ZERO-RATE THRESHOLDS & NEW CAPACITY BOUNDS FOR LIST-DECODING AND LIST-RECOVERY

Nicolas Resch
CWI $\rightarrow$ University of Amsterdam

Chen Yuan
Shanghai Jiao Tong University

Yihan Zhang
Institute of Science and Technology Austria

ZERO-RATE THRESHOLDS & NEW CAPACITY BOUNDS FOR LIST-DECODING ~~AND LIST-RECOVERY~~

Nicolas Resch

CWI → University of Amsterdam

Chen Yuan

Shanghai Jiao Tong University

Yihan Zhang

Institute of Science and Technology Austria

LIST-DECODABILITY

KEY DEFINITIONS

KEY DEFINITIONS

- A *code* is a subset $\mathscr{C} \subseteq [q]^n$ with $[q] = \{0,1,\ldots,q-1\}$

KEY DEFINITIONS

- A ***code*** is a subset $\mathscr{C} \subseteq [q]^n$ with $[q] = \{0,1,\ldots,q-1\}$
- ***Rate*** is $R := \log_q(|\mathscr{C}|)/n$ (so $|\mathscr{C}| = q^{Rn}$): measures ***efficiency***

KEY DEFINITIONS

- A *code* is a subset $\mathscr{C} \subseteq [q]^n$ with $[q] = \{0,1,\ldots,q-1\}$
- *Rate* is $R := \log_q(|\mathscr{C}|)/n$ (so $|\mathscr{C}| = q^{Rn}$): measures *efficiency*
- *Distance* is $\delta := \min\{d(\mathbf{c},\mathbf{c}') : \mathbf{c} \neq \overrightarrow{\mathbf{c}}' \in \mathscr{C}\}$: measures *error-resilience*

KEY DEFINITIONS

- A *code* is a subset $\mathscr{C} \subseteq [q]^n$ with $[q] = \{0,1,\ldots,q-1\}$
- *Rate* is $R := \log_q(|\mathscr{C}|)/n$ (so $|\mathscr{C}| = q^{Rn}$): measures *efficiency*
- *Distance* is $\delta := \min\{d(\mathbf{c},\mathbf{c}') : \mathbf{c} \neq \overrightarrow{\mathbf{c}}' \in \mathscr{C}\}$: measures *error-resilience*
  - Here, $d(\mathbf{x},\mathbf{y}) = |\{i \in [n] : x_i \neq y_i\}|/n$ is *(Hamming) distance*

KEY DEFINITIONS

- A *code* is a subset $\mathscr{C} \subseteq [q]^n$ with $[q] = \{0,1,\ldots,q-1\}$
- *Rate* is $R := \log_q(|\mathscr{C}|)/n$ (so $|\mathscr{C}| = q^{Rn}$): measures *efficiency*
- *Distance* is $\delta := \min\{d(\mathbf{c},\mathbf{c}') : \mathbf{c} \neq \overrightarrow{\mathbf{c}}' \in \mathscr{C}\}$: measures *error-resilience*
  - Here, $d(\mathbf{x},\mathbf{y}) = |\{i \in [n] : x_i \neq y_i\}|/n$ is *(Hamming) distance*
  - Let $\mathrm{wt}(\mathbf{x}) = d(\mathbf{x},\mathbf{0})$ be its *weight*

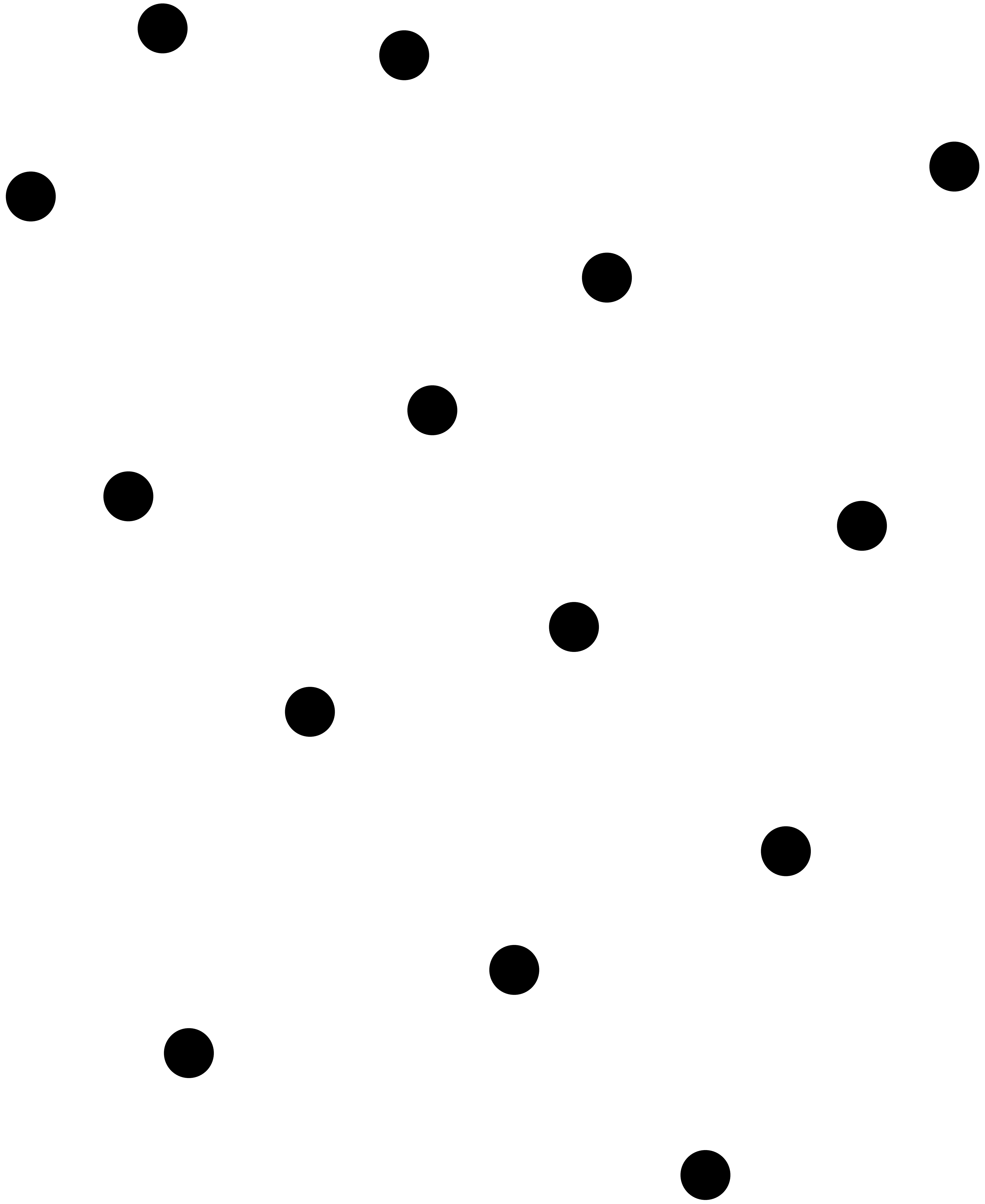
KEY DEFINITIONS

- A *code* is a subset $\mathscr{C} \subseteq [q]^n$ with $[q] = \{0, 1, \ldots, q-1\}$
- *Rate* is $R := \log_q(|\mathscr{C}|)/n$ (so $|\mathscr{C}| = q^{Rn}$): measures *efficiency*
- *Distance* is $\delta := \min\{d(\mathbf{c}, \mathbf{c}') : \mathbf{c} \neq \overrightarrow{\mathbf{c}}' \in \mathscr{C}\}$: measures *error-resilience*
  - Here, $d(\mathbf{x}, \mathbf{y}) = |\{i \in [n] : x_i \neq y_i\}|/n$ is *(Hamming) distance*
  - Let $\mathrm{wt}(\mathbf{x}) = d(\mathbf{x}, \mathbf{0})$ be its *weight*
- **Goal:** design codes with large distance and large rate

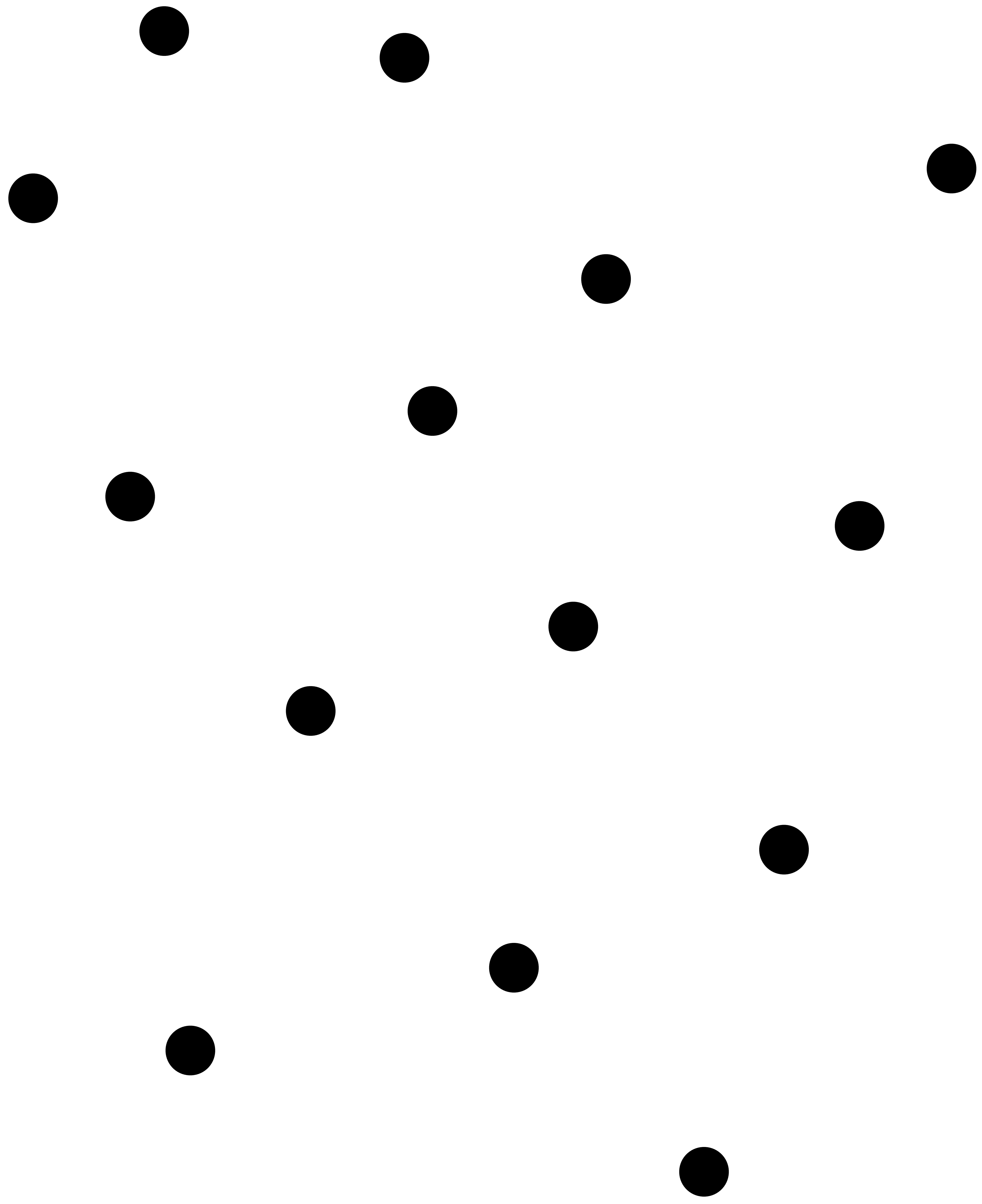
KEY DEFINITIONS

- A *code* is a subset $\mathscr{C} \subseteq [q]^n$ with $[q] = \{0,1,\ldots,q-1\}$
- *Rate* is $R := \log_q(|\mathscr{C}|)/n$ (so $|\mathscr{C}| = q^{Rn}$): measures *efficiency*
- *Distance* is $\delta := \min\{d(\mathbf{c},\mathbf{c}') : \mathbf{c} \neq \vec{\mathbf{c}}' \in \mathscr{C}\}$: measures *error-resilience*
  - Here, $d(\mathbf{x},\mathbf{y}) = |\{i \in [n] : x_i \neq y_i\}|/n$ is *(Hamming) distance*
  - Let $\mathrm{wt}(\mathbf{x}) = d(\mathbf{x},\mathbf{0})$ be its *weight*
- Goal: design codes with large distance and large rate
- However, these desiderata are at odds

RATE-DISTANCE TRADEOFF

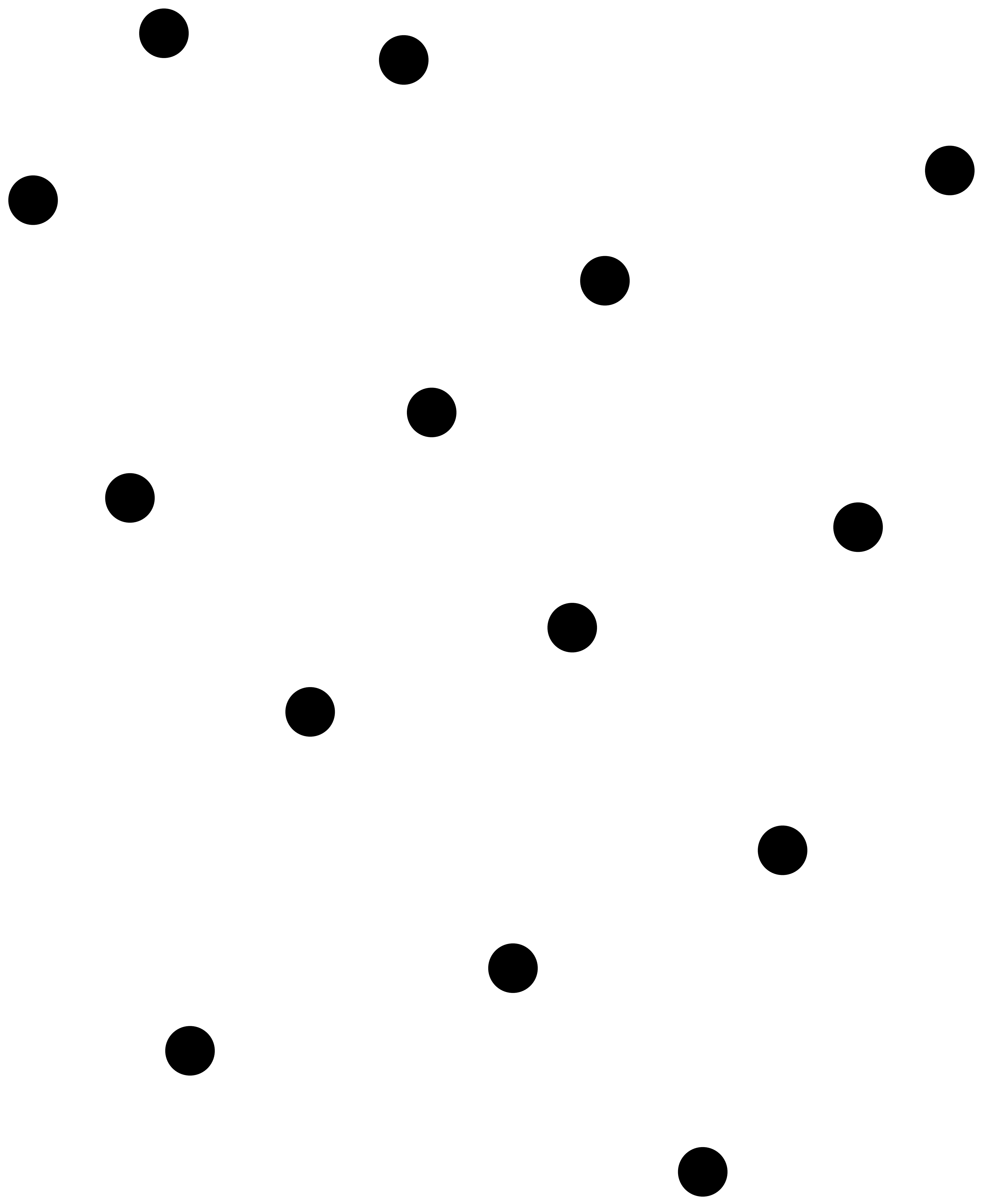


RATE-DISTANCE TRADEOFF



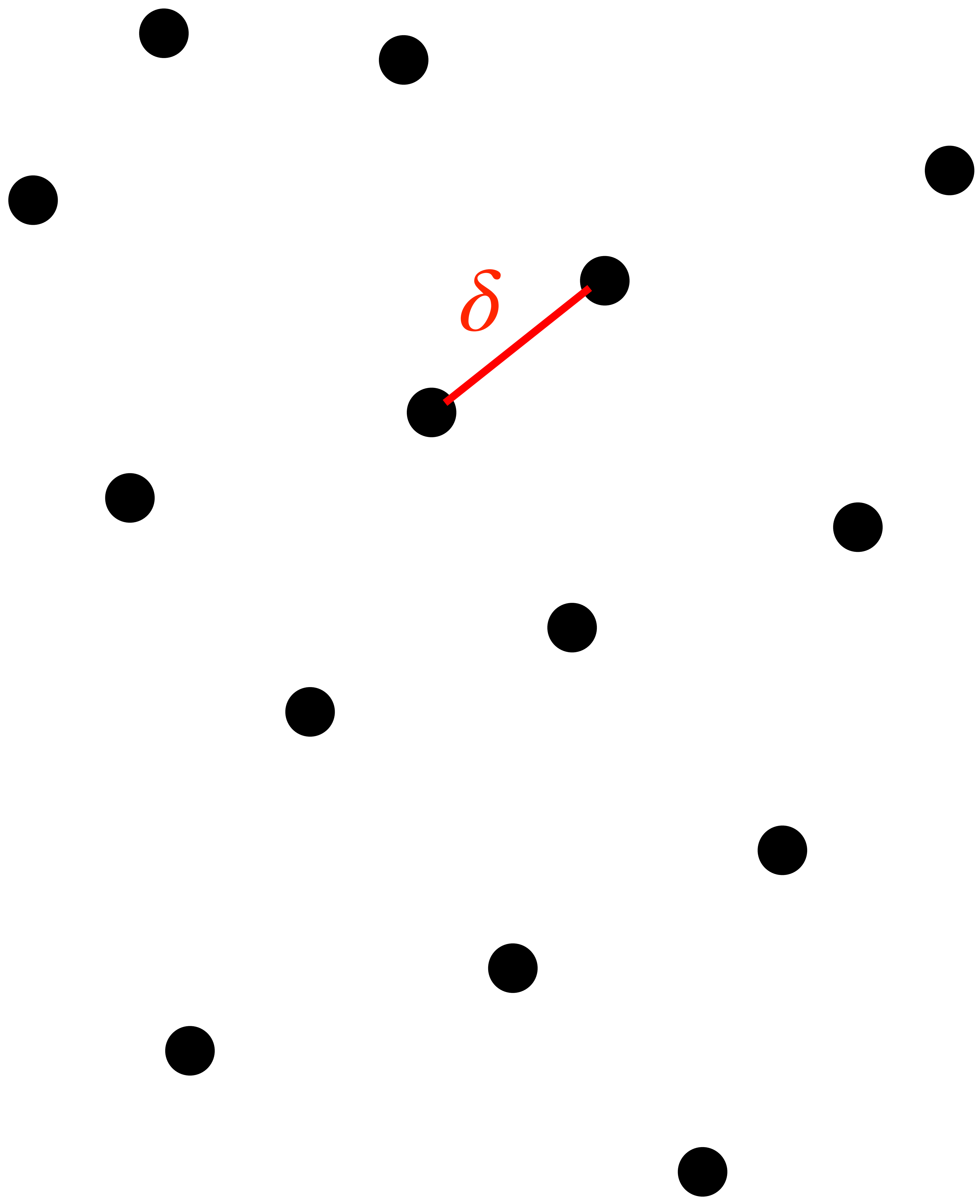
- Larger rate: more points

RATE-DISTANCE TRADEOFF



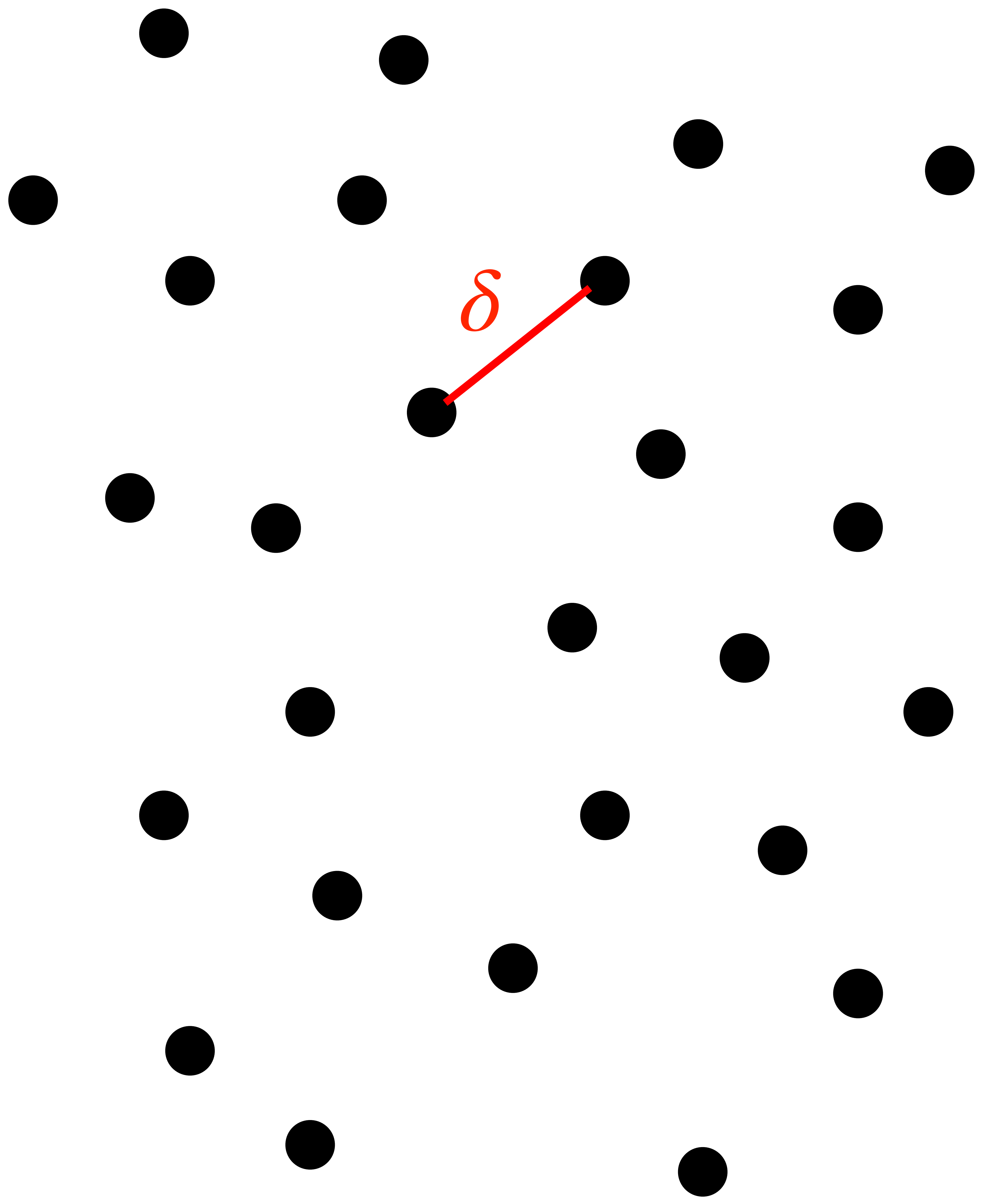
- Larger rate: more points
- Larger distance: further apart points

