Codes and Modular Curves

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Overview

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Question 2 (And Asymptotically?)

For a sequence $(\mathcal{C}_s)_{s\in\mathbb{N}}$ of $[n_s, k_s, d_s]_q$ codes with $n_s \to +\infty$ and $\frac{k_s}{n_s} \to R$, $\frac{d_s}{n_s} \to \delta$; which pairs (δ, R) are achievable?

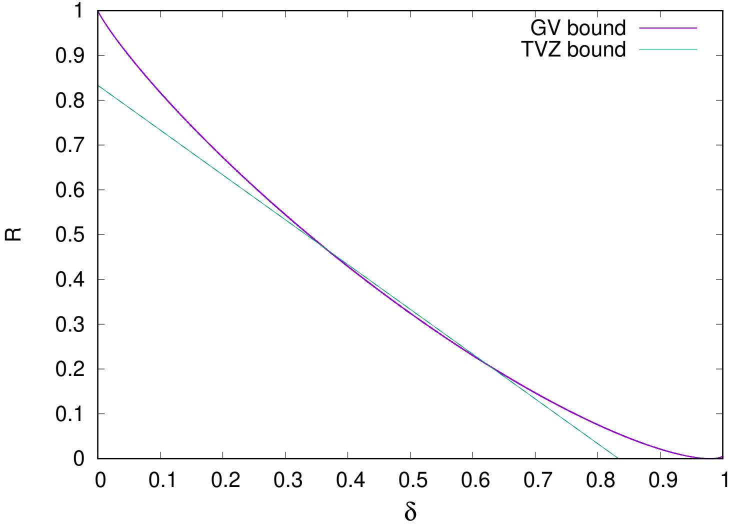
The unbelievable theorem...

Theorem 1 (Tsfasman, Vlăduț, Zink 1982)

Let $q = p^{2m}$ for p prime. There exists a sequence of codes $(\mathcal{C}_s)_s$ over $\mathbb{F}_q$ with parameters $[n_s, k_s, d_s]_q$ such that $R \stackrel{\text{def}}{=} \lim_{s\to+\infty} \frac{k_s}{n_s}, \delta \stackrel{\text{def}}{=} \lim_{s\to+\infty} \frac{d_s}{n_s}$ and

$$R + \delta \geqslant 1 - \frac{1}{\sqrt{q}-1}.$$

The unbelievable picture (for $q = 49$)



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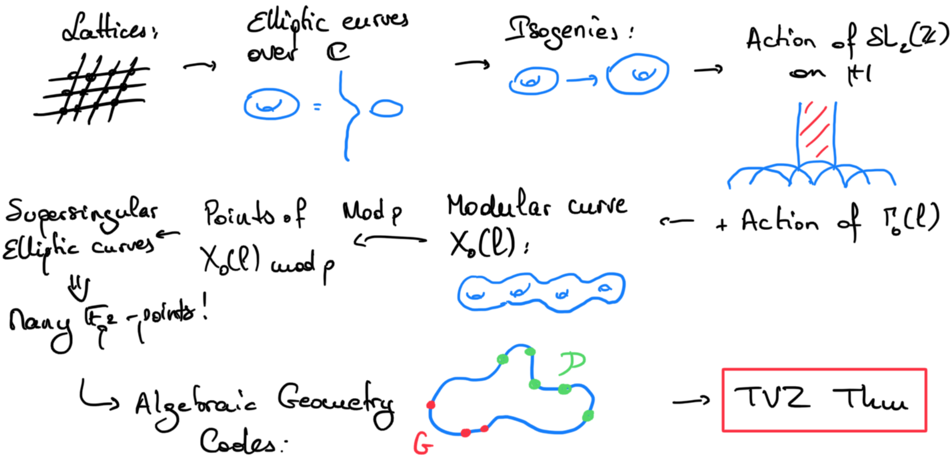

My advice

Let us start

Definition 1

A code is a linear subspace $\mathcal{C} \subseteq \mathbb{F}_q^n$. Its parameters $[n, k, d]_q$ are

- *its* length *n;*
- *its dimension $k \stackrel{\text{def}}{=} \dim_{\mathbb{F}_q} \mathcal{C}$;*
- *its minimum distance $d \stackrel{\text{def}}{=} \min_{\boldsymbol{c} \in \mathcal{C} \setminus \{0\}} \{w_H(\boldsymbol{c})\}$, where $w_H(\cdot)$ denotes the* Hamming weight.

