably long,and a better continuation is as follows.

> Differentiate their equation twice at the cusp infinity(i.e.with E2*,E4,E6);the first time we get E4q,and the second E6q.

Homogeneous properties of U

Notation:

$$\partial_\sigma = \frac{\partial U}{\partial \sigma}, \partial_4 = \frac{\partial U}{\partial E_4}, \partial_6 = \frac{\partial U}{\partial E_6}, \text{etc..}$$

U is homogeneous with weights, so that (generalized Euler theorem)

$$(\ell+1)U = \sigma\partial_\sigma + 2E_4\partial_4 + 3E_6\partial_6. \tag{4}$$

Note that partial derivatives are also homogeneous:

$$\begin{aligned} \ell\partial_\sigma &= \sigma\partial_{\sigma\sigma} + 2E_4\partial_{\sigma 4} + 3E_6\partial_{\sigma 6}, & (5)\\ (\ell-1)\partial_4 &= \sigma\partial_{\sigma 4} + 2E_4\partial_{44} + 3E_6\partial_{46}, & (6)\\ (\ell-2)\partial_6 &= \sigma\partial_{\sigma 6} + 2E_4\partial_{46} + 3E_6\partial_{66}. & (7)\end{aligned}$$

Getting the isogenous curve (1/4)

Differentiate $U(\sigma, E_4, E_6) = 0$ to get

$$\sigma' \partial_\sigma + E_4' \partial_4 + E_6' \partial_6 = 0, \tag{8}$$

$\sigma = \frac{\ell}{2}\,(\ell \tilde{E}_2 - E_2)$ leading to

$$\sigma' = \frac{\ell}{2}\,(\ell^2 \tilde{E}_2' - E_2') = \frac{\ell}{24}\,(\ell^2(\tilde{E}_2^2 - \tilde{E}_4) - (E_2^2 - E_4)).$$

Replace $\ell \tilde{E}_2$ by $2\sigma/\ell + E_2$ to get

$$\sigma' = \frac{\ell}{24}\,\left(\frac{4\sigma^2}{\ell^2} + \frac{4\sigma}{\ell} E_2 - (\ell^2 \tilde{E}_4 - E_4)\right),$$

that we plug in (8) together with the expressions for ${E_4}'$ and ${E_6}'$ from equation (3) to get a polynomial of degree 1 in E_2 whose coefficient of E_2 is

$$\sigma \partial_\sigma + 2E_4 \partial_4 + 3E_6 \partial_6,$$

which we recognize in (4).

Getting the isogenous curve (2/4)

$$(\ell+1)UE_2+\frac{\ell}{4}\left(4\sigma^2/\ell^2-(\ell^2\tilde{E}_4-E_4)\right)\partial_\sigma-2E_6\partial_6-3E_4^2\partial_4=0 \quad (9)$$

from which we deduce $\tilde{E}_4$ since $U(\sigma,E_4,E_6)=0$.

Finding $\tilde{E}_6$: we differentiate (8)

$$\begin{array}{rl} & \sigma''\partial_\sigma+\sigma'(\sigma'\partial_{\sigma\sigma}+E_4'\partial_{\sigma 4}+E_6'\partial_{\sigma 6}) \\ + & E_4''\partial_4+E_4'(\sigma'\partial_{4\sigma}+E_4'\partial_{44}+E_6'\partial_{46}) \\ + & E_6''\partial_6+E_6'(\sigma'\partial_{6\sigma}+E_4'\partial_{64}+E_6'\partial_{66})=0 \end{array}$$

We compute in sequence

$$12E_2''=2E_2E_2'-E_4'=E_2\,(E_2^2-E_4)/6-(E_2E_4-E_6)/3,$$

$$12\tilde{E}_2''=2\tilde{E}_2\tilde{E}_2'-\tilde{E}_4'=\tilde{E}_2\,(\tilde{E}_2^2-\tilde{E}_4)/6-\,(\tilde{E}_2\tilde{E}_4-\tilde{E}_6)/3,$$

$$\to\sigma''=\frac{\ell}{2}\,(\ell^3\tilde{E}_2''-E_2'')$$

Getting the isogenous curve (3/4)

Differentiate Ramanujan's relations:

$$E_4'' = \frac{1}{3}(E_2'E_4 + E_2E_4' - E_6'), \quad E_6'' = \frac{1}{2}(E_2'E_6 + E_2E_6' - 2E_4E_4'),$$

Finally yields an expression as polynomial in E_2:

$$C_2E_2^2 + C_1E_2 + C_0 = 0.$$

The unknown $\tilde{E}_6$ is to be found in C_0 only.

Prop. (By luck ?) The coefficients C_1 and C_2 vanish for a triplet such that $U_\ell(\sigma, E_4, E_6) = 0$.

Sketch of the proof: Replace $\partial_{\sigma\sigma}$, ∂_{44} and ∂_{66} by their values from (5). Factoring the resulting expressions yields the same factor $\sigma\partial_\sigma + 2E_4\partial_4 + 3E_6\partial_6$, which cancels C_1 and C_2.□

Getting the isogenous curve (4/4)

We are left with

$$\tilde{E}_6 = -\frac{N}{\ell^6 \, \partial_\sigma^3}$$

where N is a polynomial in degree 3 in ℓ

$$N = -E_6 \partial_\sigma^3 \ell^3 + c_2 \ell^2 + 12 \partial_\sigma^2 \sigma (3E_4^2 \partial_6 + 2E_6 \partial_4) \ell - \partial_\sigma^3 \sigma^3.$$

The coefficient c_2 has an ugly expression (that may be simplified??).