RATE-DISTANCE TRADEOFF



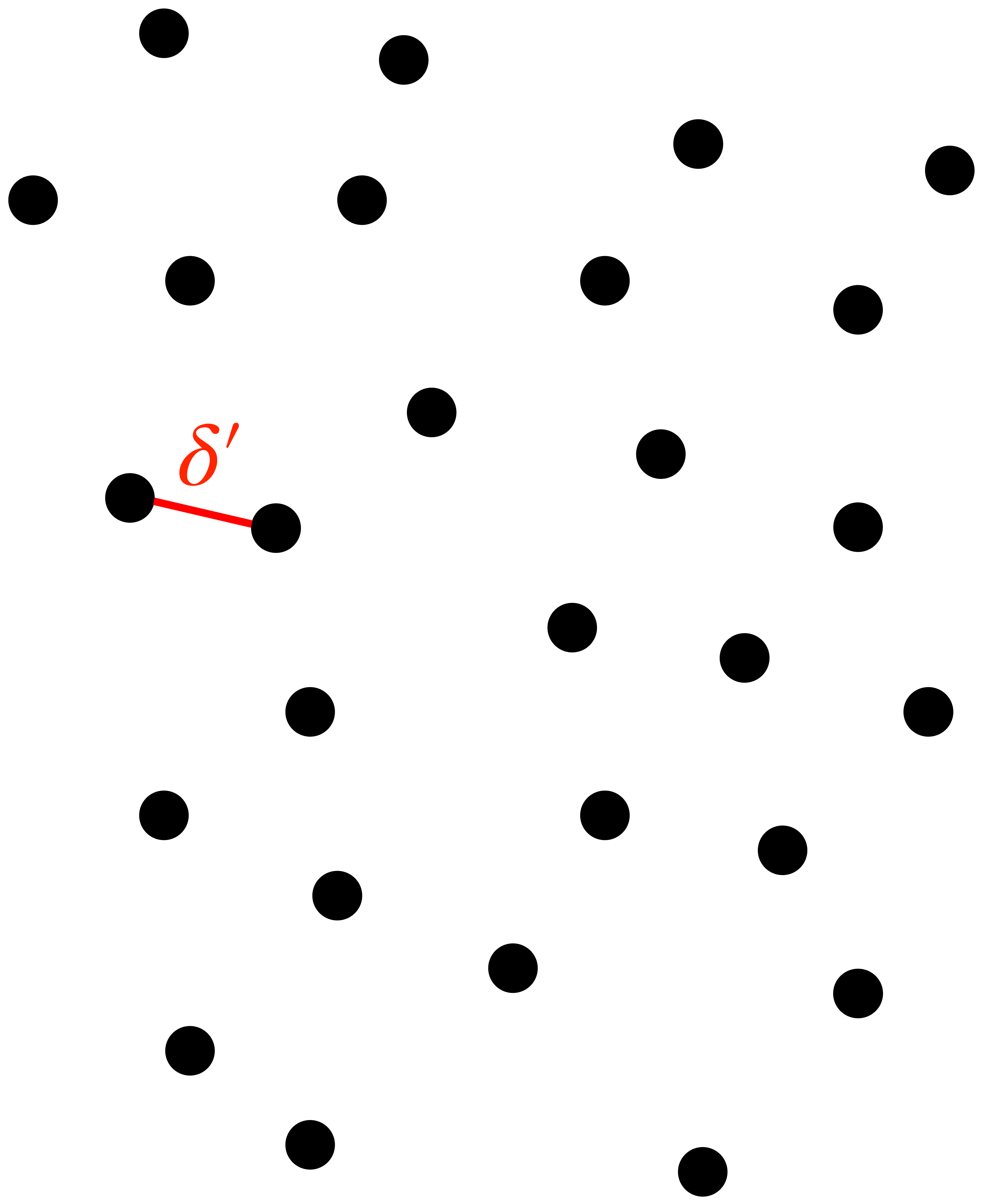
- Larger rate: more points
- Larger distance: further apart points

RATE-DISTANCE TRADEOFF



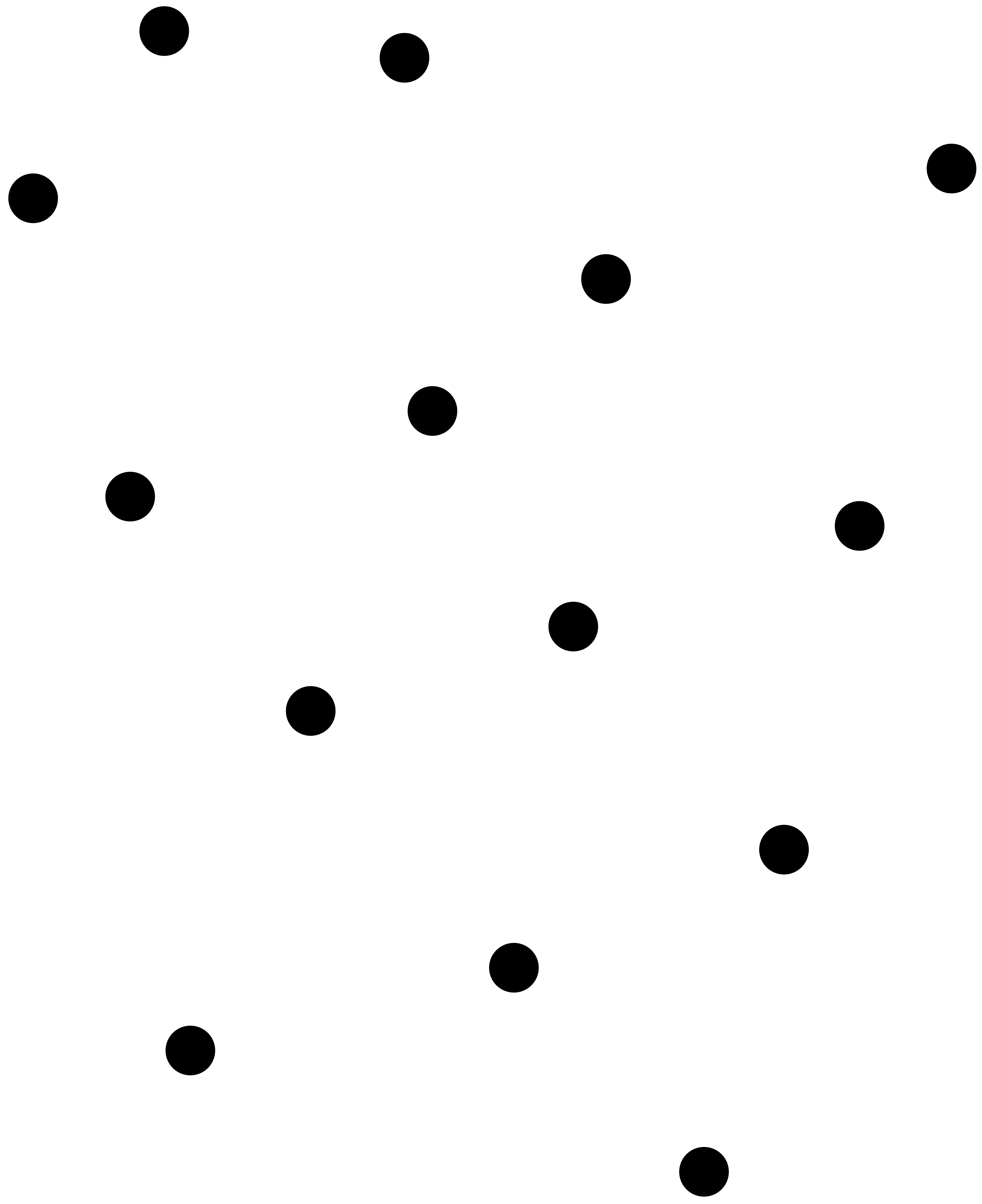
- Larger rate: more points
- Larger distance: further apart points

RATE-DISTANCE TRADEOFF



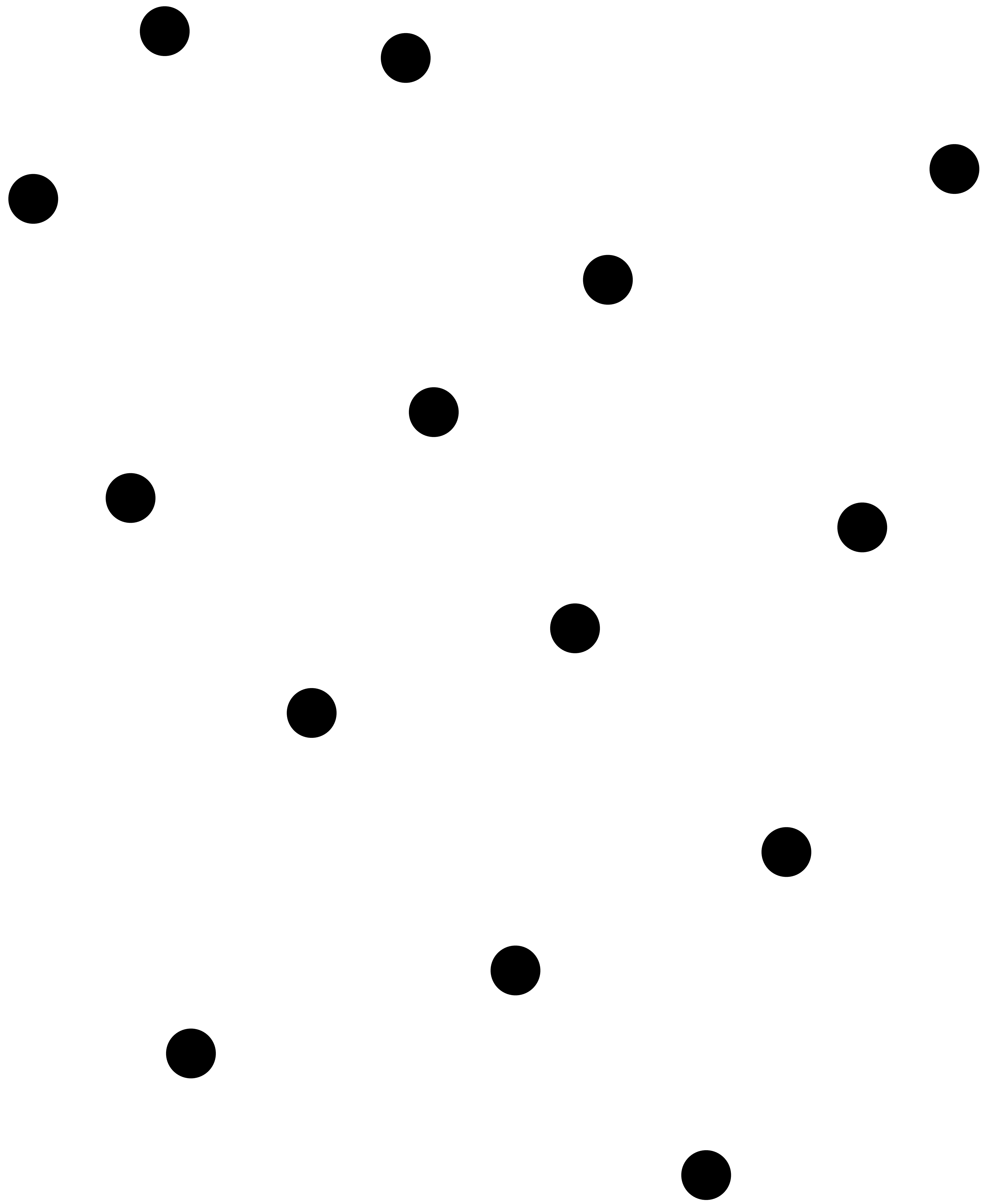
- Larger rate: more points
- Larger distance: further apart points

RATE-DISTANCE TRADEOFF



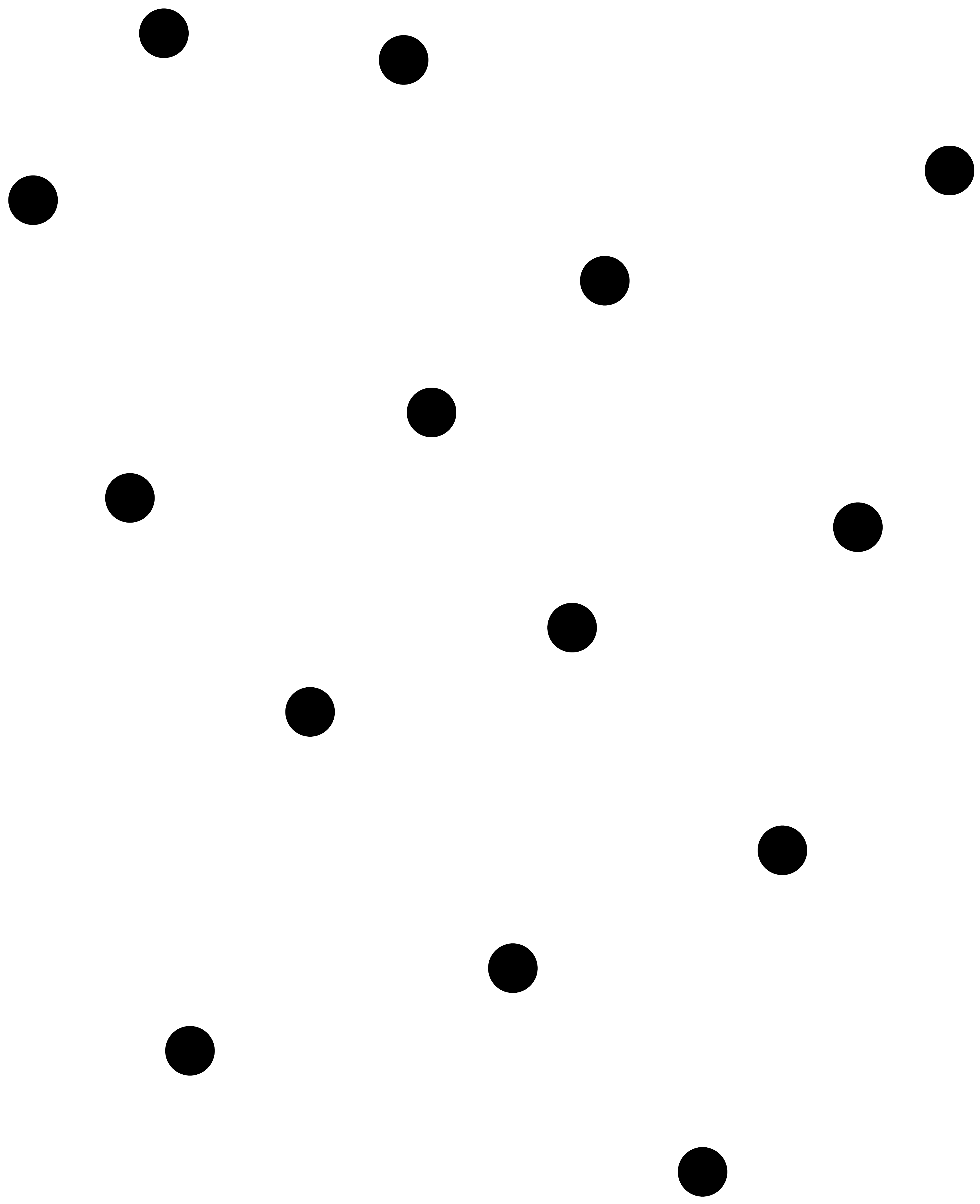
- Larger rate: more points
- Larger distance: further apart points

RATE-DISTANCE TRADEOFF



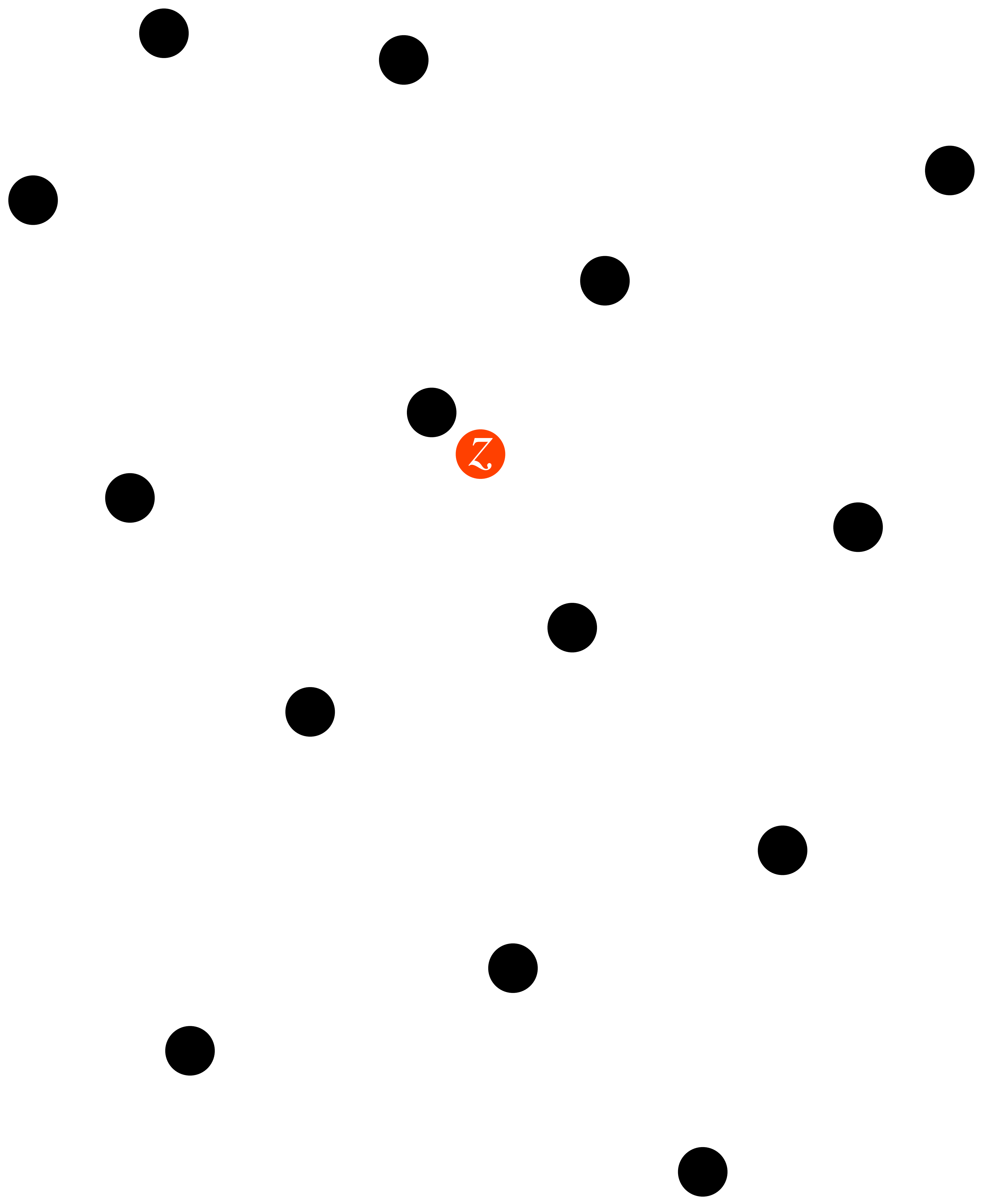
- Larger rate: more points
- Larger distance: further apart points
- Suppose channel corrupts ρ-fraction of symbols