It is well–known that any $[n, k, d]_q$–code satisfies

$$k + d \leqslant n + 1 \qquad \text{(Singleton Bound)}$$

The Gilbert Varshamov bound

Theorem 2

Let n, d be positive integers, then there exists a (possibly nonlinear) code $\mathcal{C} \subseteq \mathbb{F}_q^n$ with minimum distance d such that

$$\sharp\mathcal{C} \cdot \underbrace{\left(\sum_{j=0}^{d-1} \binom{n}{j} (q-1)^j\right)}_{\text{Volume of a ball of radius } d-1} \geqslant q^n.$$

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Asymptotically we get the existence of sequences of codes with parameters (δ, R) such that

$$R = 1 - H_q(\delta),$$

where $H_q(\cdot)$ denotes the q–ary entropy function (see Lecture Notes p. 4)

Facts

1. For a long time, Gilbert Varshamov bound was supposed to be optimal;
2. Actually, at least for large enough q, there is room for improvement.

My advice if you wish to discover algebraic geometry

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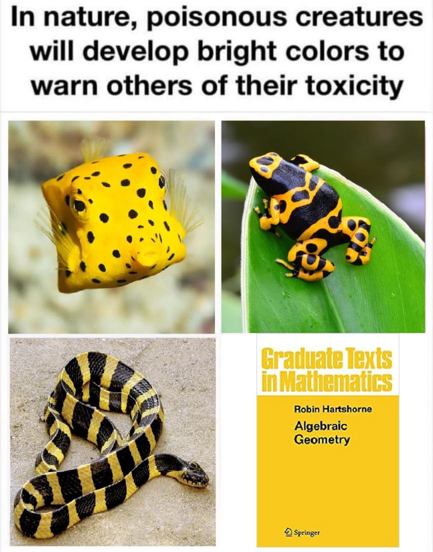


Curves

In the sequel, $\mathbb{K}$ denotes a perfect field.

Definition 2

An affine curve $\mathscr{X} \subseteq \mathbb{A}^2(\overline{\mathbb{K}})$ is the vanishing locus of a polynomial $F \in \mathbb{K}[x, y]$. If F is irreducible over $\overline{\mathbb{K}}$, then the curve is said to be absolutely irreducible.

Definition 3

A rational point *or* $\mathbb{K}$–point *of $\mathscr{X}$ is an element of $\mathscr{X} \subseteq \mathbb{A}^2(\overline{\mathbb{K}})$ whose coordinates lie in $\mathbb{K}$. For any extension $\mathbb{L}/\mathbb{K}$, an $\mathbb{L}$–point is an element of $\mathscr{X}$ whose coordinates lie in $\mathbb{L}$. The set of $\mathbb{K}$– (resp. $\mathbb{L}$–) points is denoted $\mathscr{X}(\mathbb{K})$ (resp. $\mathscr{X}(\mathbb{L})$).*

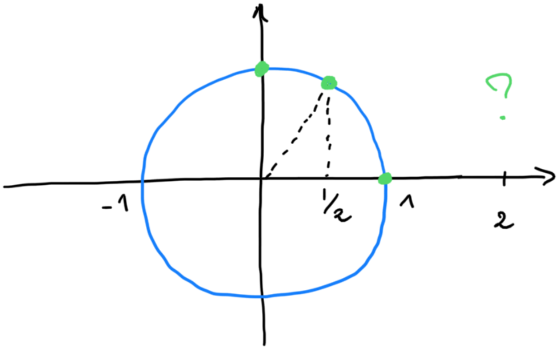
Remark

In particular, $\mathscr{X} = \mathscr{X}(\overline{\mathbb{K}})$. Moreover $\mathscr{X}(\mathbb{K})$ is the subset of $\mathscr{X}$ invariant under the action of Gal$(\overline{\mathbb{K}}/\mathbb{K})$.

Example I

Example 1

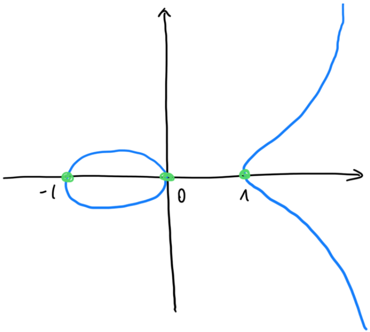
$\mathbb{K} = \mathbb{Q}$ and $F(x, y) = x^2 + y^2 - 1$. The points $(1, 0), (0, 1)$ are $\mathbb{Q}$–points. The point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ is a $\mathbb{Q}(\sqrt{3})$–point. The point $(2, -i\sqrt{3})$ is a $\mathbb{C}$–point (actually also a $\mathbb{Q}(i\sqrt{3})$–point).



Example II

Example 2

$\mathbb{K} = \mathbb{Q}$ and $F(x,y) = y^2 - x(x-1)(x+1)$. The points $(-1,0), (0,0)$ and $(1,0)$ are $\mathbb{Q}$–points.