The case $\ell \equiv 11 \bmod 12$

Atkin (cont’d):

> The number and size of the terms in their modular equation are also larger than those in mine, especially when q=11(mod 12). In that case, the cuspform eta**2(tau)*eta**2(q*tau) could be used instead of E2* to form the modular equation. This both saves on size and number of coefficients,and has convenient derivatives; the reader can by now easily work out the precise algorithm.

Properties

In this case, Atkin suggests to replace σ with $f(q) = \eta(q)^2 \eta(q^\ell)^2$ another modular form of weight 2.

Ex.

$$CCRA_{11}(X) = X^{12} - 990\,\Delta X^6 + 440\,\Delta E_4 X^4 - 165\,\Delta E_6 X^3$$
$$+22\,\Delta E_4^2 X^2 - \Delta E_4 E_6 X - 11\,\Delta^2,$$

which is sparser $U_{11}(X)$.

CCRA is homogeneous:

$$(\ell+1) CCRA_\ell = f \partial_f + 2E_4 \partial_4 + 3E_6 \partial_6. \tag{10}$$

We have $f^{12} = \Delta(z)\Delta(\ell z)$ and therefore we deduce the discriminant $\tilde{\Delta} = f^{12}/\Delta$, yielding a relation for $\tilde{E}_4$ and $\tilde{E}_6$.

Computing σ

Write

$$\frac{f'}{f} = 2\frac{\eta'}{\eta} + 2\ell\frac{\tilde{\eta}'}{\tilde{\eta}} = \frac{1}{12}(\ell\tilde{E}_2 + E_2),$$

from which we deduce f'.

Starting from $f'\partial_f + E_4'\partial_4 + E_6'\partial_6 = 0$, and replacing by the known values, we find

$$(f\partial_f + 4E_4\partial_4 + 6E_6\partial_6)\,E_2 + f\ell\tilde{E}_2\partial_f - 6E_4^2\partial_6 - 4E_6\partial_4 = 0,$$

which is

$$f\ell\partial_f(\ell\tilde{E}_2 - E_2) - 6E_4^2\partial_6 - 4E_6\partial_4 = 0,$$

which gives us

$$\boxed{\sigma = \frac{\ell\left(3\,\partial_6\,E_4^2 + 2\,\partial_4\,E_6\right)}{f\,\partial_f}.}$$

Computing $\tilde{E}_4$

We differentiate f' to obtain:

$$f'' = \frac{1}{12}\left(f'(\ell\tilde{E}_2+E_2)+f(\ell^2\tilde{E}_2'+E_2')\right)$$
$$= \frac{f}{12^2}\left((\ell\tilde{E}_2+E_2)^2+\ell^2(\tilde{E}_2^2-\tilde{E}_4)+(E_2^2-E_4)\right).$$

We inject this together with $\tilde{E}_2 = (E_2+2\sigma/\ell)/\ell$ into

$$\begin{array}{rl} & f''\partial_f + f'(f'\partial_{ff}+E_4'\partial_{f4}+E_6'\partial_{f6}) \\ + & E_4''\partial_4+E_4'(f'\partial_{4f}+E_4'\partial_{44}+E_6'\partial_{46}) \\ + & E_6''\partial_6+E_6'(f'\partial_{6f}+E_4'\partial_{64}+E_6'\partial_{66}) = 0 \end{array}$$

This yields a polynomial of degree 2 in E_2 whose coefficients of degree 2 and 1 turn out to vanish. We are left with

$$\tilde{E}_4 = -\frac{M}{\ell^2 f^2 E_4 E_6 \partial_f^3}$$

with a bulky expression for M.

Computing $\tilde{E}_6$

Prop. (applying Atkin-Lehner involution)

$$U_\ell(-\ell\sigma, A^*, B^*) = 0, \quad V_\ell(\ell^4 A, A^*, B^*) = 0, \quad W_\ell(\ell^6 B, A^*, B^*) = 0,$$

with $A^* = \ell^4 \tilde{E}_4$, $B^* = \ell^6 \tilde{E}_6$.

Also:

$$\tilde{\Delta} = \frac{\tilde{E}_4^3 - \tilde{E}_6^2}{1728}$$

So that $\tilde{E}_6$ is a root of the gcd of the two polynomials.
In practice, there is one root. Otherwise, use a heavy further differential!!!

V. Conclusions

When is this useful?

- you don't like using Atkin's laundry hammer;
- (technical, rare) when some $\partial_X = 0$, the triplet (U, V, W) is useful;
- for small ℓ, either use sparse formulas $(U_\ell, N_{\ell,A}, D_{\ell,A})$ or only U_ℓ and the ugly formulas.

Working ugly formulas can be done using multipliers for Borweins' like modular polynomials as explained by R. Dupont. But this is another story. . . !