RATE-DISTANCE TRADEOFF



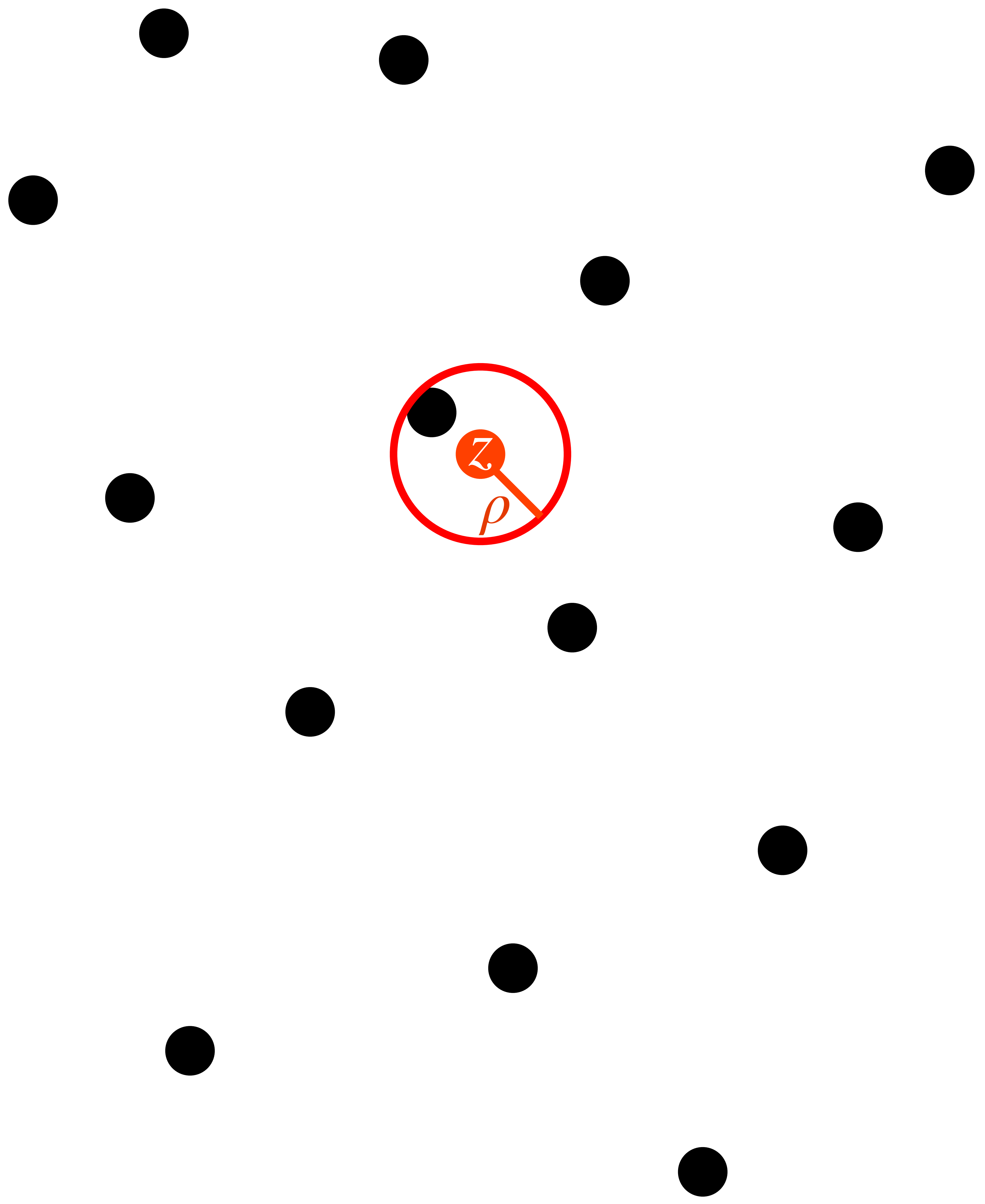
- Larger rate: more points
- Larger distance: further apart points
- Suppose channel corrupts ρ-fraction of symbols
  - $\rho < \delta/2$: can uniquely decode

RATE-DISTANCE TRADEOFF



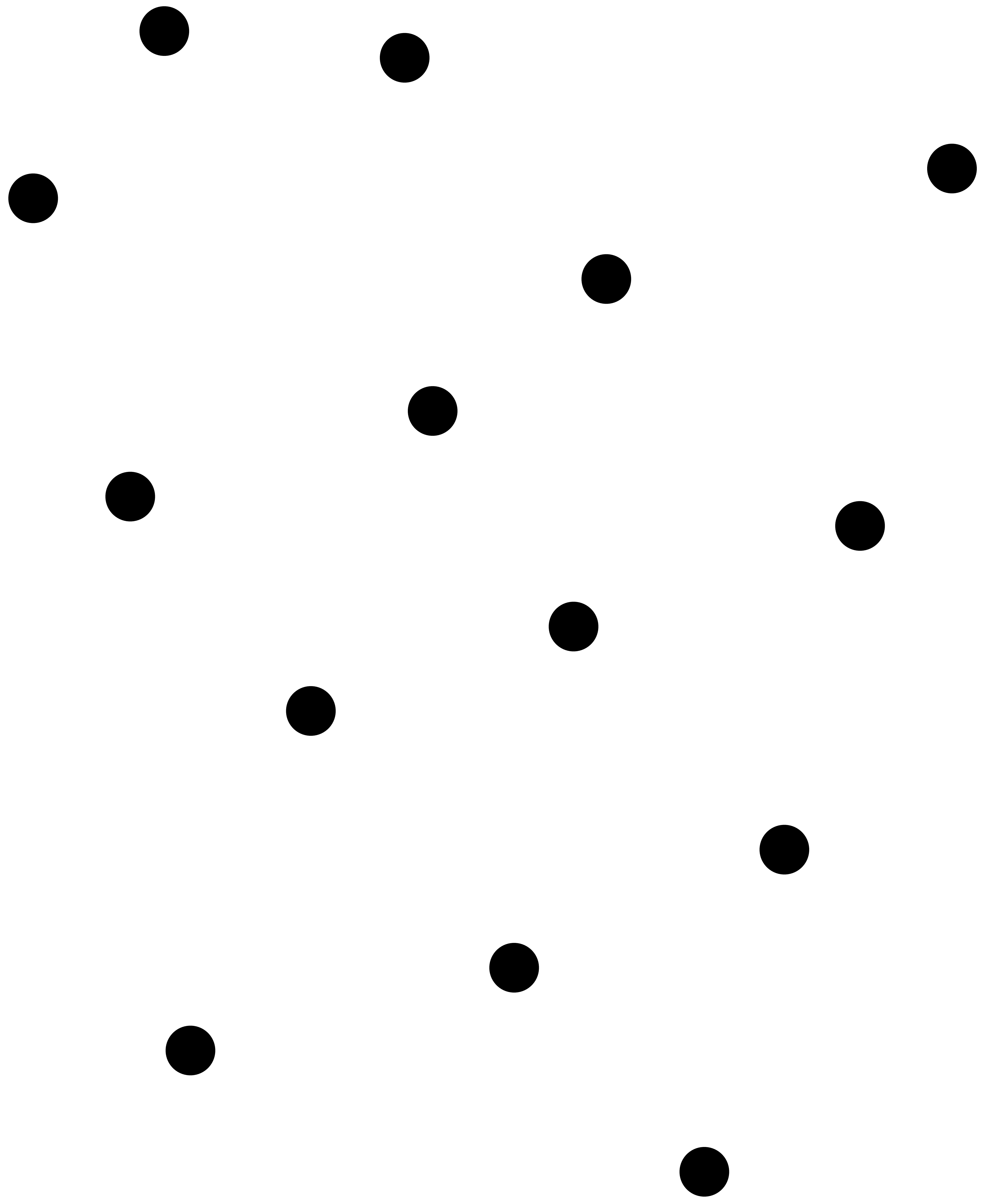
- Larger rate: more points
- Larger distance: further apart points
- Suppose channel corrupts ρ-fraction of symbols
  - $\rho < \delta/2$: can uniquely decode

RATE-DISTANCE TRADEOFF



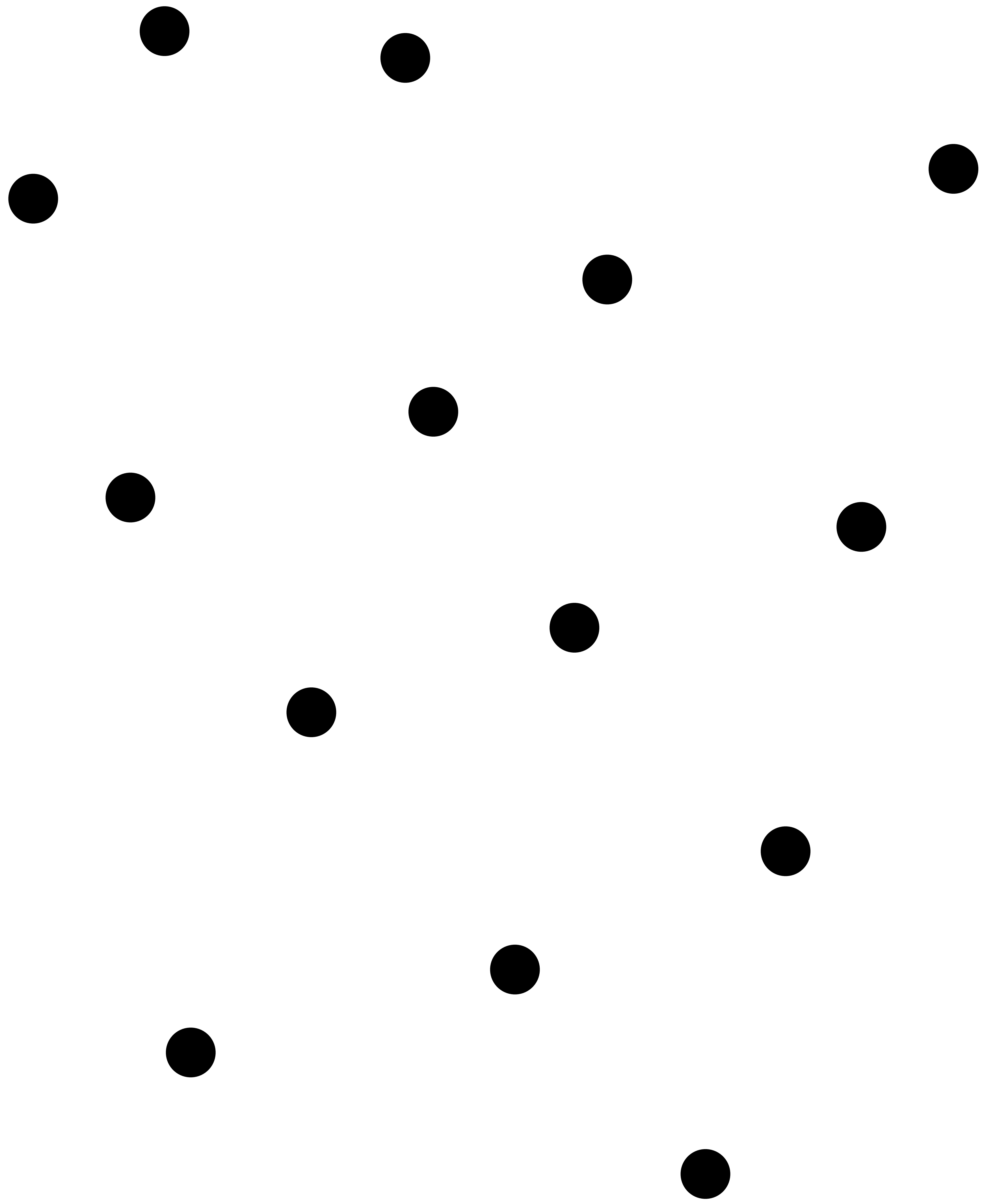
- Larger rate: more points
- Larger distance: further apart points
- Suppose channel corrupts ρ-fraction of symbols
  - $\rho < \delta/2$: can uniquely decode

RATE-DISTANCE TRADEOFF



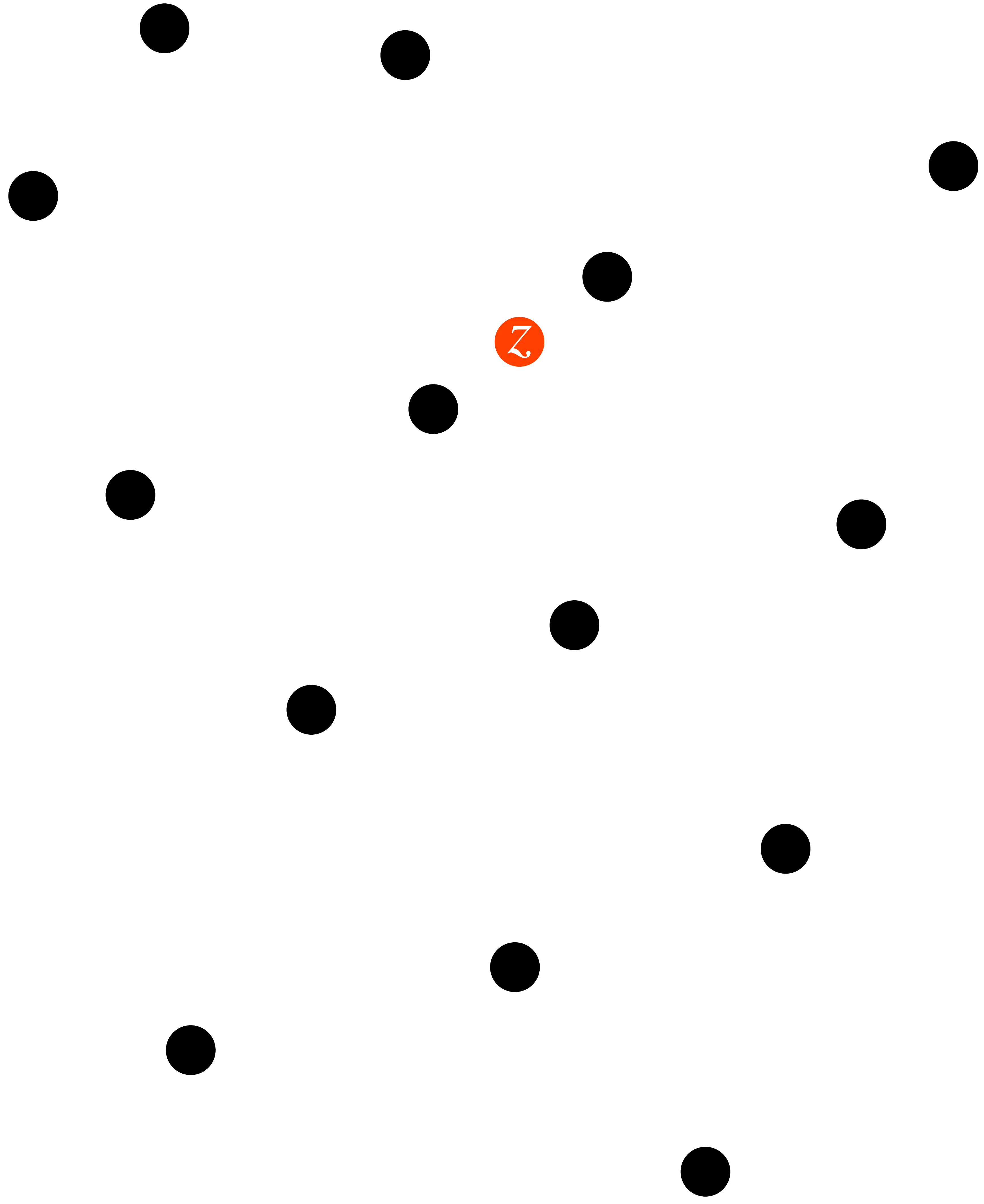
- Larger rate: more points
- Larger distance: further apart points
- Suppose channel corrupts ρ-fraction of symbols
  - $\rho < \delta/2$: can uniquely decode

RATE-DISTANCE TRADEOFF



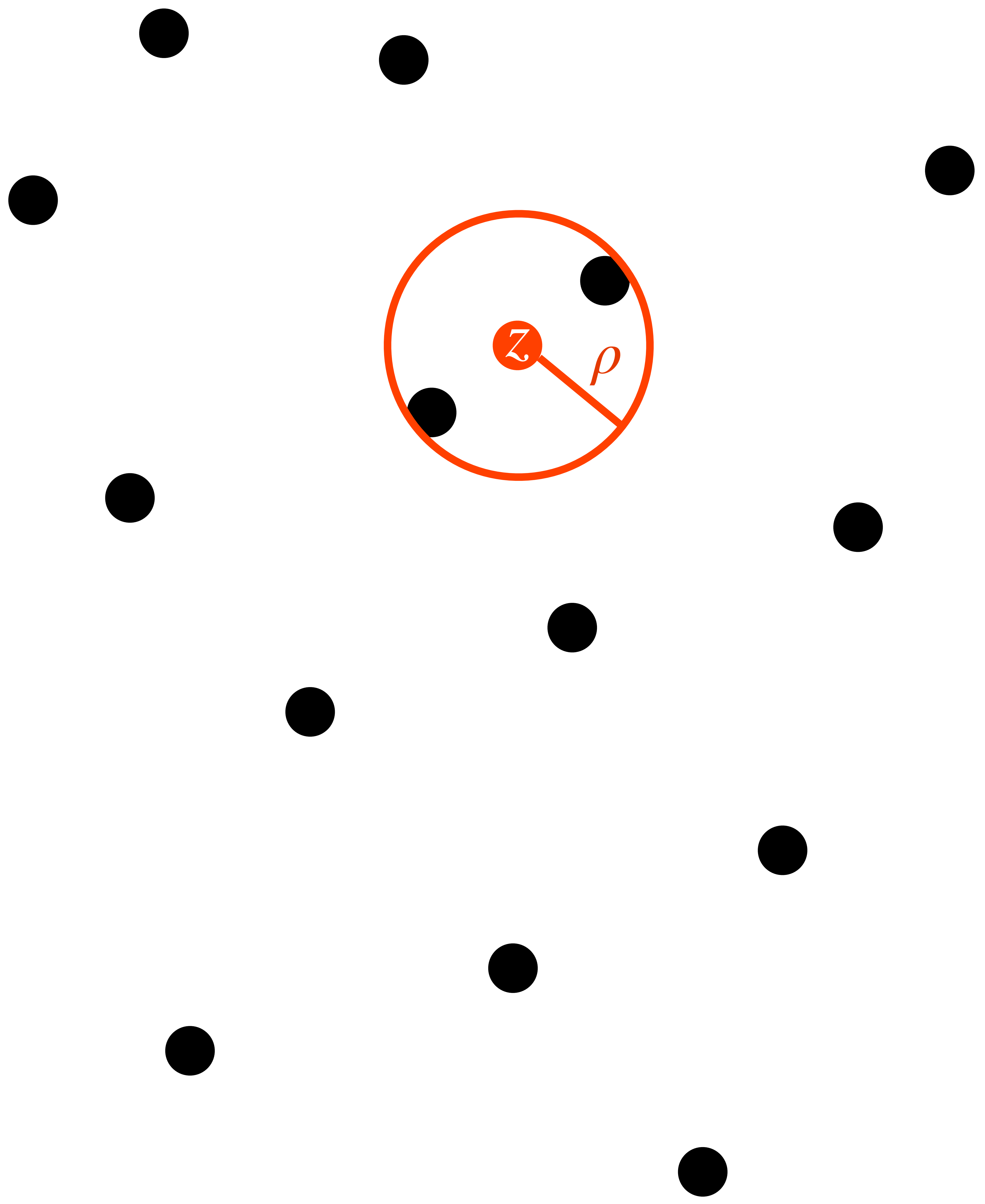
- Larger rate: more points
- Larger distance: further apart points
- Suppose channel corrupts ρ-fraction of symbols
  - $\rho < \delta/2$: can uniquely decode
  - $\rho \geq \delta/2$: cannot uniquely decode