Singularities

Definition 4

Let $\mathscr{X} \subseteq \mathbb{A}^2(\mathbb{K})$ be a curve defined as the vanishing locus of $F \in \mathbb{K}[x, y]$. A point P of $\mathscr{X}$ is said to be singular *if both $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ vanish at P. A curve with no singular points is said to be* smooth*.*

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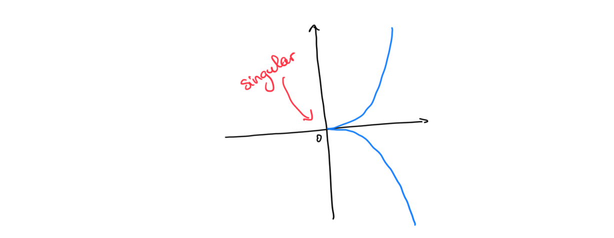
Example 3

The curve of equation $y^2 = x^3$ is singular at $(0, 0)$.



From now on...

From now on, any “curve” is smooth and absolutely irreducible.

Regular and rational functions

Definition 5 (Regular functions)

Let $\mathscr{X} \subseteq \mathbb{A}^2(\mathbb{K})$ *be a curve defined as the vanishing locus of* $F \in \mathbb{K}[x, y]$*. A* regular function *on* $\mathscr{X}$ *is the restriction to* $\mathscr{X}$ *of an element of* $\mathbb{K}[x, y]$*. The ring of regular functions on* $\mathscr{X}$ *is noting but*

$$\mathbb{K}[x, y]/(F).$$

Definition 6 (Rational functions)

A rational function *on* $\mathscr{X}$ *is the restriction to* $\mathscr{X}$ *of an element of* $\mathbb{K}(x, y)$ *whose denominator is prime to* F*. Since* $\mathscr{X}$ *is irreducible (*i.e. F *is irreducible), then the* function field *of* $\mathscr{X}$ *is defined as*

$$\mathbb{K}(\mathscr{X}) \stackrel{\text{def}}{=} \mathit{Frac}\left(\mathbb{K}[x, y]/(F)\right).$$

Regular and rational maps

Definition 7

Let $\mathscr{X}, \mathscr{Y}$ be two curves. A regular *(resp.* rational*)* map *from $\mathscr{X}$ to $\mathscr{Y}$ is a map*

$$\phi : \left\{ \begin{array}{ccc} \mathscr{X} & \longrightarrow & \mathscr{Y} \\ (x,y) & \longmapsto & (\phi_1(x,y), \phi_2(x,y)) \end{array} \right.$$

where ϕ_1, ϕ_2 are regular *(resp.* rational*) functions on $\mathscr{X}$.*
If there is $\psi : \mathscr{Y} \to \mathscr{X}$ such that $\psi \circ \phi = Id_{\mathscr{X}}$ and $\phi \circ \psi = Id_{\mathscr{Y}}$, then ϕ is said to be an isomorphism *(resp.* a birational map*).*

Remark

Such a function might be defined only on $\mathscr{X}$ minus a finite number of points.

Example

Example 4

The affine line $\mathbb{A}^1$ and the circle $\mathscr{C}$ of equation $x^2 + y^2 = 1$ are birational to each other via the map:

$$\left\{ \begin{array}{ccc} \mathbb{A}^1 & \longrightarrow & \mathscr{C} \\ t & \longmapsto & \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right) \end{array} \right.$$

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Remark

The map is undefined at $\{\pm i\}$.

Link with function fields

Proposition 1

Given a rational map $\phi : \mathscr{X} \to \mathscr{Y}$, there is a field extension $\phi^\star : \mathbb{K}(\mathscr{Y}) \hookrightarrow \mathbb{K}(\mathscr{X})$ given by

$$\phi^\star f \stackrel{\text{def}}{=} f \circ \phi.$$

Divisors

Definition 8

On a curve $\mathscr{X}$ over $\mathbb{K}$, a rational divisor is a finite formal sum $\sum a_P P$ of $\overline{\mathbb{K}}$–points which is globally invariant under the action of Gal$(\overline{\mathbb{K}}/\mathbb{K})$. The degree *of a divisor $\sum a_P P$ is the integer $\sum a_P$.*

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Example 5

On $\mathscr{C}$ of equation $x^2 + y^2 = 1$, the following object is a rational divisor:

$$3 \cdot (1,0) - 2 \cdot (0,1) + 4(2, -i\sqrt{3}) + 4(2, i\sqrt{3})$$

and that one is not:

$$3 \cdot (1,0) - 2 \cdot (0,1) + 4\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) + 2\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

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Valuations

Definition 9 (Informal)

Let $\mathscr{X}$ be a curve. To any point $P \in \mathscr{X}(\overline{\mathbb{K}})$ we associate a map $\nu_P : \overline{\mathbb{K}}(\mathscr{X})^\times \to \mathbb{Z}$ called the valuation at P *describing the order of pole or zero of any function on $\mathscr{X}$. A* local parameter at P *is a function of valuation 1 at P.*

Principal divisors

Definition 10

To any function $f \in \mathbb{K}(\mathscr{X})^{\times}$ *one associates a divisor called* principal divisor *and defined as*

$$(f) = \sum_{P \in \mathscr{X}(\overline{\mathbb{K}})} \nu_P(f) \cdot P.$$

Proposition 2

The degree of a principal divisor is always 0.

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Example 6

On the blackboard.

Riemann–Roch spaces

Definition 11

The divisor group on $\mathscr{X}$ *is endowed with a partial order. Let* $A = \sum a_P P$ *and* $B = \sum b_P P$,

$$A \geqslant B \iff \forall P \in \mathscr{X}(\overline{\mathbb{K}}), a_P \geqslant b_P$$

Definition 12

Let $\mathscr{X}$ *be a curve over* $\mathbb{K}$ *and* G *be a divisor on* $\mathscr{X}$, *we define the space*

$$L(G) \stackrel{\text{def}}{=} \left\{ f \in \mathbb{K}(\mathscr{X})^\times \mid (f) + G \geqslant 0 \right\} \cup \{0\},$$

Properties of Riemann–Roch spaces

Proposition 3

- *If* $G < 0$, *then* $L(G) = \{0\}$
- *For any* G, *the space* $L(G)$ *has finite dimension and* $\dim L(G) \leqslant \deg G + 1$.

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Definition 13 (Genus)

The genus of a curve $\mathscr{X}$ *is defined as*

$$g \stackrel{\textbf{\textit{def}}}{=} 1 - \min_{D}\{\dim L(D) - \deg D\}.$$

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Theorem 3 (Riemann–Roch)

Let G be a divisor on a curve, then

$$\dim L(G) \geqslant \deg G + 1 - g \qquad \textit{with equality if} \quad \deg G > 2g - 2.$$

Construction

Definition 14 (Goppa 1981, Vlăduț–Manin 1984)

Let $\mathscr{X}$ be a curve over a finite field $\mathbb{F}_q$, G be a divisor on $\mathscr{X}$ and $\mathcal{P} = (P_1, \ldots, P_n)$ an n–tuple of distinct rational points of $\mathscr{X}$. We define

$$\mathcal{C}_L(\mathscr{X}, \mathcal{P}, G) \stackrel{\textbf{def}}{=} \{(f(P_1), \ldots, f(P_n)) \mid f \in L(G)\}.$$

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Theorem 4

The code $\mathcal{C}_L(\mathscr{X}, \mathcal{P}, G)$ as parameters

$$\begin{aligned} k &\geqslant \deg G + 1 - g \\ d &\geqslant n - \deg G. \end{aligned}$$

How to provide excellent codes from curves?

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$$n + 1 - g \leqslant k + d \qquad (\leqslant n + 1, \quad \text{By Singleton bound})$$

Definition 15 (Ihara constant)

$$A(q) \stackrel{\textit{def}}{=} \limsup_{g \to +\infty} \left(\max_{\mathscr{X} \text{ of genus } g} \frac{\sharp X(\mathbb{F}_q)}{g} \right)$$

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Corollary 1

There exists a sequence of codes $(\mathcal{C}_s)_s$ over $\mathbb{F}_q$ with parameters $[n_s, k_s, d_s]_q$ such that $R \stackrel{\textbf{def}}{=} \lim_{s\to+\infty} \frac{k_s}{n_s}, \delta \stackrel{\textbf{def}}{=} \lim_{s\to+\infty} \frac{d_s}{n_s}$ and

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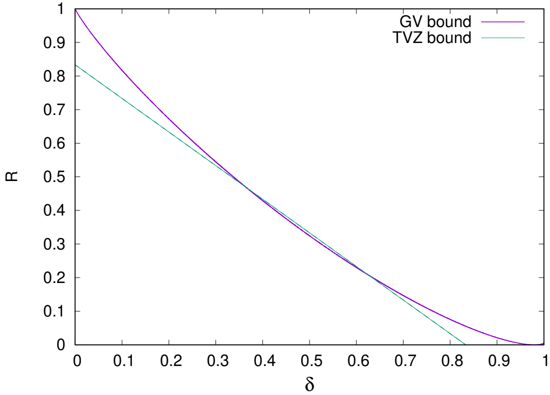
$$R + \delta \geqslant 1 - \frac{1}{\sqrt{q} - 1}.$$

Theorem 6 (Drinfeld Vlăduț 1894)

For any prime power q, $A(q) \leqslant \sqrt{q} - 1$. i.e. *TVZ is optimal.*

How to get this?