RATE-DISTANCE TRADEOFF



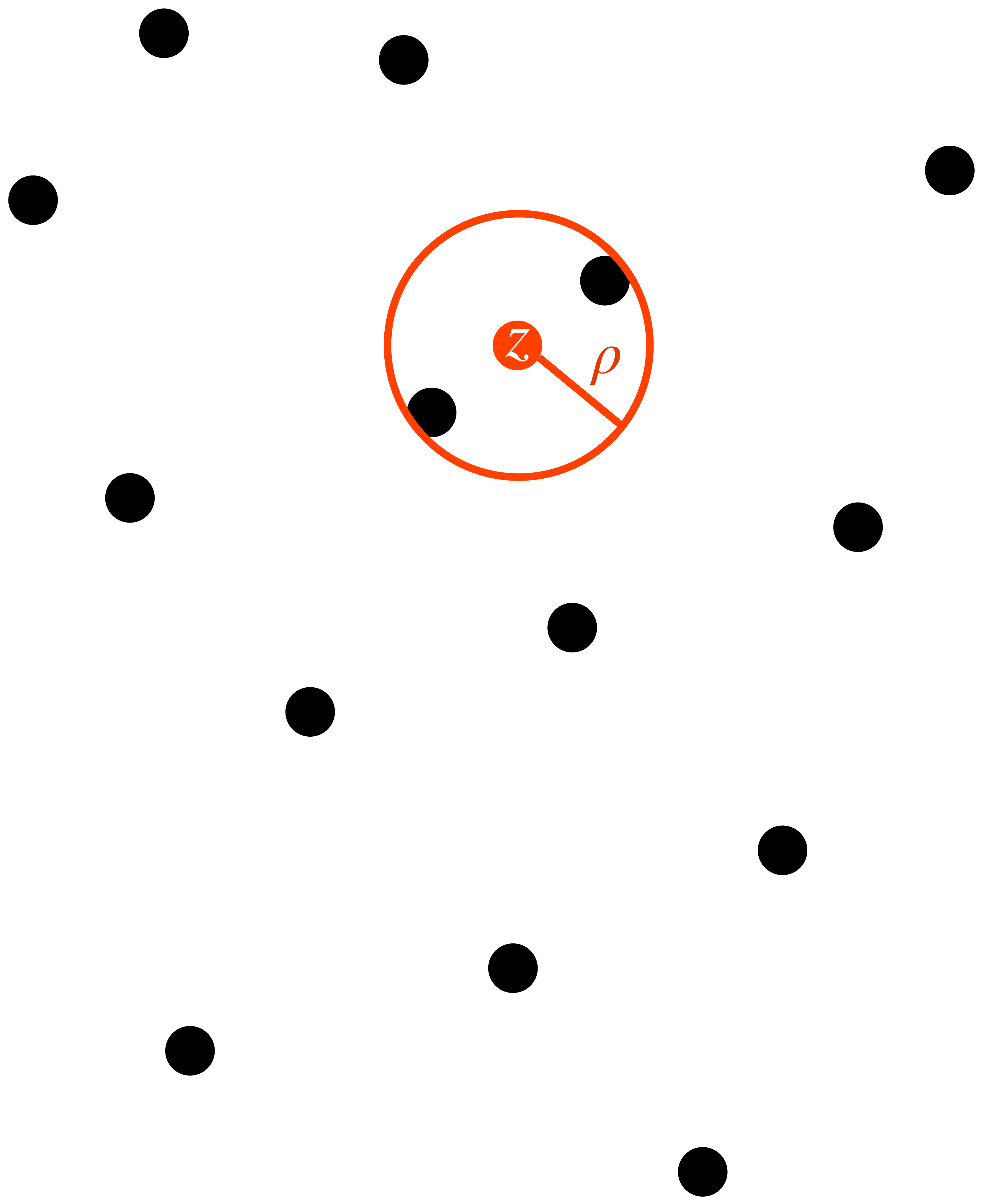
- Larger rate: more points
- Larger distance: further apart points
- Suppose channel corrupts ρ-fraction of symbols
  - $\rho < \delta/2$: can uniquely decode
  - $\rho \geq \delta/2$: cannot uniquely decode

RATE-DISTANCE TRADEOFF



- Larger rate: more points
- Larger distance: further apart points
- Suppose channel corrupts ρ-fraction of symbols
  - $\rho < \delta/2$: can uniquely decode
  - $\rho \geq \delta/2$: cannot uniquely decode

RATE-DISTANCE TRADEOFF



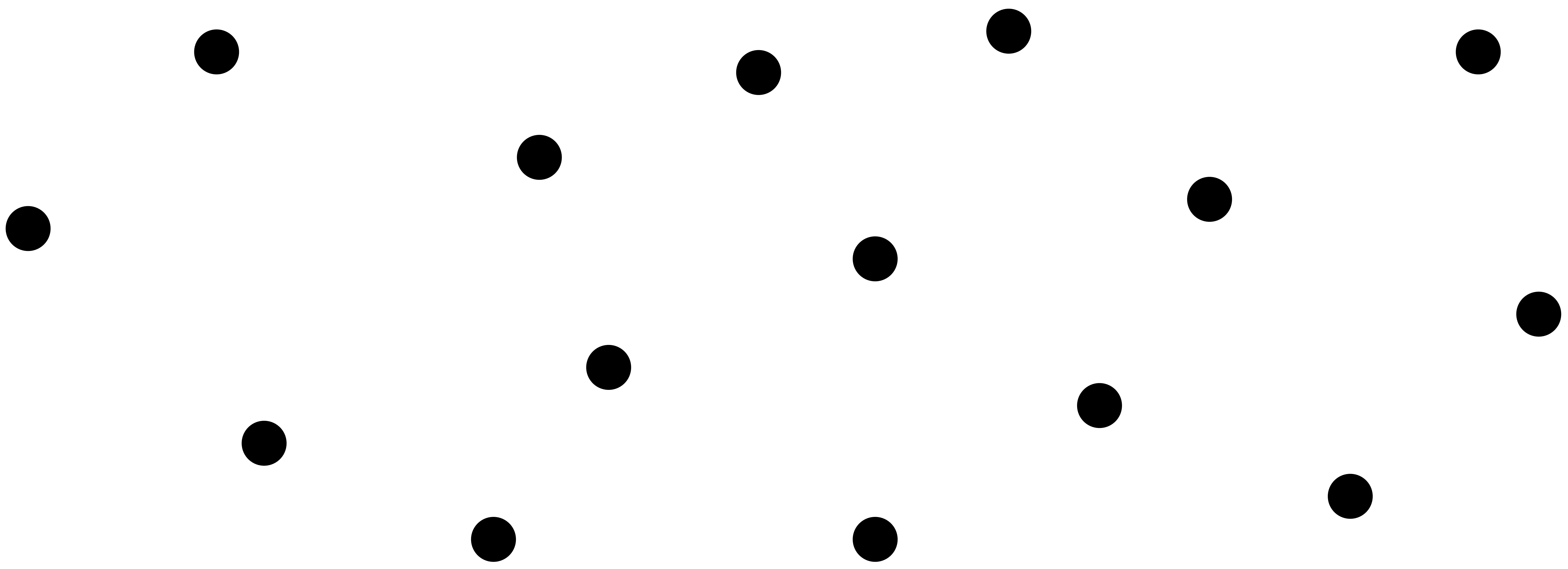
- Larger rate: more points
- Larger distance: further apart points
- Suppose channel corrupts ρ-fraction of symbols
  - $\rho < \delta/2$: can uniquely decode
  - $\rho \geq \delta/2$: cannot uniquely decode
- "Half-the-distance limit"

LIST-DECODABILITY

[Elias'57, Wozencraft'58]

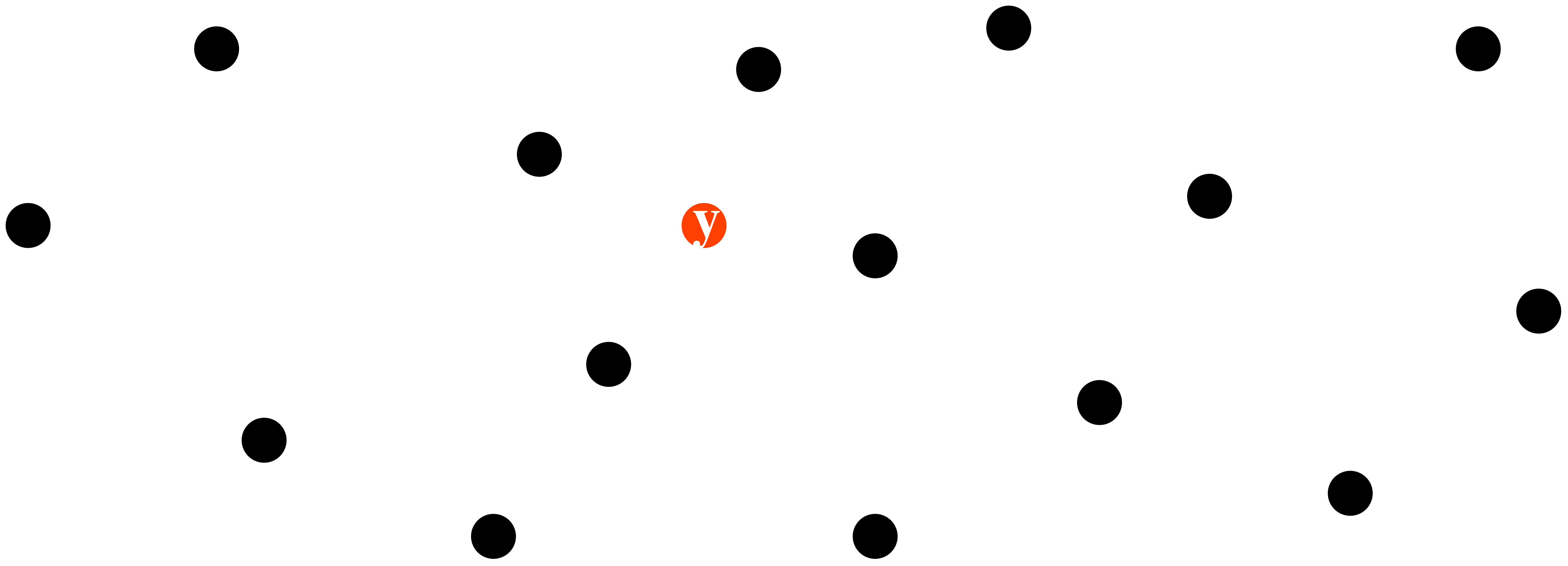
LIST-DECODABILITY

[Elias'57, Wozencraft'58]



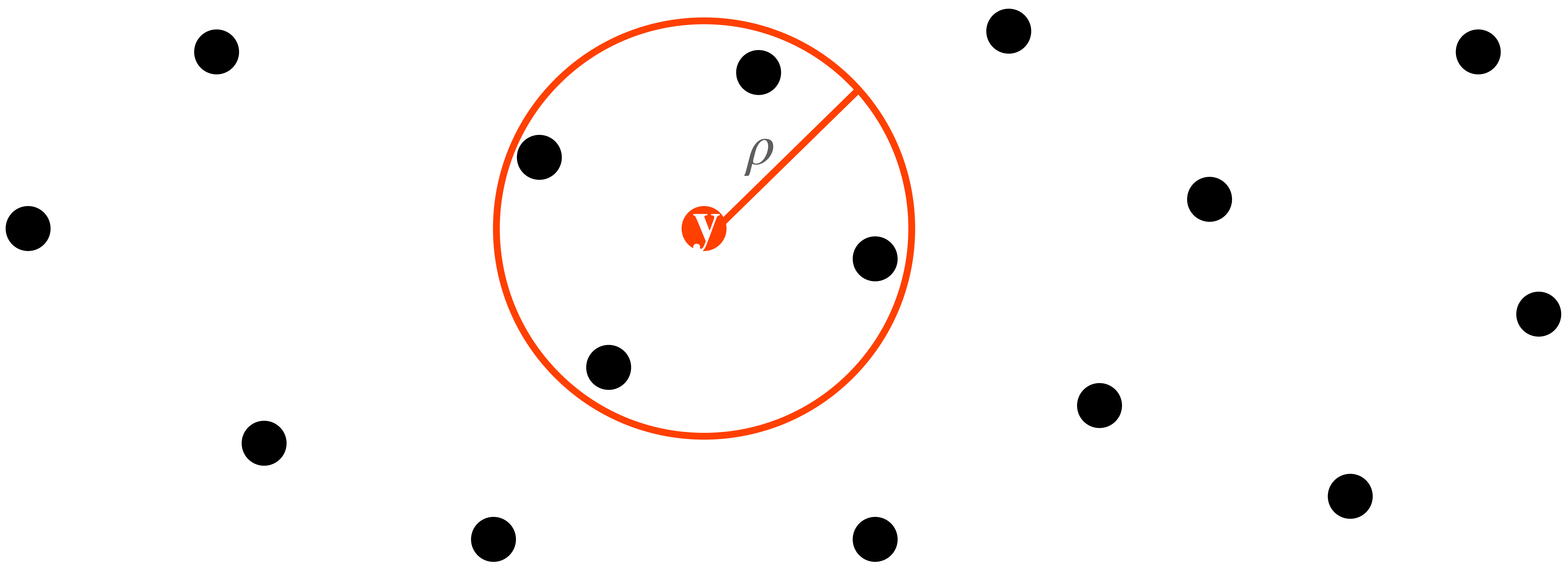
LIST-DECODABILITY

[Elias'57, Wozencraft'58]



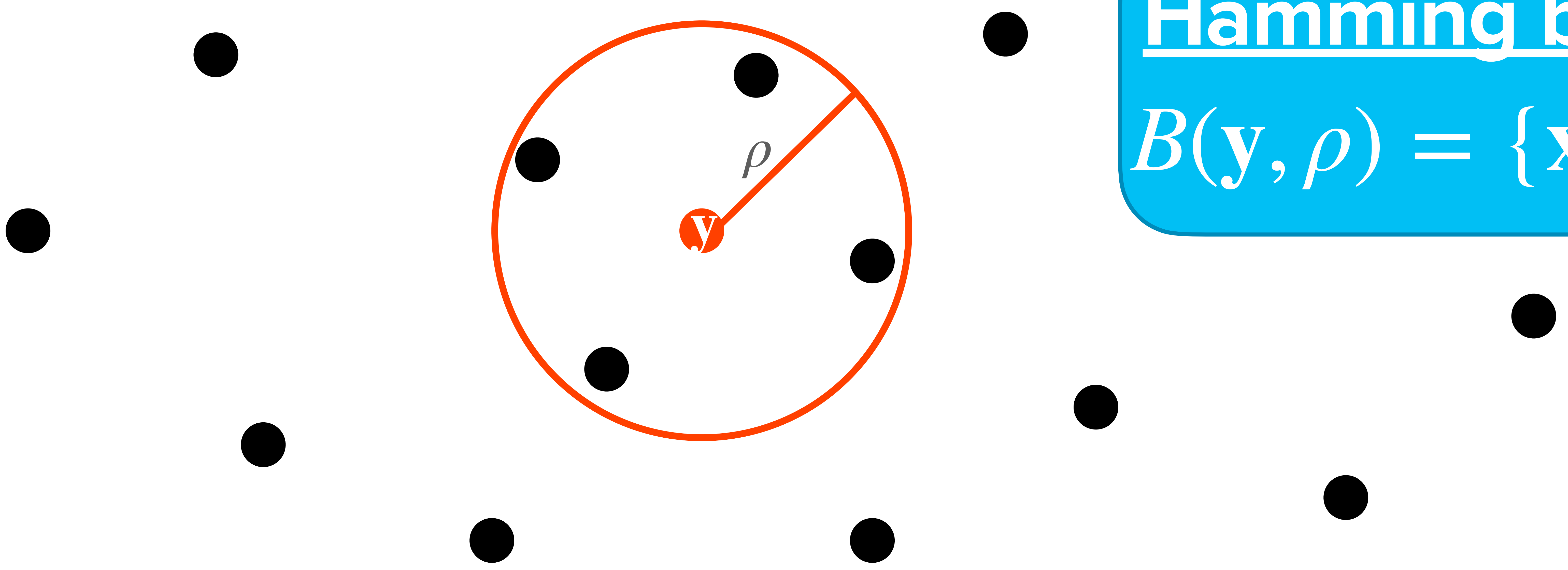
LIST-DECODABILITY

[Elias'57, Wozencraft'58]



LIST-DECODABILITY

[Elias'57, Wozencraft'58]

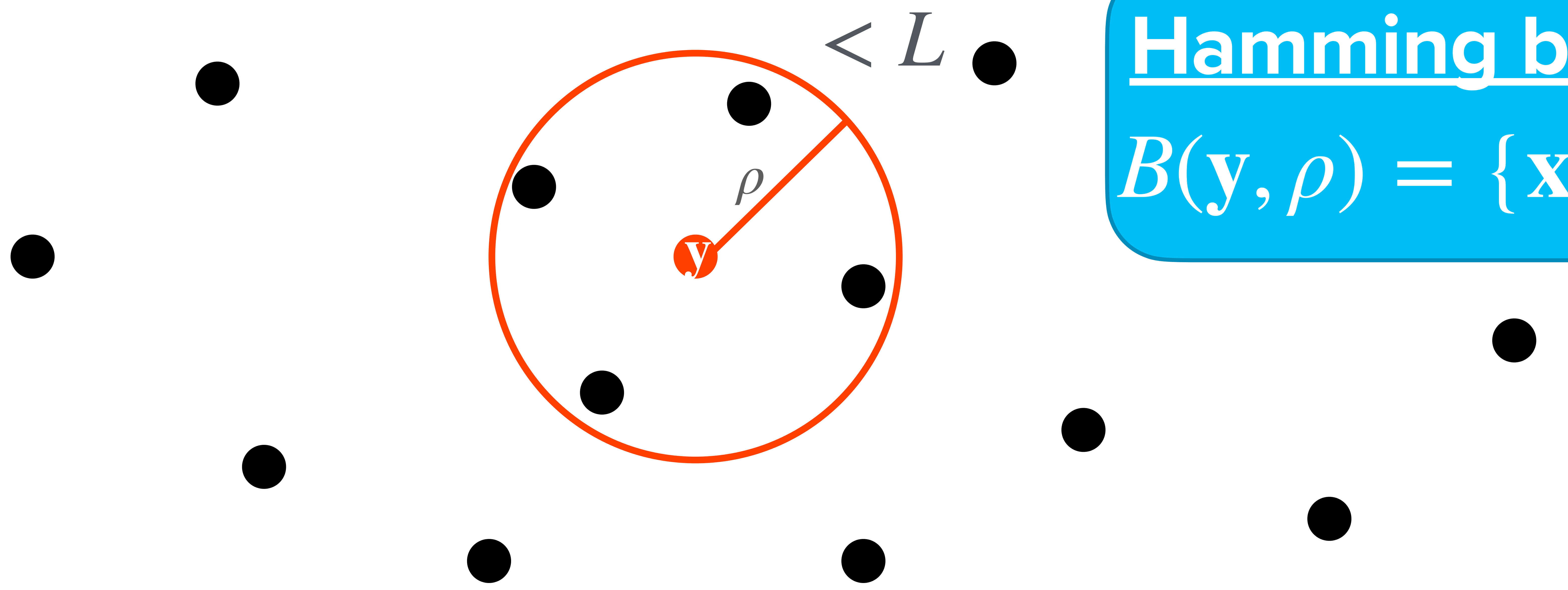


Hamming balls:

$$B(\mathbf{y},\rho) = \{\mathbf{x} \in [q]^n : d(\mathbf{x},\mathbf{y}) \leq \rho\}$$

LIST-DECODABILITY

[Elias'57, Wozencraft'58]



Hamming balls:

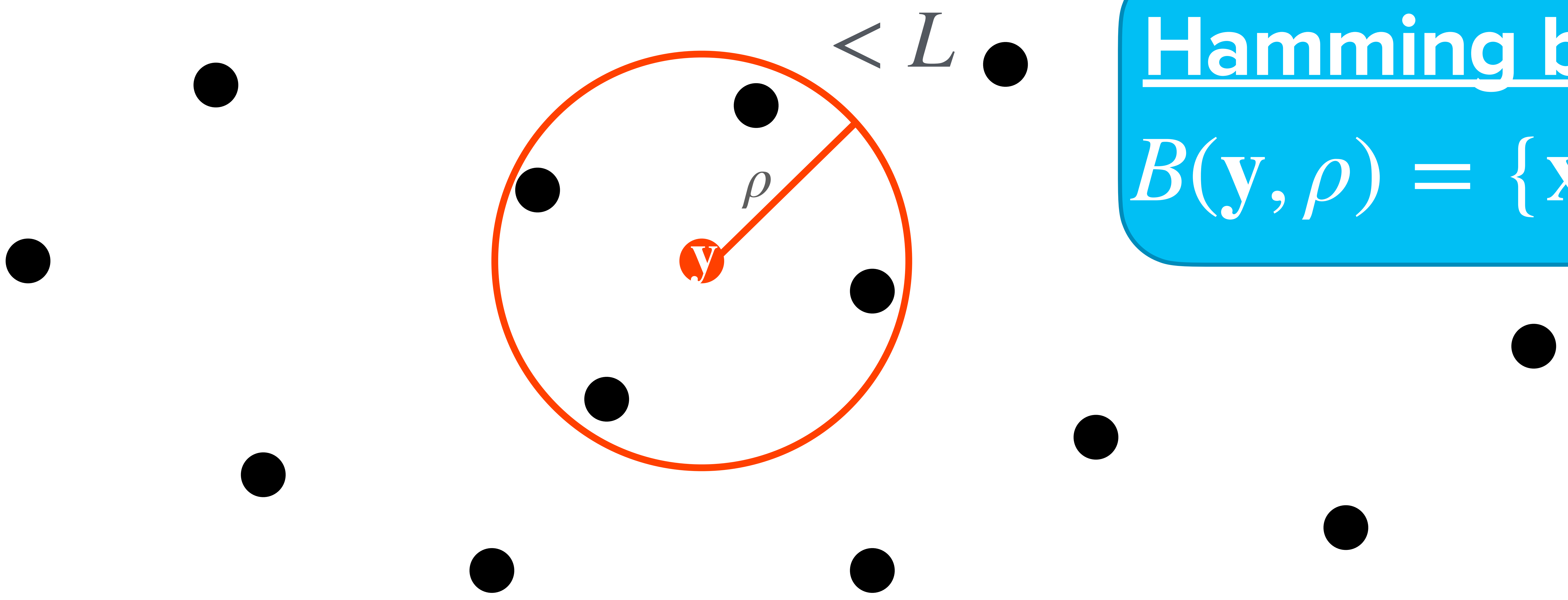
$B(\mathbf{y}, \rho) = \{\mathbf{x} \in [q]^n : d(\mathbf{x}, \mathbf{y}) \leq \rho\}$