Goal. Exhibit a family of curves $(\mathscr{X}_s)_{s\in\mathbb{N}}$ over $\mathbb{F}_q$ such that

$$\limsup_{s\to+\infty} \frac{\sharp X_s(\mathbb{F}_q)}{g(\mathscr{X}_s)} = \sqrt{q} - 1.$$

Elliptic curves

Definition 16

Let $\mathbb{K}$ *be a field of characteristic* $\neq 2, 3$. *An* elliptic curve $\mathscr{E}$ *over* $\mathbb{K}$ *is a genus one curve with one rationa point. Such a curve can be represented with an equation* $y^2 = f(x)$ *where* $f \in \mathbb{K}[x]$ *is squarefree of degree* 3. *In addition, it can be put in* Weierstrass *form*

$$y^2 = x^3 + Ax + B \qquad \textit{for some } A, B \in \mathbb{K}.$$

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Remark

Weierstrass form is not unique. For instance, a change of variables:

$$\begin{array}{rcl} x & \mapsto & u^2 x \\ y & \mapsto & u^3 y \end{array}$$

for some $u \in \mathbb{K}^\times$ provides an equation of an isomorphic curve with Weierstrass equation:

$$y^2 = x^3 + Au^{-4}x + u^{-6}B.$$

The j–invariant

Definition 17

Let $\mathscr{E}$ be an elliptic curve with Weierstrass equation: $y^2 = x^3 + Ax + B$. The j–invariant of $\mathscr{E}$ is defined as

$$j \stackrel{\text{def}}{=} 1728 \frac{4A^3}{4A^3 + 27B^2}.$$

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Two elliptic curves are isomorphic over $\overline{\mathbb{K}}$ if and only if they have the same j–invariant. Conversely, for any $j_0 \in \overline{\mathbb{K}}$ there is an elliptic curve defined over $\mathbb{K}(j_0)$ with j–invariant j_0.

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$\overline{\mathbb{K}}$–isomorphism classes of elliptic curves are parameterised by $\mathbb{A}^1(\overline{\mathbb{K}})$.

The group law

Theorem 8

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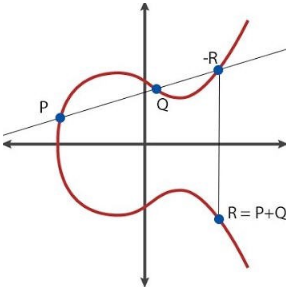

The group structure is inherited from that of $\mathrm{Pic}^0(\mathscr{E}) \simeq \mathrm{Div}^0_{\mathbb{K}}(\mathscr{E})/\mathrm{Princ}(\mathscr{E})$.

Torsion

Definition 7

Given an elliptic curve $\mathscr{E}$ over $\mathbb{K}$ and an integer $m > 1$, the m–torsion of $\mathscr{E}$ is defined as

$$\mathscr{E}[m] \stackrel{\text{def}}{=} \{P \in \mathscr{E}(\overline{\mathbb{K}}) \mid mP = 0\}.$$

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$\mathscr{E}[m]$ is not necessarily composed of rational points but is **globally** *stable under the action of $Gal(\overline{\mathbb{K}}/\mathbb{K})$.*

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Proposition 4

For an elliptic curve $\mathscr{E}$ and $m \in \mathbb{N}$

$$\mathscr{E}[m] \simeq \begin{cases} \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} & \text{if } m \text{ is prime to } Char(\mathbb{K}); \\ 0 \text{ or } \mathbb{Z}/m\mathbb{Z} & \text{otherwise} \end{cases}$$

Isogenies

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An isogeny $\phi : \mathscr{E} \to \mathscr{E}'$ induces a group morphism. Moreover, when ϕ is separable, then

$$\sharp ker \phi = \deg \phi$$

Conversely, for any finite subgroup $K \subseteq \mathscr{E}(\overline{\mathbb{K}})$ which is globally invariant under Gal$(\overline{\mathbb{K}}/K)$, there exists an isogeny $\psi : \mathscr{E} \to \mathscr{E}'$ defined over $\mathbb{K}$ with kernel K. We denote

$$\mathscr{E}' \stackrel{\text{def}}{=} \mathscr{E}/K.$$

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Proposition 6

For any degree m isogeny $\phi : \mathscr{E} \to \mathscr{E}'$, there exists a unique $\hat{\phi} : \mathscr{E}' \to \mathscr{E}$ such that

$$\hat{\phi} \circ \phi = [m]_{\mathscr{E}} : \left\{ \begin{array}{ccc} \mathscr{E} & \longrightarrow & \mathscr{E} \\ P & \longmapsto & mP \end{array} \right. \quad \text{and} \quad \phi \circ \hat{\phi} = [m]_{\mathscr{E}'} : \left\{ \begin{array}{ccc} \mathscr{E}' & \longrightarrow & \mathscr{E}' \\ P & \longmapsto & mP. \end{array} \right.$$

The idea behind dual isogenies

- On a curve $\mathscr{E}$, the map $P \mapsto mP$ induces an isomorphism $\mathscr{E} \xrightarrow{\sim} \mathscr{E}/\mathscr{E}[m]$;
- Let $K \subseteq \mathscr{E}[m]$ of cadinality m and $\phi : \mathscr{E} \to \mathscr{E}' = \mathscr{E}/K$ the corresponding isogeny;
- The dual isogeny is $\hat{\phi} : \mathscr{E}' \to \mathscr{E}'/\phi(\mathscr{E}[m]) \simeq \mathscr{E}'/(\mathscr{E}[m]/K) \simeq \mathscr{E}$

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$$\begin{tikzcd} \mathscr{E} \arrow[r, "\phi"] \arrow[rr, bend right, "P \mapsto mP"'] & \mathscr{E}' \arrow[r, "\hat{\phi}"] & \mathscr{E} \end{tikzcd}$$

Lattices and Elliptic curves over $\mathbb{C}$

Theorem 9

Let $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$. Then, the quotient $\mathbb{C}/\Lambda$ is biholomorphic to a complex elliptic curve. Conversely, for any elliptic curve $\mathscr{E}$ over $\mathbb{C}$, there exists a lattice $\Lambda \subseteq \mathbb{C}$ such that $\mathscr{E}$ is biholomorphic to $\mathbb{C}/\Lambda$.

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Sketch of proof for $\mathbb{C}/\Lambda \to \mathscr{E}$.

The connection is made by the *Weierstrass $\wp_\Lambda$–function*:

$$\forall z \in \mathbb{C} \setminus \Lambda, \quad \wp_\Lambda(z) \stackrel{\text{def}}{=} \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

together with the map $\left\{ \begin{array}{ccc} \mathbb{C}/\Lambda & \longrightarrow & \mathbb{A}^2 \\ z & \longmapsto & (\wp_\Lambda(z), \wp'_\Lambda(z)) \end{array} \right.$ (Λ being sent onto $O_{\mathscr{E}}$). $\square$

Lattices and Elliptic curves over $\mathbb{C}$

Sketch of proof for $\mathbb{C}/\Lambda \to \mathscr{E}$.

The series $\wp_\Lambda(z) - \frac{1}{z^2} = \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$ is even an vanishes at 0

$$\wp_\Lambda(z) = \frac{1}{z^2} + O(z^2)$$

$\square$

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\wp'_\Lambda(z)^2 - 4\wp_\Lambda(z)^3 - g_2\wp_\Lambda(z) &= O(1) \quad \text{(for some } g_2 \text{ in } \mathbb{C})
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Liouville $\implies$ $\wp'_\Lambda(z)^2 = 4\wp_\Lambda(z)^3 + g_2\wp_\Lambda(z) + g_4$ for some $g_2, g_4 \in \mathbb{C}$. □

Things happen well!

Theorem 10

For any lattice $\Lambda \subset \mathbb{C}$, The biholomorphic map $\mathbb{C}/\Lambda \xrightarrow{\sim} \mathscr{E}$ is a group isomorphism from $(\mathbb{C}/\Lambda, +)$ to $(\mathscr{E}, +_{\mathscr{E}})$.

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Remark

For any $m > 0$, the structure of $\mathscr{E}[m]$ can be understood from that of $\left(\frac{1}{m}\Lambda\right)/\Lambda$

Isogenies regarded from tori

Theorem 11

Let $\Lambda, \Lambda' \subset \mathbb{C}$ *be two lattices and* $f : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ *sending* 0 *to* 0*. Then,* f *lifts to a holomorphic ma* $f_0 : \mathbb{C} \to \mathbb{C}$ *which is a similitude,* i.e. *there exists* $a \in \mathbb{C}$ *such that* $\forall z \in \mathbb{C}$*,* $f_0(z) = az$*.*

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Remark

(Up to some similitude) For ℓ *prime, a degree* ℓ*–isogeny between two tori corresponds to the data of two lattices* Λ, Λ' *such that* $\Lambda \subseteq \Lambda'$ *and* $\sharp(\Lambda'/\Lambda) = \ell$.