LIST-DECODABILITY

[Elias'57, Wozencraft'58]

$\mathscr{C} \subseteq [q]^n$ is (ρ, L)-*list-decodable* if $\forall \mathbf{y} \in [q]^n$,

$$|\mathscr{C} \cap B(\mathbf{y}, \rho)| < L$$



Hamming balls:

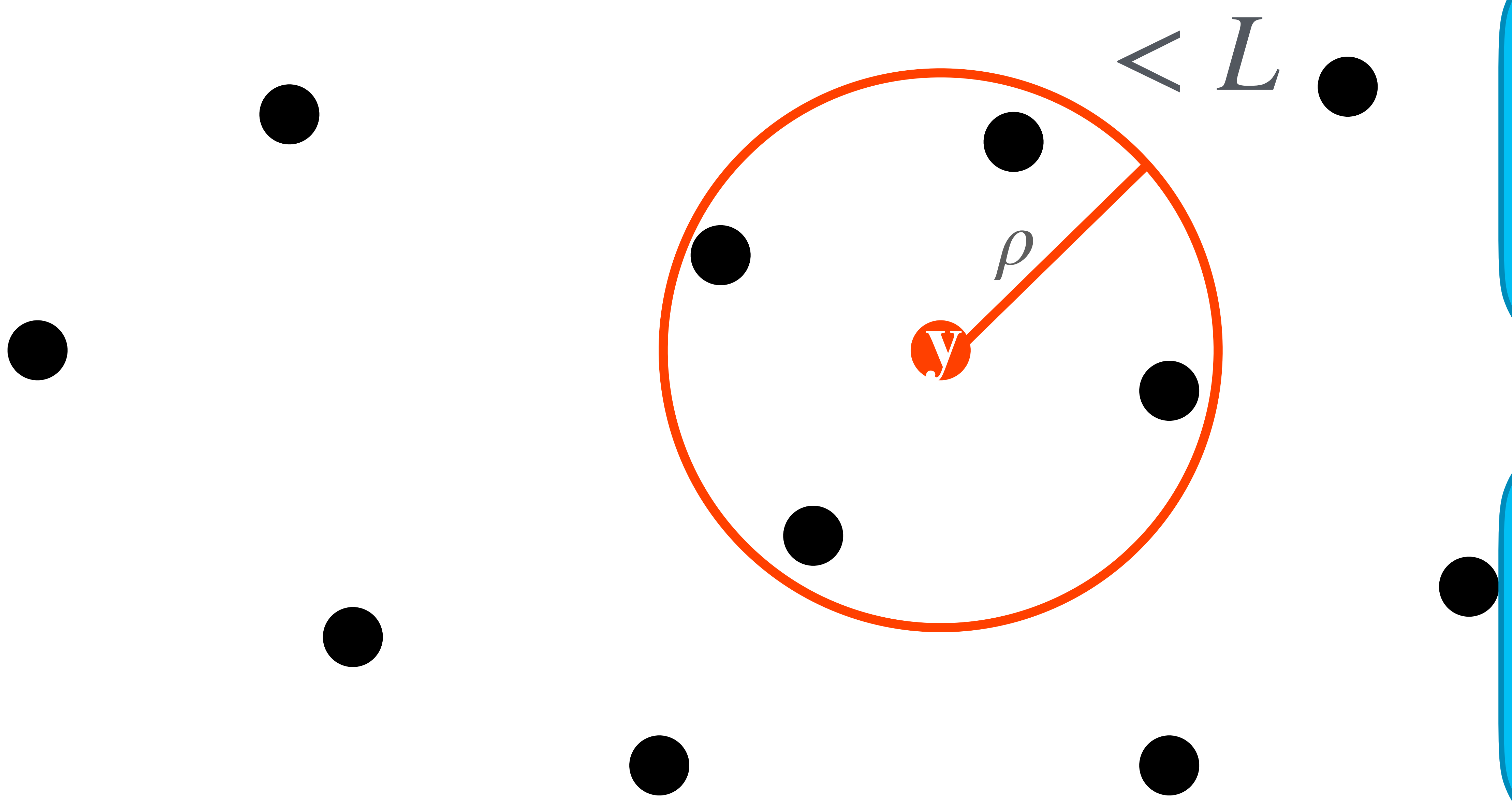
$$B(\mathbf{y}, \rho) = \{\mathbf{x} \in [q]^n : d(\mathbf{x}, \mathbf{y}) \leq \rho\}$$

LIST-DECODABILITY

[Elias'57, Wozencraft'58]

$\mathscr{C} \subseteq [q]^n$ is (ρ, L)-*list-decodable* if $\forall \mathbf{y} \in [q]^n$,

$$|\mathscr{C} \cap B(\mathbf{y}, \rho)| < L$$



Hamming balls:

$$B(\mathbf{y}, \rho) = \{\mathbf{x} \in [q]^n : d(\mathbf{x}, \mathbf{y}) \leq \rho\}$$

Tradeoffs between R, ρ, L?

DREAM

DREAM

$R_{\text{cap}} =$ **max rate of (ρ, L)-list-dec. code**

DREAM

$R_{\text{cap}} \to 1 - h_q(\rho)$ **(GV) as** $L \to \infty$

$R_{\text{cap}} =$ **max rate of** (ρ, L)**-list-dec. code**

DREAM

$R_{\text{cap}} \to 1 - h_q(\rho)$ **(GV) as** $L \to \infty$

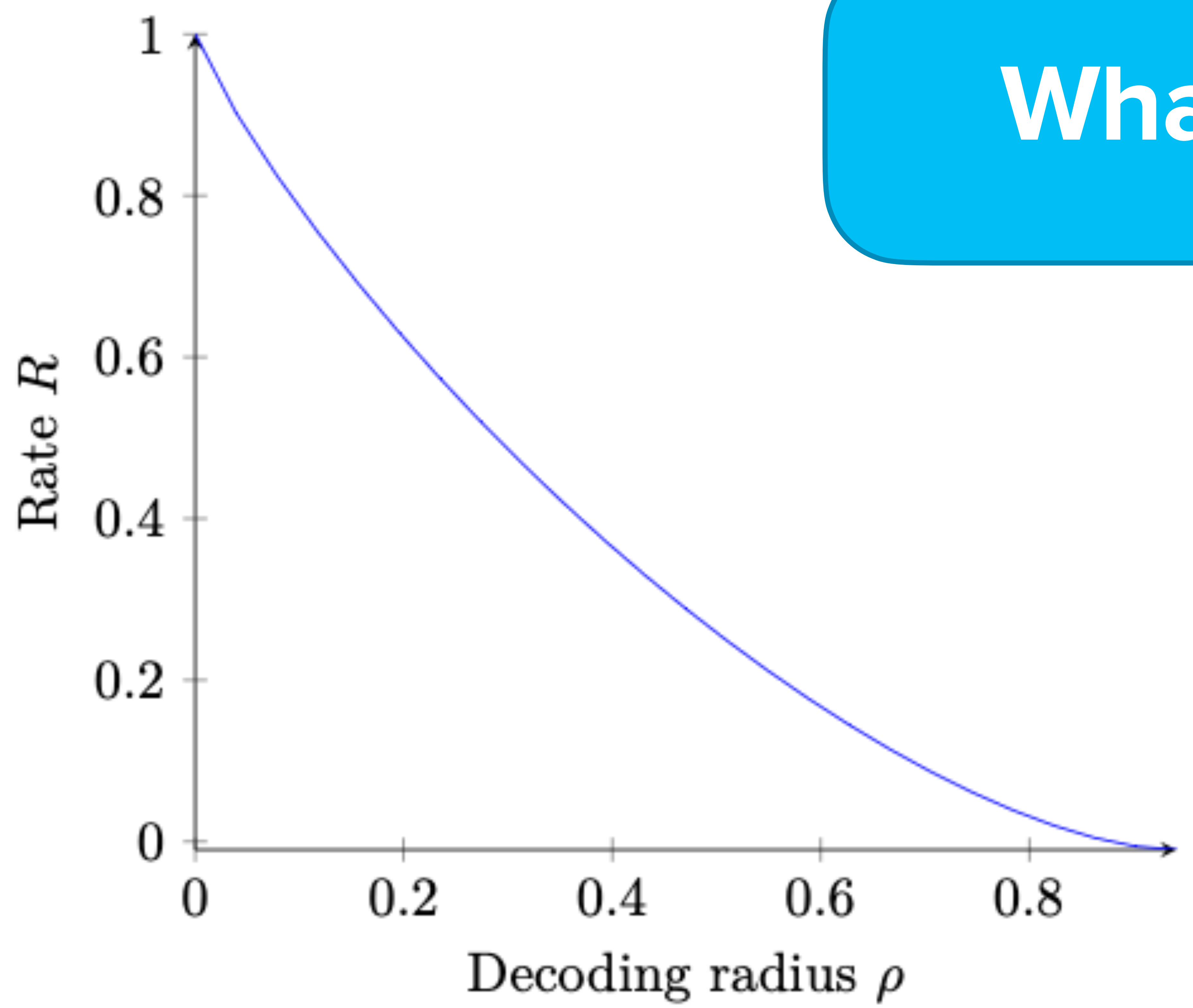
$R_{\text{cap}} =$ **max rate of** (ρ, L)**-list-dec. code**

What about fixed L**? E.g.,** $L = 10$**?**

DREAM

$R_{\text{cap}} \to 1 - h_q(\rho)$ **(GV) as** $L \to \infty$

$R_{\text{cap}} =$ **max rate of** (ρ, L)**-list-dec. code**



What about fixed L**? E.g.,** $L = 10$**?**

Describe this curve?

LET'S BE MORE MODEST...

LET'S BE MORE MODEST...

- $q = 2, L = 2$ (unique-decoding) case:

LET'S BE MORE MODEST...

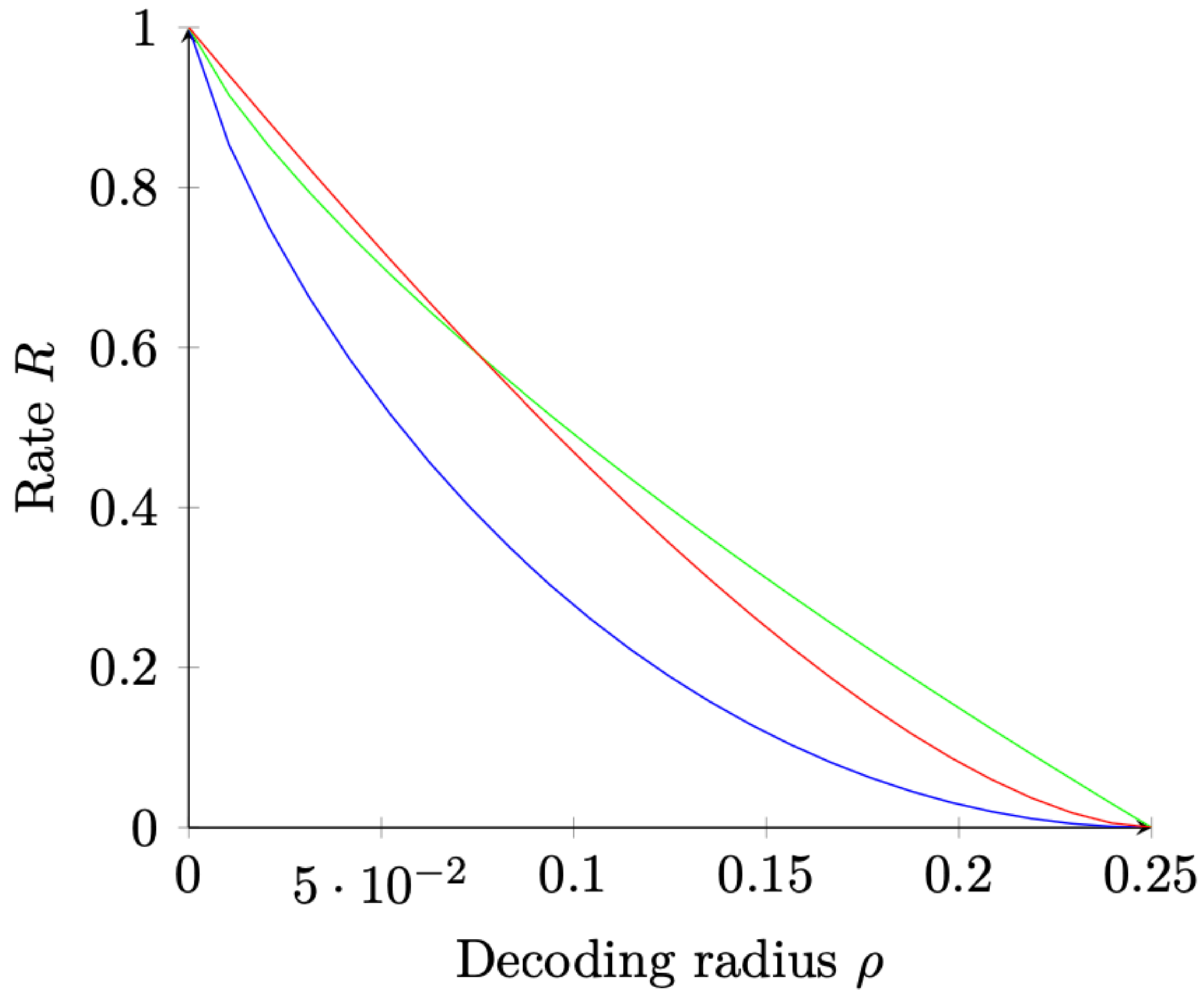
- $q = 2, L = 2$ (unique-decoding) case:

$$\text{GV} \leq R \leq \min\{\text{LP}, \text{EB}\}$$

LET'S BE MORE MODEST...

- $q = 2, L = 2$ (unique-decoding) case:

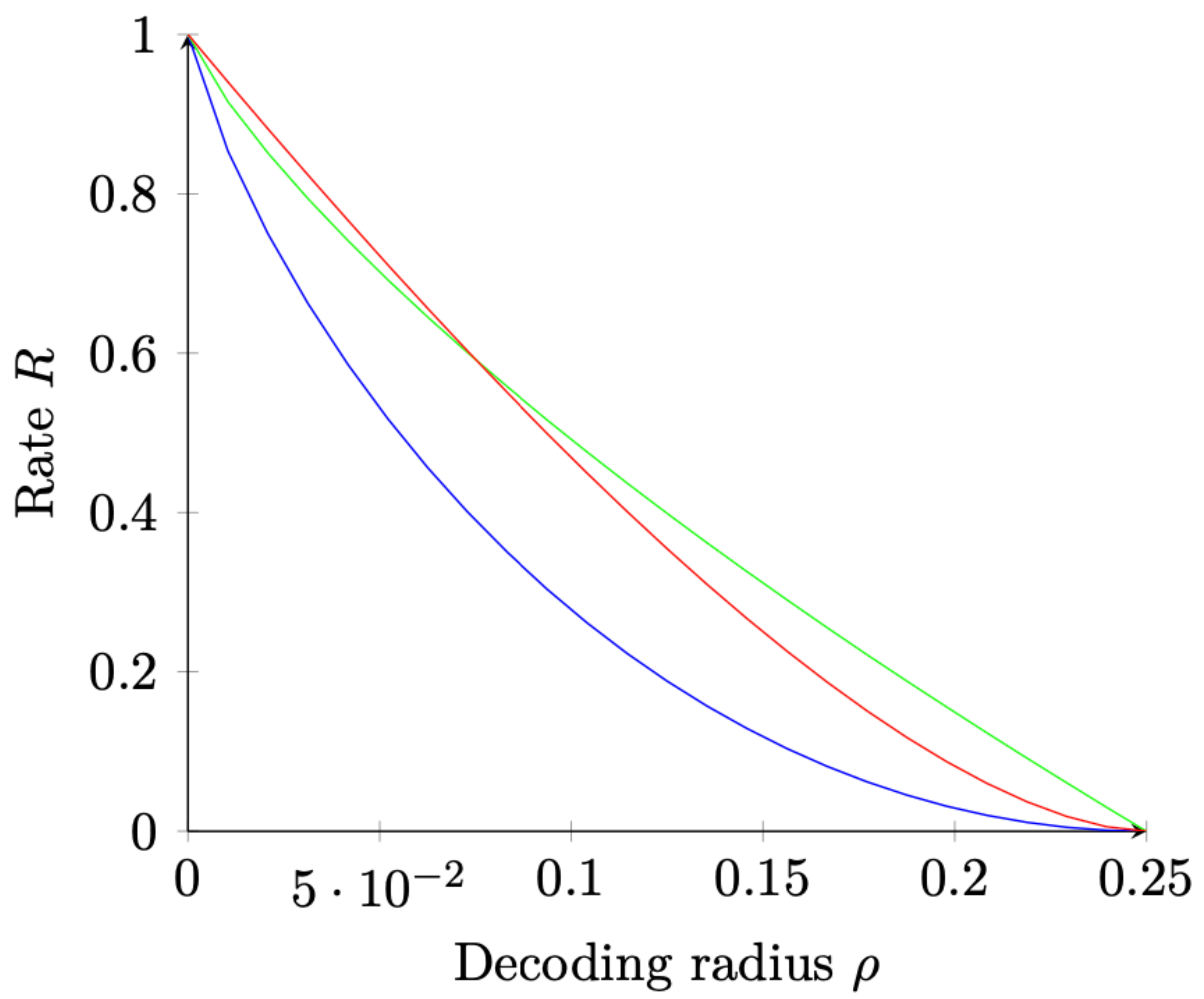
$$\text{GV} \leq R \leq \min\{\text{LP}, \text{EB}\}$$



LET'S BE MORE MODEST...

- $q = 2, L = 2$ (unique-decoding) case:

$$\text{GV} \leq R \leq \min\{\text{LP}, \text{EB}\}$$



- $q = 49, L = 2$:

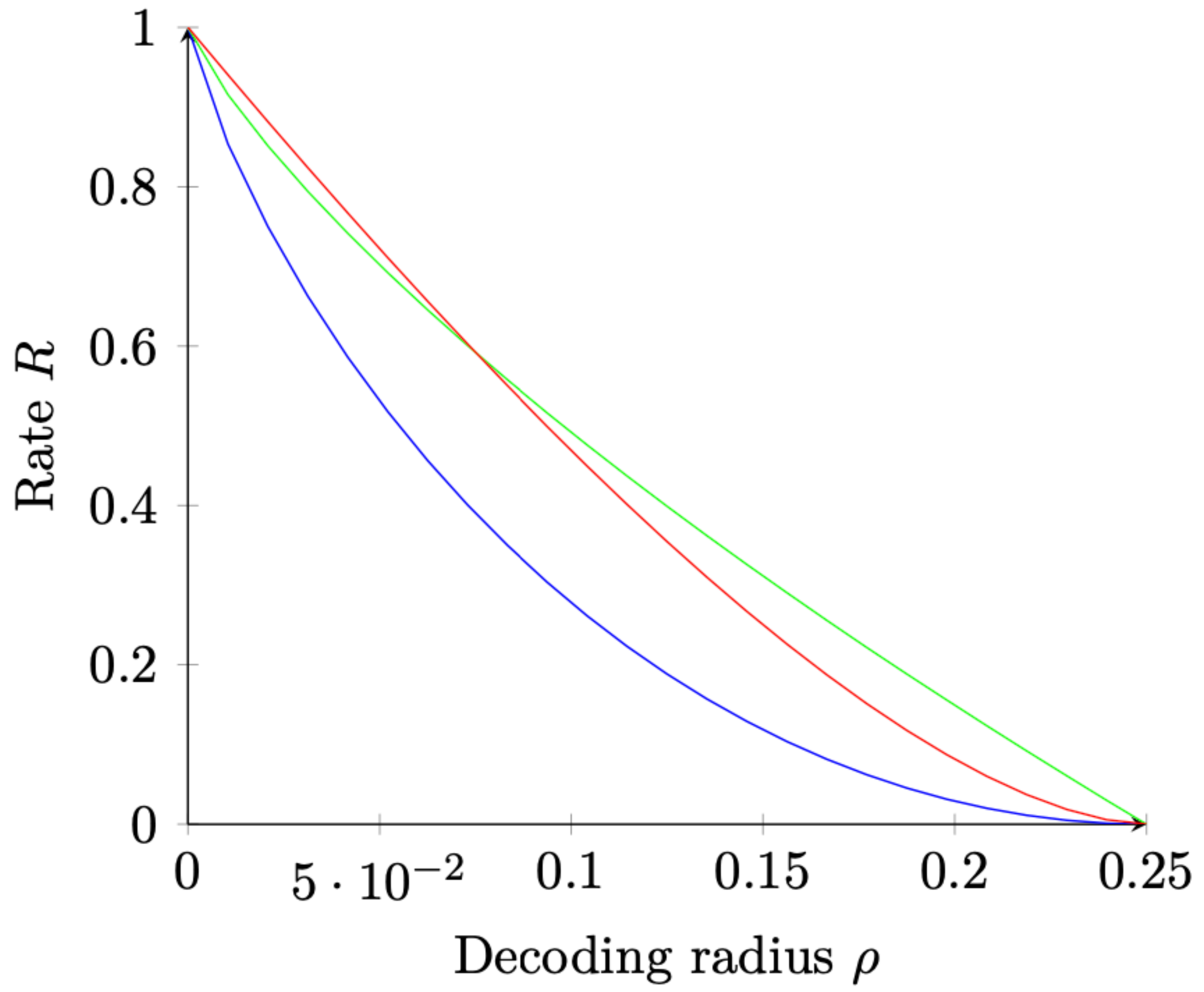
LET'S BE MORE MODEST...