Complex elliptic curves with nontrival automorphisms

Theorem 12

Let $\mathbb{C}/\Lambda$ be a complex torus with an automorphism $a \mapsto az$ and $|a| = 1$, $a \neq \pm 1$. Then, up to a similitude, Λ equals either $\mathbb{Z} \oplus \mathbb{Z}i$ or $\mathbb{Z} \oplus \mathbb{Z}\rho$, where $\rho = e^{\frac{2i\pi}{6}}$

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The corresponding elliptic curves can be proved to have respective equations:

$$
\begin{array}{rclcrcll}
y^2 & = & x^3 + x & \text{for} & \Lambda & = & \mathbb{Z} \oplus \mathbb{Z}i & (j\text{–invariant } 1728) \\
y^2 & = & x^3 + 1 & \text{for} & \Lambda & = & \mathbb{Z} \oplus \mathbb{Z}\rho & (j\text{–invariant } 0).
\end{array}
$$

The corresponding automorphisms being respectively

$$
\begin{array}{rcl}
(x, y) & \longmapsto & (-x, iy) \\
(x, y) & \longmapsto & (\rho x, -y).
\end{array}
$$

The Poincaré upper half plane

Question 3

How to classify complex elliptic curves up to isogeny? Equivalently, how to classify lattices up to similitude?

The Poincaré upper half plane

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- Start with a basis $\Lambda = \mathbb{Z}\omega_2 \oplus \mathbb{Z}\omega_1$;
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- for any $A \in \mathrm{SL}_2(\mathbb{Z})$,

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} \stackrel{\text{def}}{=} A \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

  is another direct basis of the **same lattice**.

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is another direct basis of the **same lattice**.
- Rescale by the similitude $z \mapsto \frac{z}{\omega_1}$ to get $\mathbb{Z} \oplus \mathbb{Z}\tau$, where $\tau \stackrel{\textbf{def}}{=} \frac{\omega_2}{\omega_1}$ is in the open upper half plane ($\mathrm{Im}(\tau) > 0$).

The poincaré upper half plane

Summary: lattices are classified by elements $\tau \in \mathbb{H}$ up to this action of $\mathsf{SL}_2(\mathbb{Z})$:

$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}), \quad A \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

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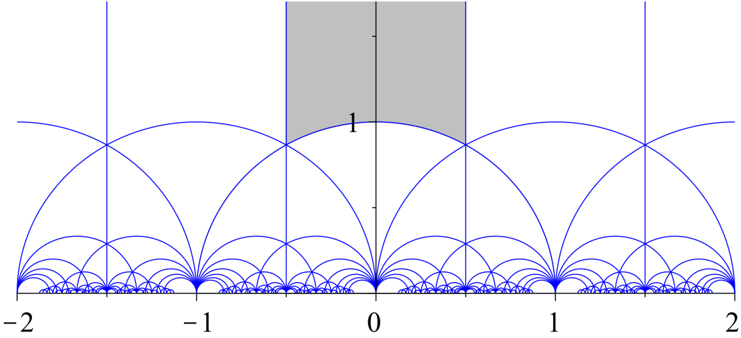

In summary

Elliptic curves up to isomorphism	$\longleftrightarrow$	Complex tori up to biholomorphic isomorphisms	$\longleftrightarrow$	Lattices of $\mathbb{C}$ up to similitudes	$\longleftrightarrow$	Points of $\mathbb{H}$ modulo the action of $\mathrm{SL}_2(\mathbb{Z})$

The curve $X_0(1)$

Theorem 13

The Riemann surface $Y_0(1) \overset{\textit{def}}{=} \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ *is biholomorphic to* $\mathbb{A}^1$ *it can be made explicit via the map* $\tau \mapsto \mathscr{E}_\tau \mapsto j(\mathscr{E}_\tau)$. *It can be compactified as*

$$X_0(1) \overset{\textit{def}}{=} \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^* \quad \textit{where} \quad \mathbb{H}^* \overset{\textit{def}}{=} \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}).$$

An the compactification is the Riemann sphere : $X_0(1) \simeq \mathbb{P}^1(\mathbb{C})$.

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Remark

$\mathrm{SL}_2(\mathbb{Z})$ *acts transitively on* $\mathbb{P}^1(\mathbb{Q})$ *as* $x \mapsto \frac{ax+b}{cx+d}$.

The curve $X_0(\ell)$

Question 4

Let ℓ be a prime number. How to classify degree ℓ isogenies $\phi : \mathscr{E} \to \mathscr{E}'$ up to isomorphism?

Remark

"up to ismomorphism" means that two isogenies $\phi_1 : \mathscr{E}_1 \to \mathscr{E}_1'$ and $\phi_2 : \mathscr{E}_2 \to \mathscr{E}_2'$ are isomorphic if there exist two isomorphisms $\eta : \mathscr{E}_1 \to \mathscr{E}_2$ and $\nu : \mathscr{E}_1' \to \mathscr{E}_2'$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathscr{E}_1 \arrow[r, "\phi_1"] \arrow[d, "\eta"'] & \mathscr{E}_2 \arrow[d, "\nu"] \\
\mathscr{E}_1' \arrow[r, "\phi_2"] & \mathscr{E}_2'
\end{tikzcd}$$

The curve $X_0(\ell)$

This leads to some "enhanced" version of $X_0(1)$ which is

$$X_0(\ell) \stackrel{\text{def}}{=} \Gamma_0(\ell)\backslash\mathbb{H}^*,$$

where

$$\Gamma_0(\ell) \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \mod \ell \right\}.$$

The modular equation

Actually, modular curves are algebraic!

Theorem 8

There exists an irreducible polynomial $\Phi_\ell \in \mathbb{Z}[x, y]$ such that for any pair $\mathscr{E}, \mathscr{E}'$ of elliptic curves related with a degree ℓ isogeny $\mathscr{E} \to \mathscr{E}'$, then $\Phi_\ell(j(\mathscr{E}), j(\mathscr{E}')) = 0$.

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Unfortunately, such a plane representation of $X_0(\ell)$ is highly singular...

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Remark

Unfortunately, such a plane representation of $X_0(\ell)$ is highly singular...

But... reduction modulo p makes sense.

The genus of $X_0(\ell)$

Theorem 9

For a prime number $\ell > 3$, the genus g_ℓ of $X_0(\ell)$ equals

$$g_\ell = \begin{cases} \frac{\ell-1}{12} - 1 & \text{if } \quad \ell \equiv 1 \mod [12] \\ \frac{\ell-5}{12} & \text{if } \quad \ell \equiv 5 \mod [12] \\ \frac{\ell-7}{12} & \text{if } \quad \ell \equiv 7 \mod [12] \\ \frac{\ell+1}{12} & \text{if } \quad \ell \equiv 11 \mod [12]. \end{cases}$$

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The proof rests on the following well–known statement.

Theorem 14 (Riemann–Hurwitz formula (tame version))

Let $\phi : \mathscr{X} \to \mathscr{Y}$ be a rational map between two curves over $\mathbb{K}$ of characteristic 0. Then, the genera $g_{\mathscr{X}}, g_{\mathscr{Y}}$ of $\mathscr{X}, \mathscr{Y}$ are related by the following formula.

$$(2g_{\mathscr{X}} - 2) = \deg \varphi \cdot (2g_{\mathscr{Y}} - 2) + \sum_{Q \in \mathscr{Y}(\overline{\mathbb{K}})} (e_Q - 1).$$

About Riemann–Hurwitz

Sketch of proof 1/2

Sketch of proof 2/2

Supersingular elliptic curves

Theorem 10

Let $\mathscr{E}$ be an elliptic curve over $\mathbb{F}_q$ (of characteristic p), then

$$\textit{either} \quad \mathscr{E}[p] \simeq \mathbb{Z}/p\mathbb{Z} \quad \textit{or} \quad \mathscr{E}[p] = \{0\}$$

In the latter case the curve is said to be supersingular.

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Theorem 11

A supersingular curve defined over some extension of $\mathbb{F}_p$ is actually always defined over $\mathbb{F}_{p^2}$ and the number of their $\overline{\mathbb{F}}_p$–isomorphism classes is

$$\left\lfloor \frac{p}{12} \right\rfloor + \begin{cases} 0 & \text{if} \quad p \equiv 1 \mod 12 \\ 1 & \text{if} \quad p \equiv 5 \mod 12 \\ 1 & \text{if} \quad p \equiv 7 \mod 12 \\ 2 & \text{if} \quad p \equiv 11 \mod 12. \end{cases}$$

Why are supersingular curves always defined over $\mathbb{F}_{p^2}$?

The main theorem

Theorem 15

The sequence of curves $\mathscr{X}_0(\ell)$ *over* $\mathbb{F}_{p^2}$ *for* $\ell \equiv 11 \mod 12$ *satisfy*

$$\lim_{\ell \to +\infty} \frac{\sharp X_0(\ell)(\mathbb{F}_{p^2})}{g_\ell} = p - 1.$$

Sketch of proof 1/2

Sketch of proof 2/2

What else?

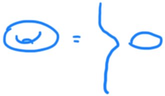
There are other approaches to provide good sequences of curves

- **Still in the modular world**: Shimira curves, Drinfeld modular curves;
- Recursive towers like Garcia Sticthenoth towers;
- Class field towers.

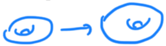
That's all, thank you!

Lattices: →

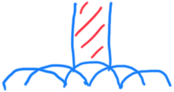
Elliptic curves over $\mathbb{C}$ →



Isogenies: →



Action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$



Supersingular Elliptic curves ← Points of $X_0(\ell)$ mod p ← Mod p ← Modular curve $X_0(\ell)$: ← + Action of $\Gamma_0(\ell)$

$\Downarrow$

Many $\mathbb{F}_{p^2}$-points!

↳ Algebraic Geometry Codes:

→ $\boxed{\text{TVZ Thm}}$