- $q = 2, L = 2$ (unique-decoding) case:

$$\text{GV} \leq R \leq \min\{\text{LP}, \text{EB}\}$$

- $q = 49, L = 2$:

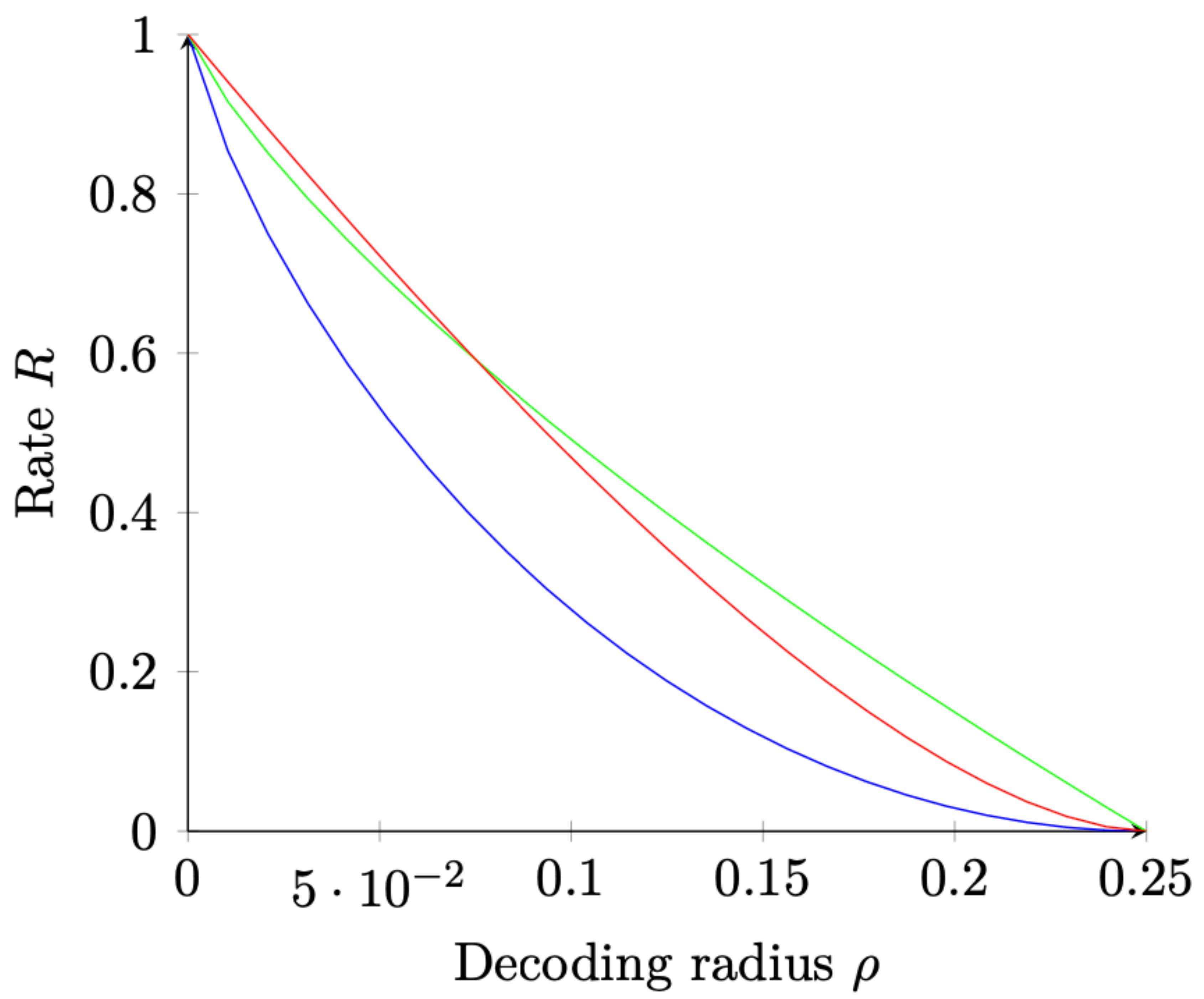
$$\max\{\text{GV}, \text{AG}\} \leq R$$



LET'S BE MORE MODEST...

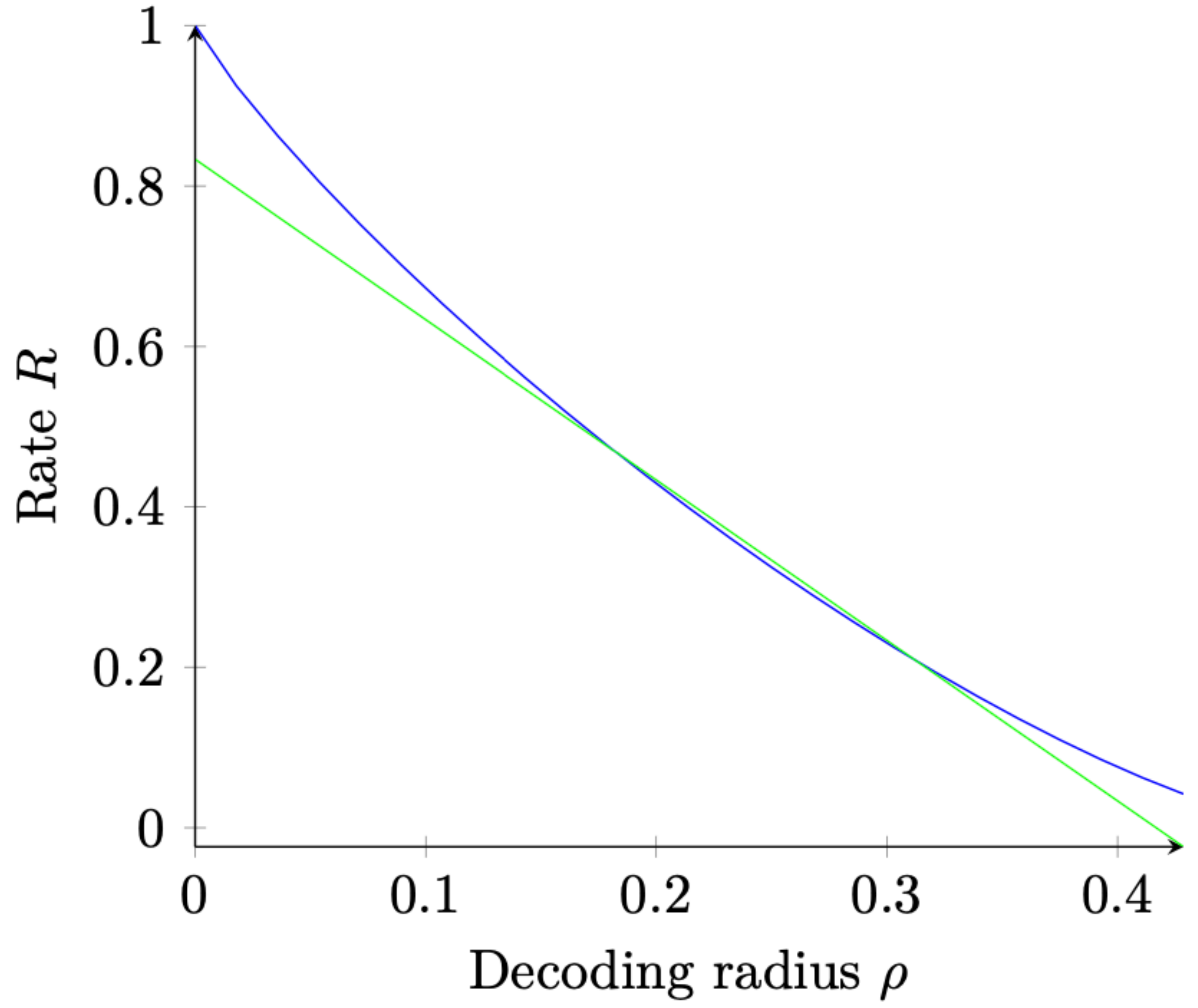
- $q = 2, L = 2$ (unique-decoding) case:

$$\text{GV} \leq R \leq \min\{\text{LP}, \text{EB}\}$$



- $q = 49, L = 2$:

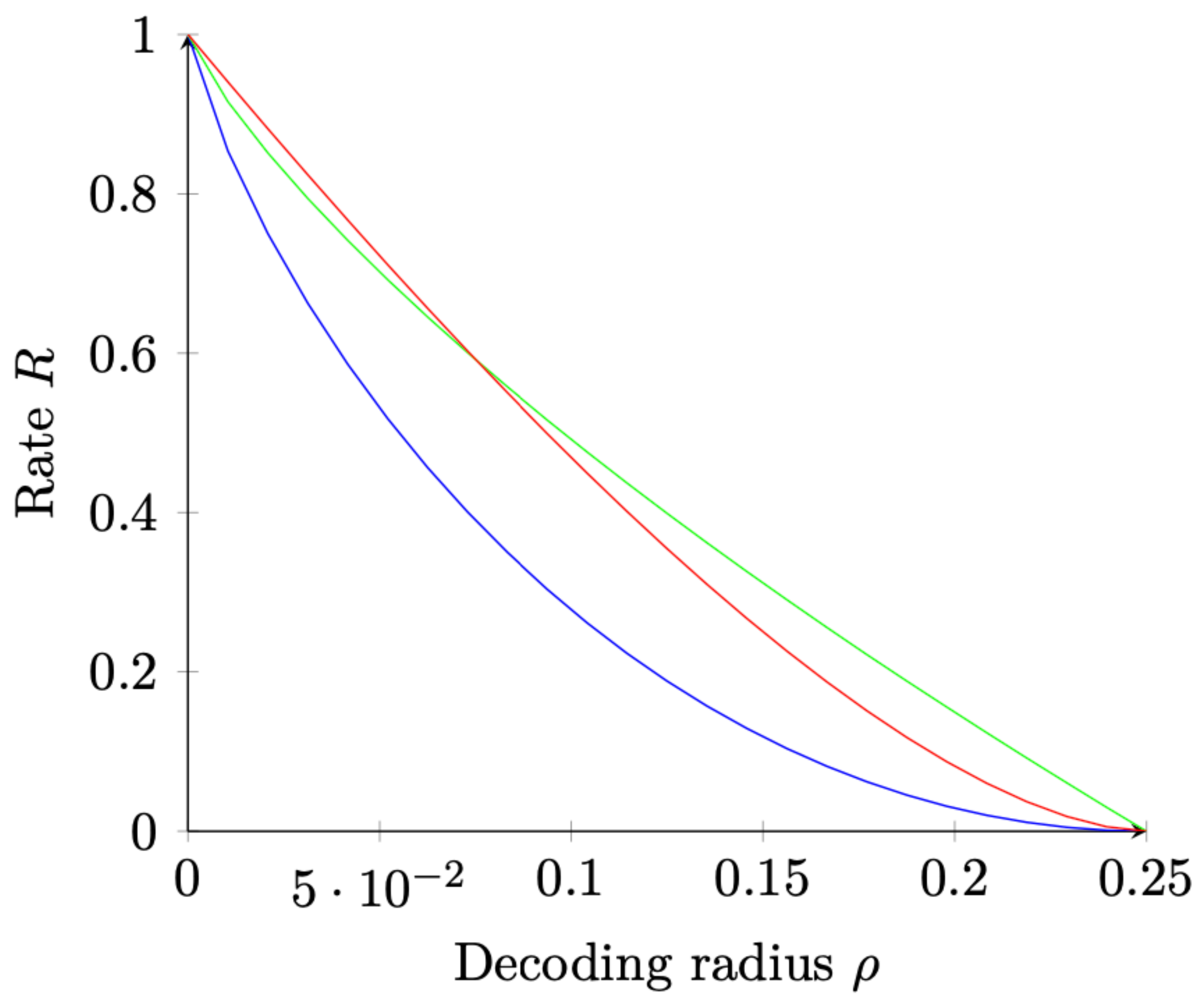
$$\max\{\text{GV}, \text{AG}\} \leq R$$



LET'S BE MORE MODEST...

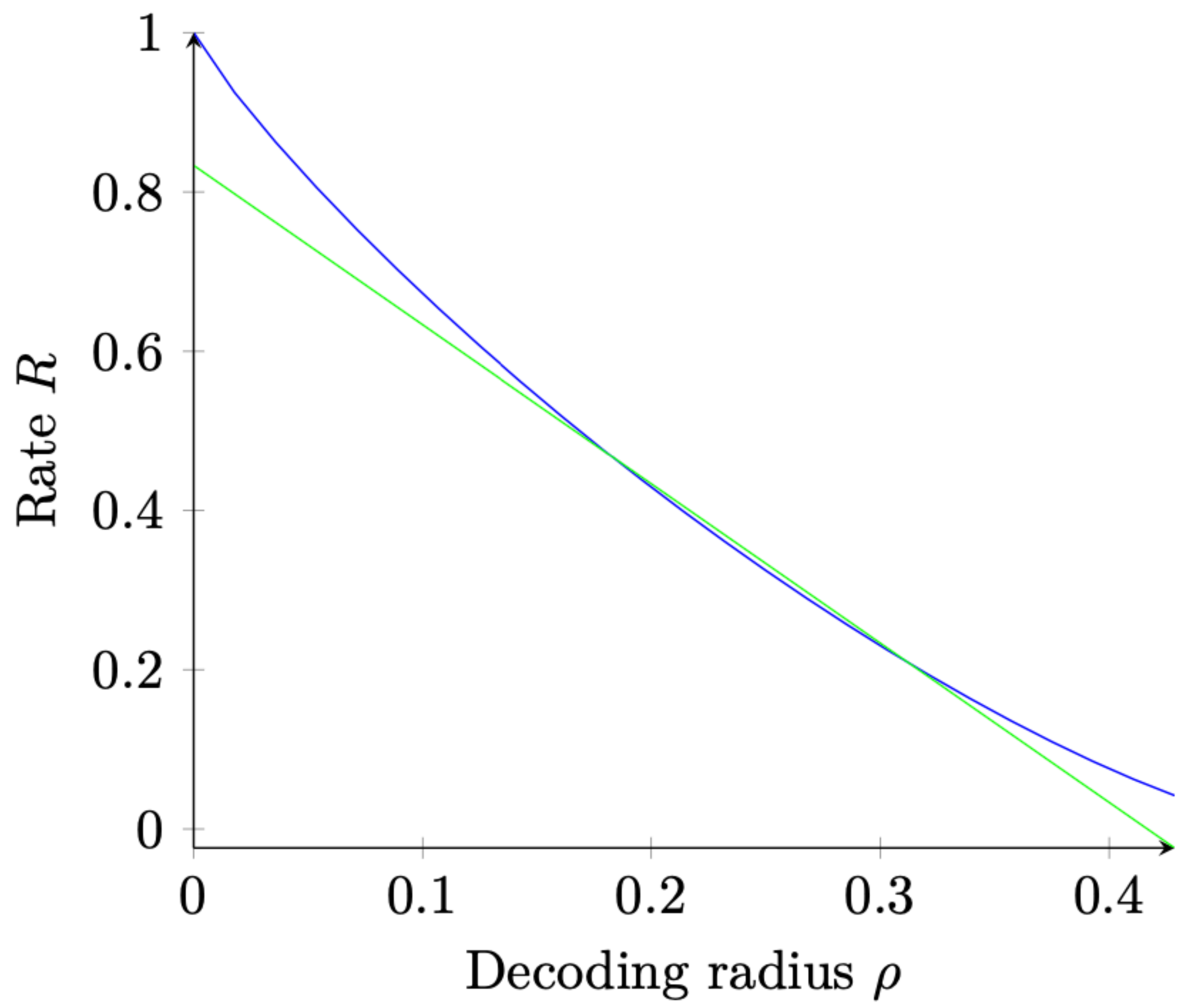
- $q = 2, L = 2$ (unique-decoding) case:

$$\text{GV} \leq R \leq \min\{\text{LP}, \text{EB}\}$$



- $q = 49, L = 2$:

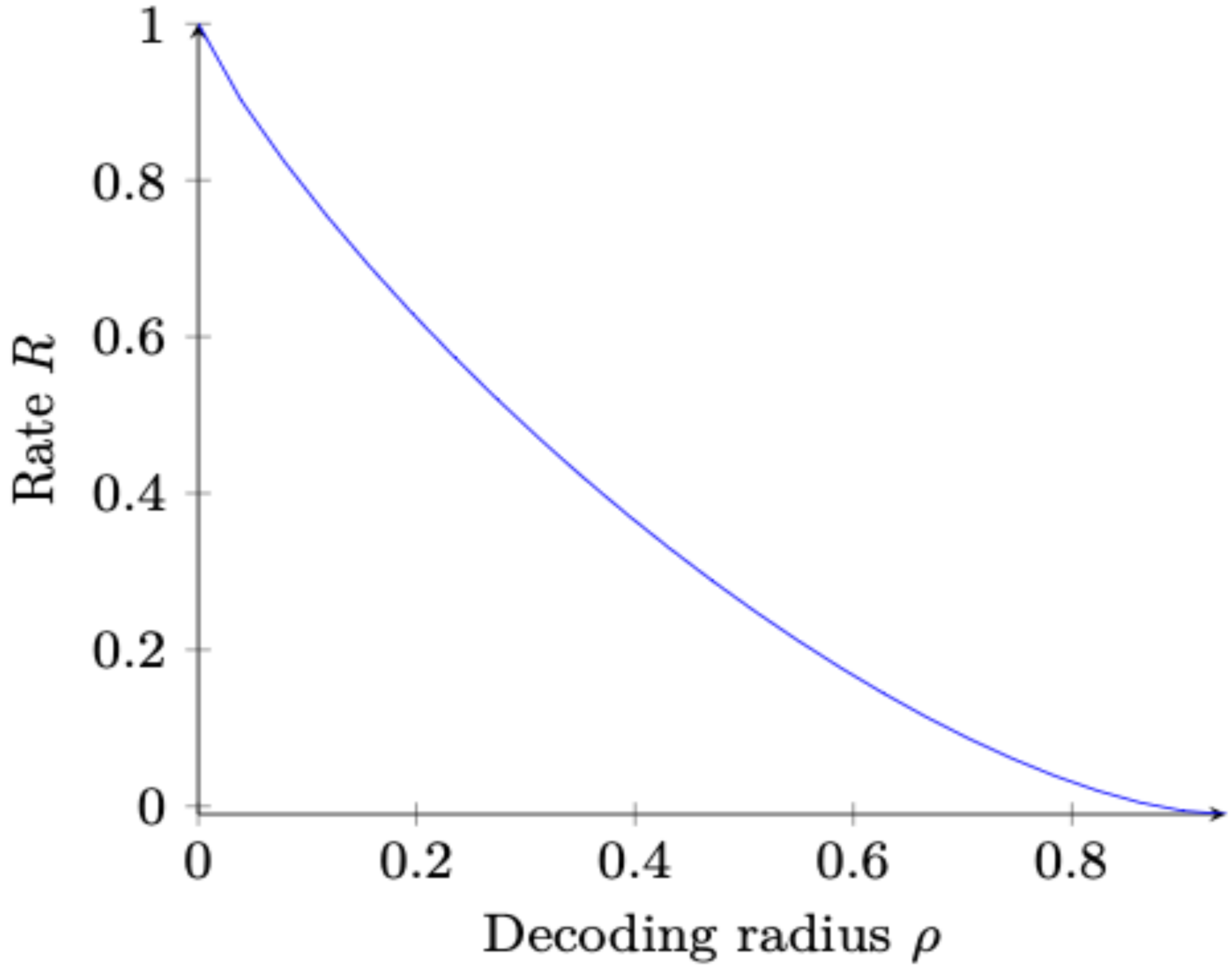
$$\max\{\text{GV}, \text{AG}\} \leq R$$



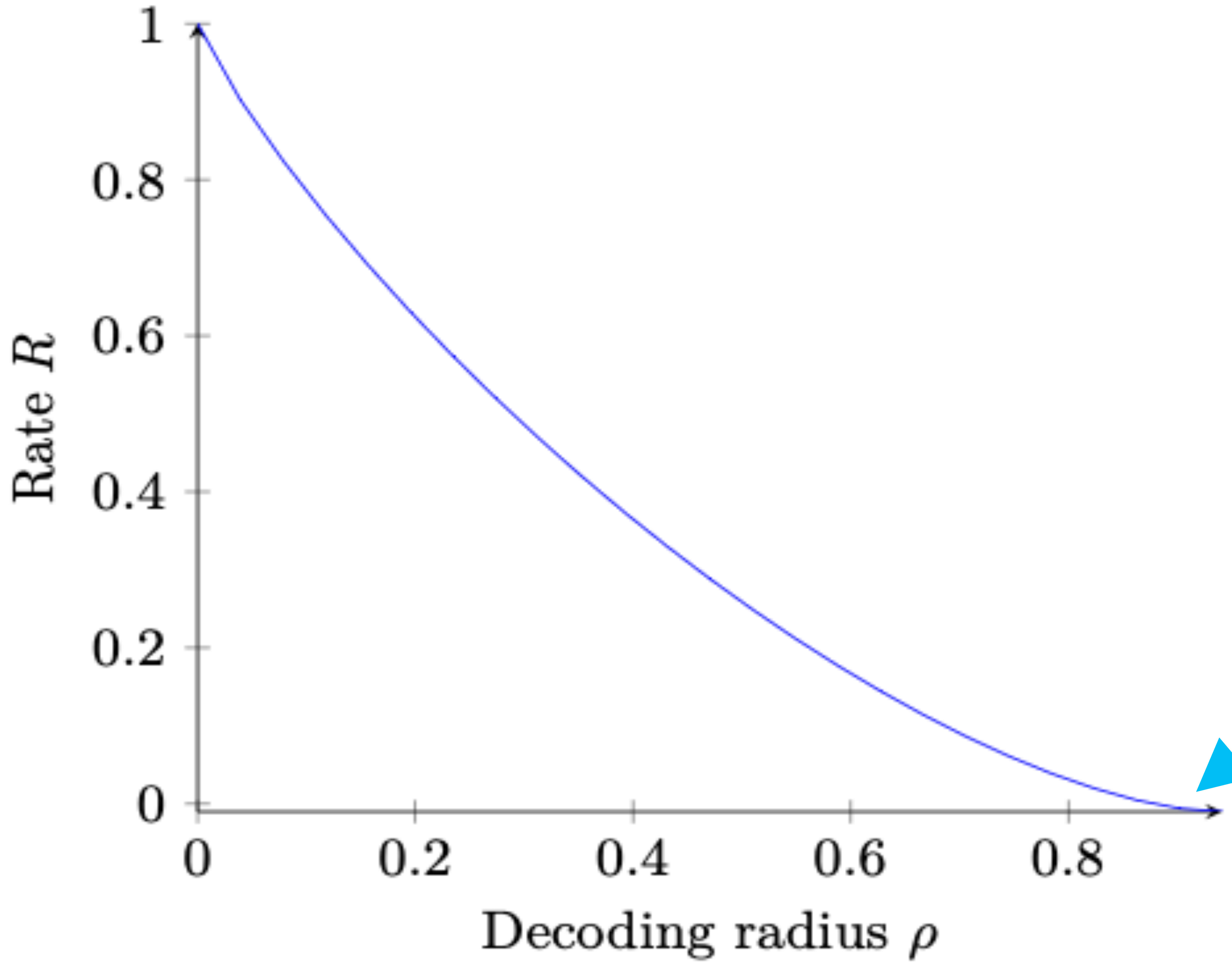
- Larger L, q: seems very difficult...

ZERO-RATE THRESHOLD

ZERO-RATE THRESHOLD

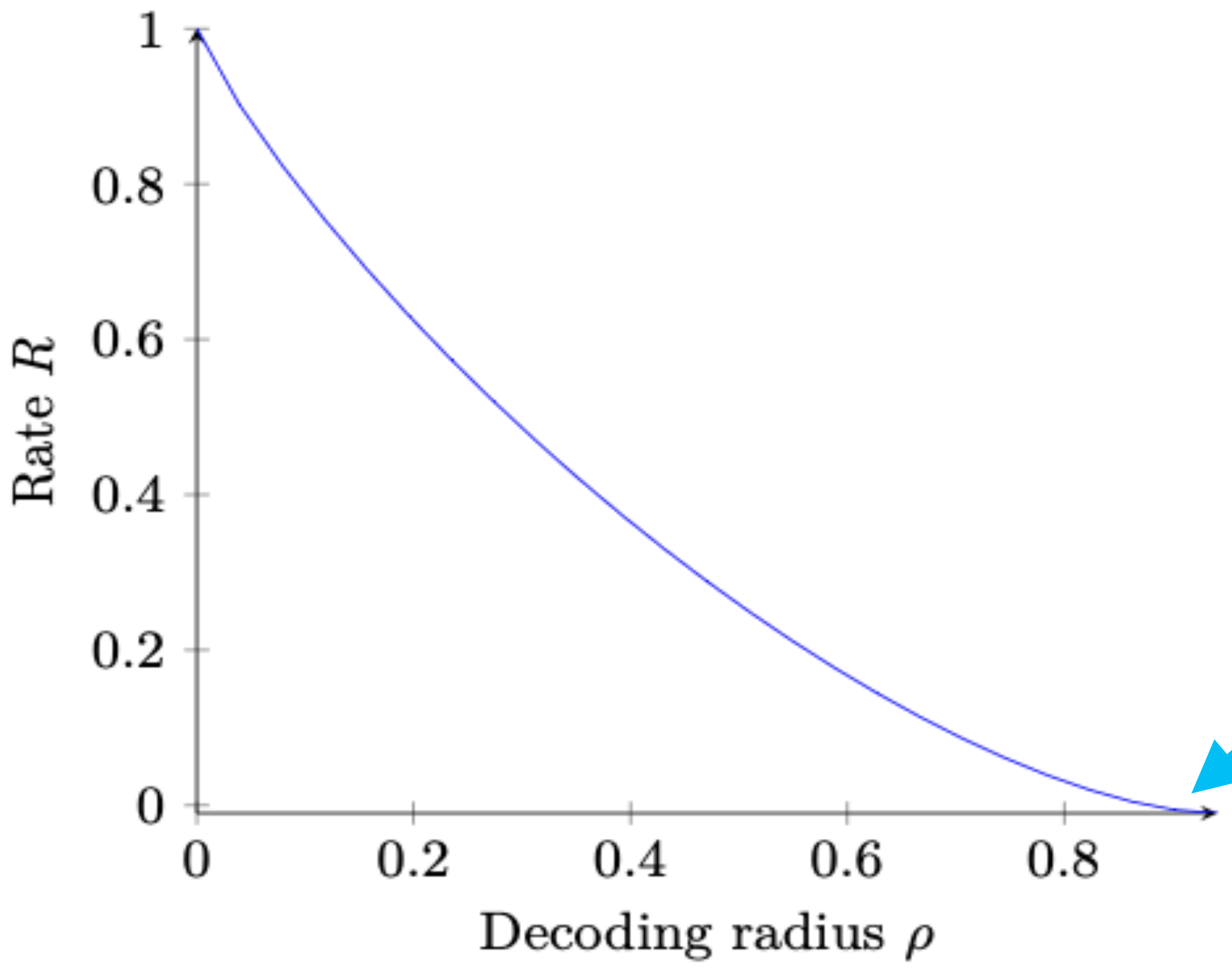


ZERO-RATE THRESHOLD



Determine this point?

ZERO-RATE THRESHOLD



Determine this point?

Zero-rate threshold $\rho_*(q, L)$

PLOTKIN BOUND

PLOTKIN BOUND

- Plotkin '60's proved that if $\mathscr{C} \subseteq \{0,1\}^n$ is uniquely-decodable from $(1/4 + \epsilon)n$ errors then $|\mathscr{C}| \leq O(1/\epsilon)$

PLOTKIN BOUND

- Plotkin '60's proved that if $\mathscr{C} \subseteq \{0,1\}^n$ is uniquely-decodable from $(1/4 + \epsilon)n$ errors then $|\mathscr{C}| \leq O(1/\epsilon)$
  - Thus, $\rho_*(2,2) = 1/4$

PLOTKIN BOUND

- Plotkin '60's proved that if $\mathscr{C} \subseteq \{0,1\}^n$ is uniquely-decodable from $(1/4 + \epsilon)n$ errors then $|\mathscr{C}| \leq O(1/\epsilon)$
  - Thus, $\rho_*(2,2) = 1/4$
- Similarly: $\rho_*(q,2) = \dfrac{q-1}{2q}$

PLOTKIN BOUND

- Plotkin '60's proved that if $\mathscr{C} \subseteq \{0,1\}^n$ is uniquely-decodable from $(1/4 + \epsilon)n$ errors then $|\mathscr{C}| \leq O(1/\epsilon)$
  - Thus, $\rho_*(2,2) = 1/4$
- Similarly: $\rho_*(q,2) = \frac{q-1}{2q}$

$L = 2$ case is solved!

PLOTKIN BOUND

- Plotkin '60's proved that if $\mathscr{C} \subseteq \{0,1\}^n$ is uniquely-decodable from $(1/4 + \epsilon)n$ errors then $|\mathscr{C}| \leq O(1/\epsilon)$
  - Thus, $\rho_*(2,2) = 1/4$
- Similarly: $\rho_*(q,2) = \dfrac{q-1}{2q}$
- Blinovsky '86:

$$\rho_*(2,L) = \frac{1}{2} - \frac{\binom{2k}{k}}{2^{2k+1}},\ L \in \{2k, 2k+1\}$$

$L = 2$ case is solved!

PLOTKIN BOUND

- Plotkin '60's proved that if $\mathscr{C} \subseteq \{0,1\}^n$ is uniquely-decodable from $(1/4 + \epsilon)n$ errors then $|\mathscr{C}| \leq O(1/\epsilon)$
  - Thus, $\rho_*(2,2) = 1/4$
- Similarly: $\rho_*(q,2) = \dfrac{q-1}{2q}$
- Blinovsky '86:

$$\rho_*(2,L) = \frac{1}{2} - \frac{\binom{2k}{k}}{2^{2k+1}},\ L \in \{2k, 2k+1\}$$

$L = 2$ case is solved!

$q = 2$ case is solved!

INTERPRETATION OF ZERO-RATE THRESHOLD

INTERPRETATION OF ZERO-RATE THRESHOLD

- The expression

$$\rho_*(2,L) = \frac{1}{2} - \frac{\binom{2k}{k}}{2^{2k+1}}, \; L \in \{2k, 2k+1\}$$

has the following probabilistic interpretation

INTERPRETATION OF ZERO-RATE THRESHOLD

- The expression

$$\rho_*(2,L) = \frac{1}{2} - \frac{\binom{2k}{k}}{2^{2k+1}}, \; L \in \{2k, 2k+1\}$$

has the following probabilistic interpretation

$$\rho_*(2,L) = 1 - \frac{1}{L} \mathop{\mathbb{E}}_{(X_1,\ldots,X_L)\sim \text{Unif}(\{0,1\})^L} [\text{pl}(X_1, \ldots, X_L)]$$

INTERPRETATION OF ZERO-RATE THRESHOLD

- The expression

$$\rho_*(2,L) = \frac{1}{2} - \frac{\binom{2k}{k}}{2^{2k+1}},\ L \in \{2k, 2k+1\}$$

has the following probabilistic interpretation

$$\rho_*(2,L) = 1 - \frac{1}{L} \mathop{\mathbb{E}}_{(X_1,\ldots,X_L)\sim\text{Unif}(\{0,1\})^L} [\text{pl}(X_1,\ldots,X_L)]$$

L indep. unif. bits

INTERPRETATION OF ZERO-RATE THRESHOLD

- The expression

$$\rho_*(2,L) = \frac{1}{2} - \frac{\binom{2k}{k}}{2^{2k+1}},\ L \in \{2k, 2k+1\}$$

has the following probabilistic interpretation

L indep. unif. bits

$$\rho_*(2,L) = 1 - \frac{1}{L} \mathbb{E}_{(X_1,\ldots,X_L)\sim\text{Unif}(\{0,1\})^L}[\text{pl}(X_1,\ldots,X_L)]$$

$\text{pl}(x_1,\ldots,x_L) =$ # of times a most popular bit appears

INTERPRETATION OF ZERO-RATE THRESHOLD

- The expression

$$\rho_*(2,L) = \frac{1}{2} - \frac{\binom{2k}{k}}{2^{2k+1}}, \; L \in \{2k, 2k+1\}$$

has the following probabilistic interpretation

L indep. unif. bits

$$\rho_*(2,L) = 1 - \frac{1}{L} \mathop{\mathbb{E}}_{(X_1,\ldots,X_L)\sim \text{Unif}(\{0,1\})^L} [\text{pl}(X_1, \ldots, X_L)]$$

$\text{pl}(x_1, \ldots, x_L) =$ # of times a most popular bit appears

pl() $= 4$

CONJECTURE FOR GENERAL q

CONJECTURE FOR GENERAL q

- For general q, answer should be

$$\rho_*(q, L) = 1 - \frac{1}{L} \mathop{\mathbb{E}}_{(X_1,\ldots,X_L)\sim \text{Unif}([q])^L} [\text{pl}(X_1, \ldots, X_L)]$$

CONJECTURE FOR GENERAL q

- For general q, answer should be

$$\rho_*(q, L) = 1 - \frac{1}{L} \mathop{\mathbb{E}}_{(X_1,\ldots,X_L)\sim \text{Unif}([q])^L} [\text{pl}(X_1, \ldots, X_L)]$$

L indep. unif. elements from $[q]$

CONJECTURE FOR GENERAL q

- For general q, answer should be

$$\rho_*(q,L) = 1 - \frac{1}{L} \mathop{\mathbb{E}}_{(X_1,\ldots,X_L)\sim \text{Unif}([q])^L} [\text{pl}(X_1,\ldots,X_L)]$$

L indep. unif. elements from $[q]$

pl() = 4

CONJECTURE FOR GENERAL q

- For general q, answer should be

L indep. unif. elements from $[q]$

$$\rho_*(q,L) = 1 - \frac{1}{L} \mathop{\mathbb{E}}_{(X_1,\ldots,X_L)\sim \mathrm{Unif}([q])^L} [\mathrm{pl}(X_1,\ldots,X_L)]$$

pl() = 4

- Blinovsky '05, '08 claims a proof... but it's flawed

OUR RESULT

OUR RESULT

- For all $q \geq 2$ and $L \geq 2$, we prove the equality

$$\rho_*(q, L) = 1 - \frac{1}{L} \mathop{\mathbb{E}}_{(X_1,\ldots,X_L)\sim\text{Unif}([q])^L} [\text{pl}(X_1, \ldots, X_L)]$$

OUR RESULT

- For all $q \geq 2$ and $L \geq 2$, we prove the equality

$$\rho_*(q, L) = 1 - \frac{1}{L} \mathop{\mathbb{E}}_{(X_1,\ldots,X_L)\sim \mathrm{Unif}([q])^L} [\mathrm{pl}(X_1, \ldots, X_L)]$$

- Two parts:

OUR RESULT

- For all $q \geq 2$ and $L \geq 2$, we prove the equality

$$\rho_*(q, L) = 1 - \frac{1}{L} \mathop{\mathbb{E}}_{(X_1,\ldots,X_L)\sim \text{Unif}([q])^L} [\text{pl}(X_1, \ldots, X_L)]$$

- Two parts:

Possibility Result: $\exists$ positive rate $(\rho_* - \epsilon, L)$-list-dec. codes (standard random code argument)

OUR RESULT

- For all $q \geq 2$ and $L \geq 2$, we prove the equality

$$\rho_*(q, L) = 1 - \frac{1}{L} \mathbb{E}_{(X_1,\ldots,X_L)\sim \text{Unif}([q])^L} [\text{pl}(X_1, \ldots, X_L)]$$

- Two parts:

Possibility Result: $\exists$ positive rate $(\rho_* - \epsilon, L)$-list-dec. codes (standard random code argument)

Impossibility Result: If $\mathscr{C} \subseteq [q]^n$ is $(\rho_* + \epsilon, L)$-list-dec., then $|\mathscr{C}| \leq O_\epsilon(1)$

OUR RESULT

- For all $q \geq 2$ and $L \geq 2$, we prove the equality

$$\rho_*(q,L) = 1 - \frac{1}{L} \mathop{\mathbb{E}}_{(X_1,\ldots,X_L)\sim\text{Unif}([q])^L} [\text{pl}(X_1,\ldots,X_L)]$$

- Two parts:

Possibility Result: $\exists$ positive rate $(\rho_* - \epsilon, L)$-list-dec. codes (standard random code argument)

Impossibility Result: If $\mathscr{C} \subseteq [q]^n$ is $(\rho_* + \epsilon, L)$-list-dec., then $|\mathscr{C}| \leq O_\epsilon(1)$

Today's focus

OUR RESULT

- For all $q \geq 2$ and $L \geq 2$, we prove the equality

$$\rho_*(q,L) = 1 - \frac{1}{L} \mathop{\mathbb{E}}_{(X_1,\ldots,X_L)\sim \mathrm{Unif}([q])^L} [\mathrm{pl}(X_1,\ldots,X_L)]$$

- Two parts:

Possibility Result: $\exists$ positive rate $(\rho_* - \epsilon, L)$-list-dec. codes (standard random code argument)

Impossibility Result: If $\mathscr{C} \subseteq [q]^n$ is $(\rho_* + \epsilon, L)$-list-dec., then $|\mathscr{C}| \leq O_\epsilon(1)$

From this: derive new upper bounds on rate of (ρ, L)-list-decodable codes

Today's focus

OUR RESULT

Analogous result for list-recovery

From this: derive new upper bounds on rate of (ρ, L)-list-decodable codes

- For all $q \geq 2$ and $L \geq 2$, we prove the equality

$$\rho_*(q, L) = 1 - \frac{1}{L} \mathop{\mathbb{E}}_{(X_1,\ldots,X_L)\sim \mathrm{Unif}([q])^L} [\mathrm{pl}(X_1, \ldots, X_L)]$$

Today's focus

- Two parts:

<u>Possibility Result</u>: $\exists$ positive rate $(\rho_* - \epsilon, L)$-list-dec. codes (standard random code argument)

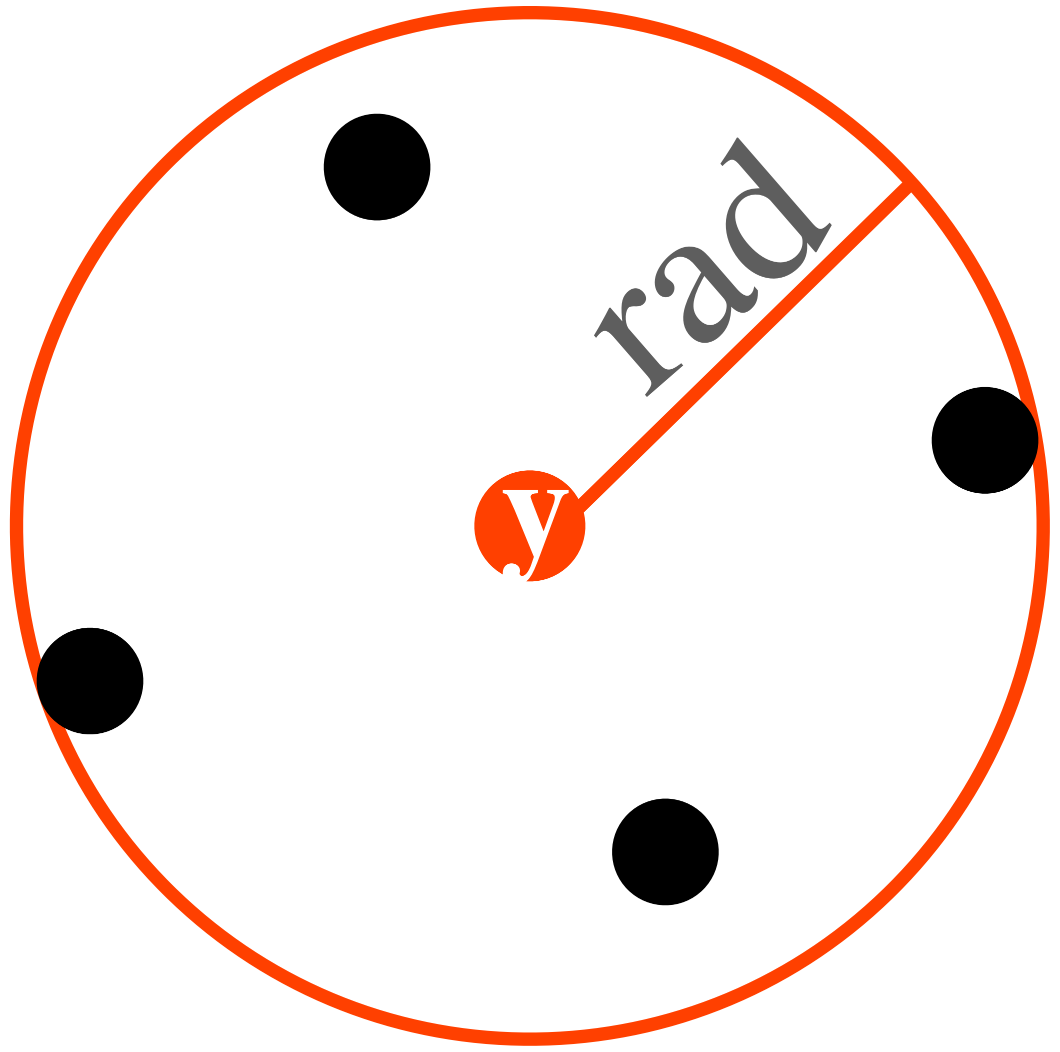
<u>Impossibility Result</u>: If $\mathscr{C} \subseteq [q]^n$ is $(\rho_* + \epsilon, L)$-list-dec., then $|\mathscr{C}| \leq O_\epsilon(1)$

HOW TO BOUND A CODE

RADIUS

- Given vectors $\mathbf{x}_1, \ldots, \mathbf{x}_L \in [q]^n$, their *radius* is

$$\mathrm{rad}(\mathbf{x}_1, \ldots, \mathbf{x}_L) = \min_{\mathbf{y} \in [q]^n} \max_{i \in [L]} d(\mathbf{x}_i, \mathbf{y})$$

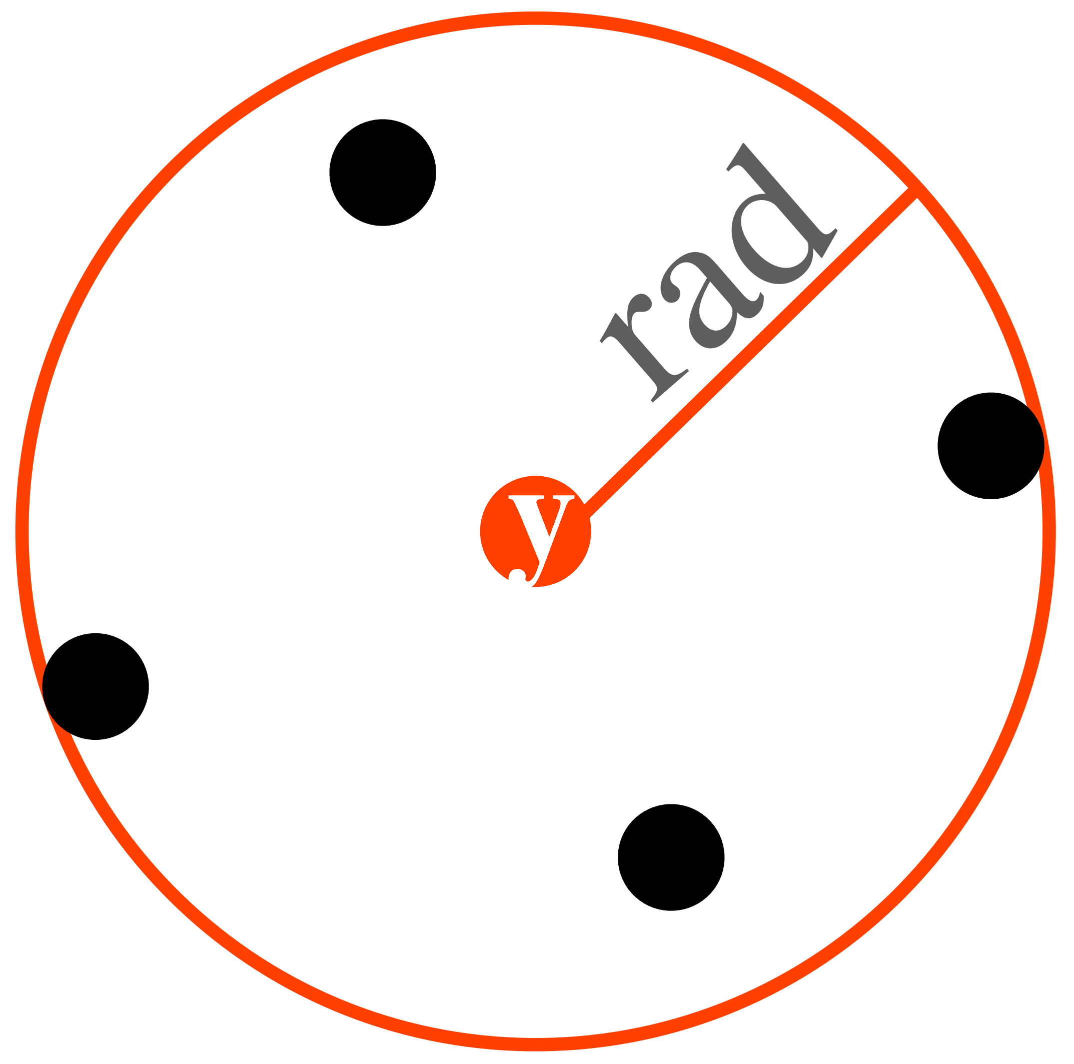


RADIUS

Radius of smallest ball containing $\{\mathbf{x}_1, \ldots, \mathbf{x}_L\}$

- Given vectors $\mathbf{x}_1, \ldots, \mathbf{x}_L \in [q]^n$, their *radius* is

$$\mathrm{rad}(\mathbf{x}_1, \ldots, \mathbf{x}_L) = \min_{\mathbf{y} \in [q]^n} \max_{i \in [L]} d(\mathbf{x}_i, \mathbf{y})$$



AVERAGE-RADIUS

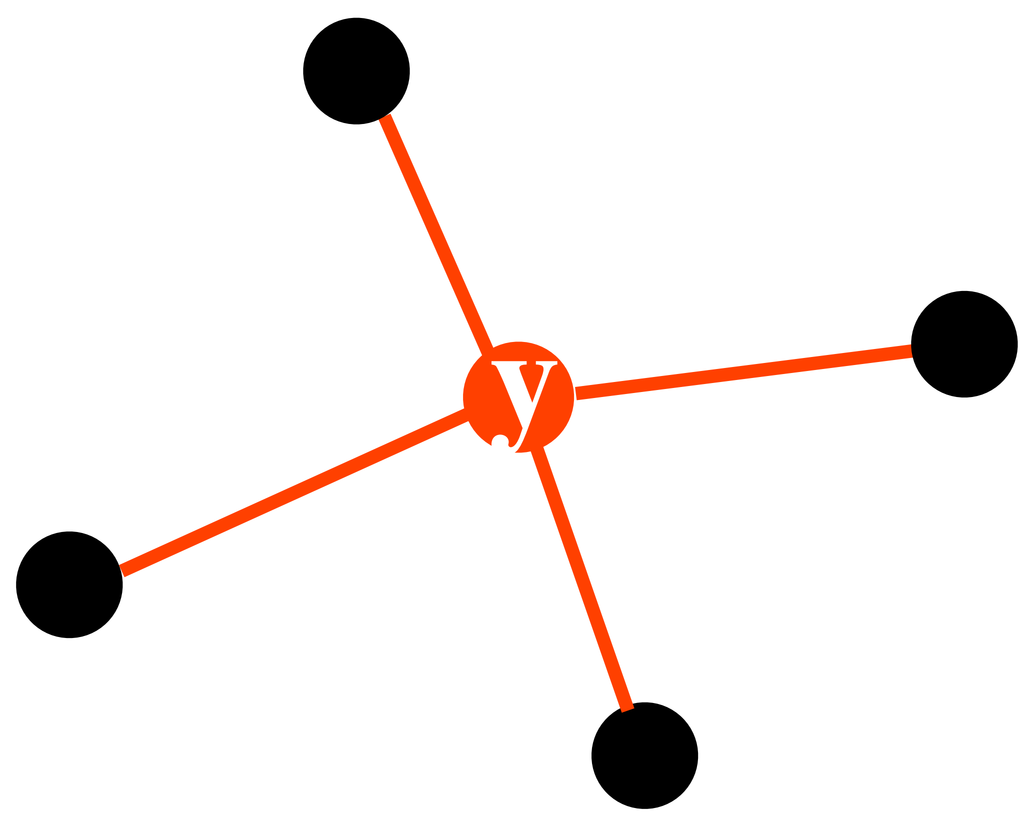
- Given vectors $\mathbf{x}_1, \ldots, \mathbf{x}_L \in [q]^n$, their *average-radius* is

$$\overline{\mathrm{rad}}(\mathbf{x}_1, \ldots, \mathbf{x}_L) = \min_{\mathbf{y} \in [q]^n} \frac{1}{L} \sum_{j \in [L]} d(\mathbf{x}_i, \mathbf{y}) = \sum_{i=1}^{n} \left(1 - \frac{1}{L} \mathsf{pl}(x_{1,i}, \ldots, x_{L,i}) \right)$$

AVERAGE-RADIUS

- Given vectors $\mathbf{x}_1, \ldots, \mathbf{x}_L \in [q]^n$, their *average-radius* is

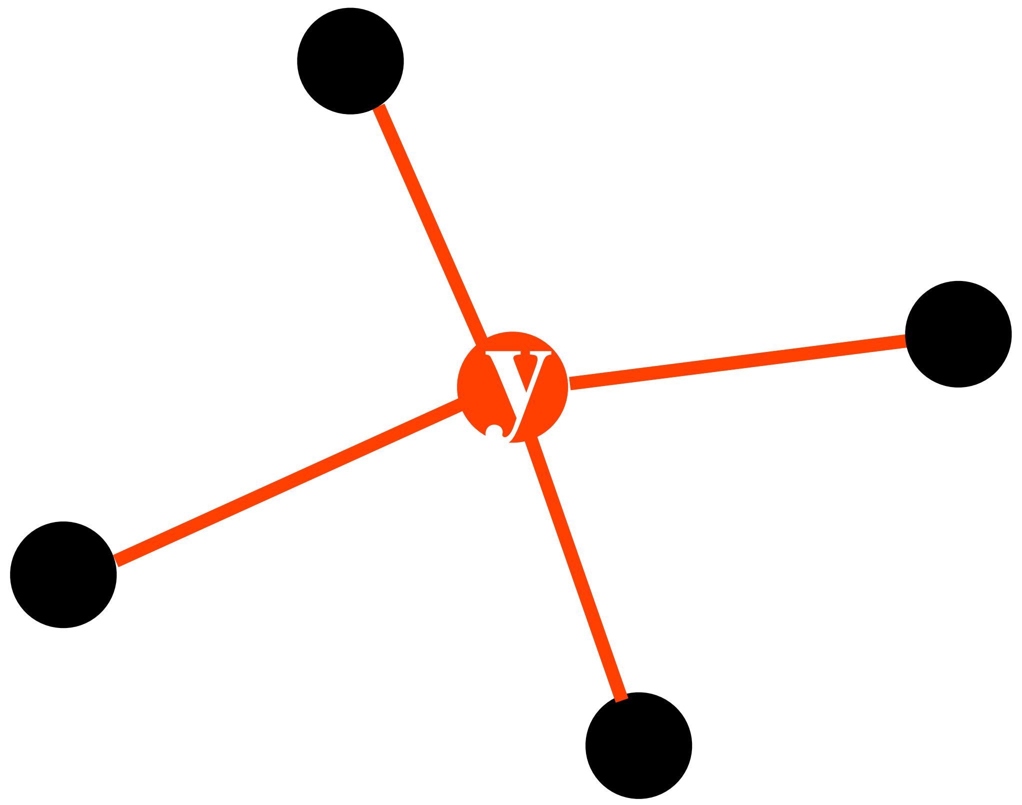
$$\overline{\mathrm{rad}}(\mathbf{x}_1, \ldots, \mathbf{x}_L) = \min_{\mathbf{y} \in [q]^n} \frac{1}{L} \sum_{j \in [L]} d(\mathbf{x}_i, \mathbf{y}) = \sum_{i=1}^{n} \left(1 - \frac{1}{L} \mathsf{pl}(x_{1,i}, \ldots, x_{L,i}) \right)$$



AVERAGE-RADIUS

- Given vectors $\mathbf{x}_1, \ldots, \mathbf{x}_L \in [q]^n$, their *average-radius* is

$$\overline{\mathrm{rad}}(\mathbf{x}_1, \ldots, \mathbf{x}_L) = \min_{\mathbf{y} \in [q]^n} \frac{1}{L} \sum_{j \in [L]} d(\mathbf{x}_i, \mathbf{y}) = \sum_{i=1}^{n} \left(1 - \frac{1}{L} \mathsf{pl}(x_{1,i}, \ldots, x_{L,i})\right)$$

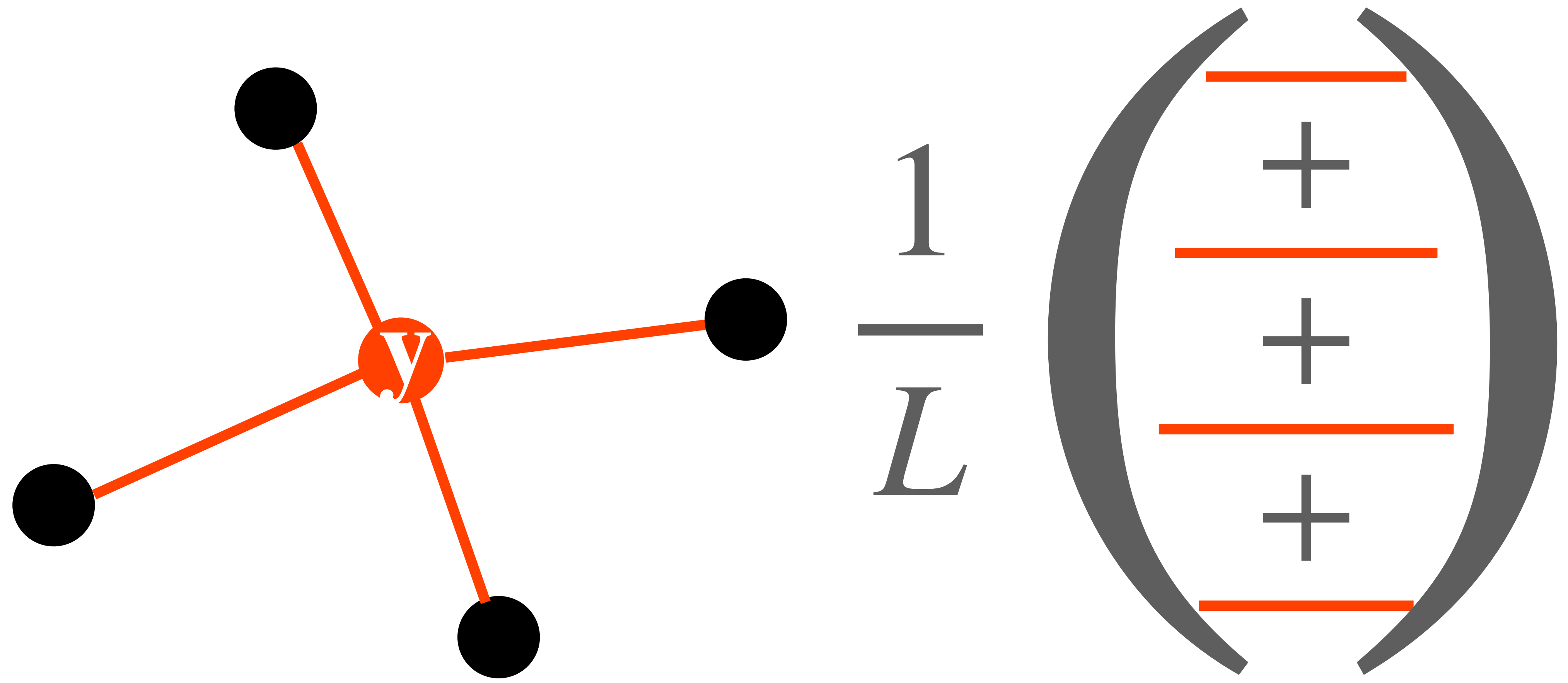


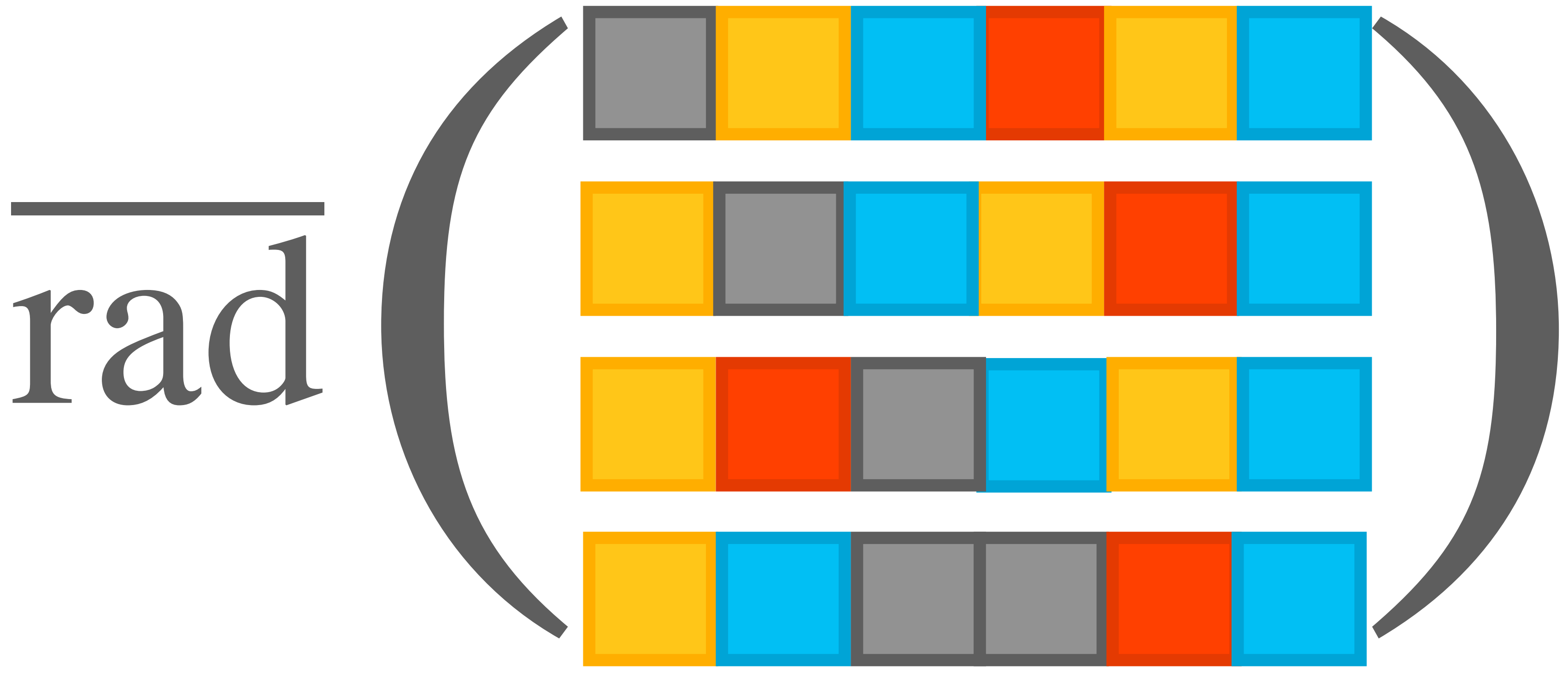
$$\frac{1}{L}\left(+ + + \right)$$

AVERAGE-RADIUS

- Given vectors $\mathbf{x}_1, \ldots, \mathbf{x}_L \in [q]^n$, their *average-radius* is

$$\overline{\mathrm{rad}}(\mathbf{x}_1, \ldots, \mathbf{x}_L) = \min_{\mathbf{y} \in [q]^n} \frac{1}{L} \sum_{j \in [L]} d(\mathbf{x}_i, \mathbf{y}) = \sum_{i=1}^{n} \left(1 - \frac{1}{L} \mathrm{pl}(x_{1,i}, \ldots, x_{L,i}) \right)$$

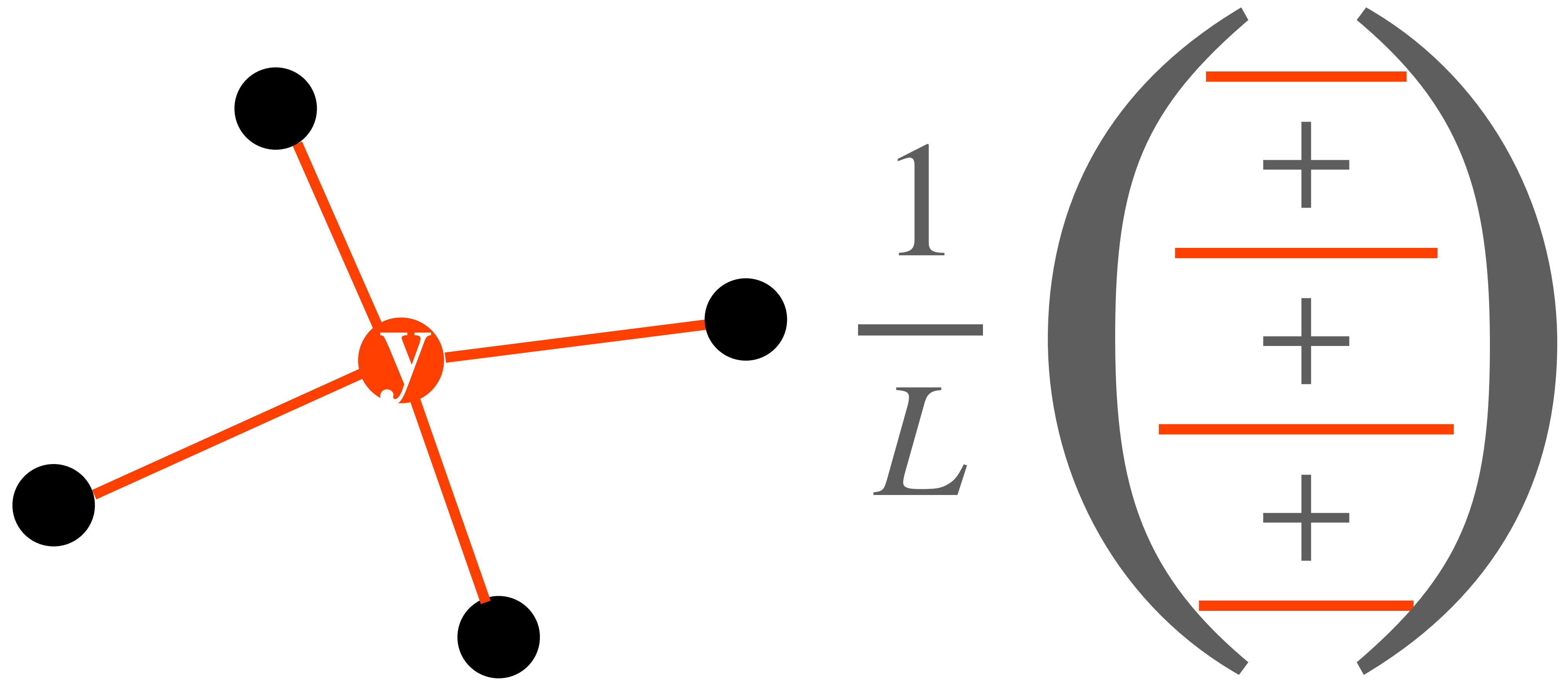


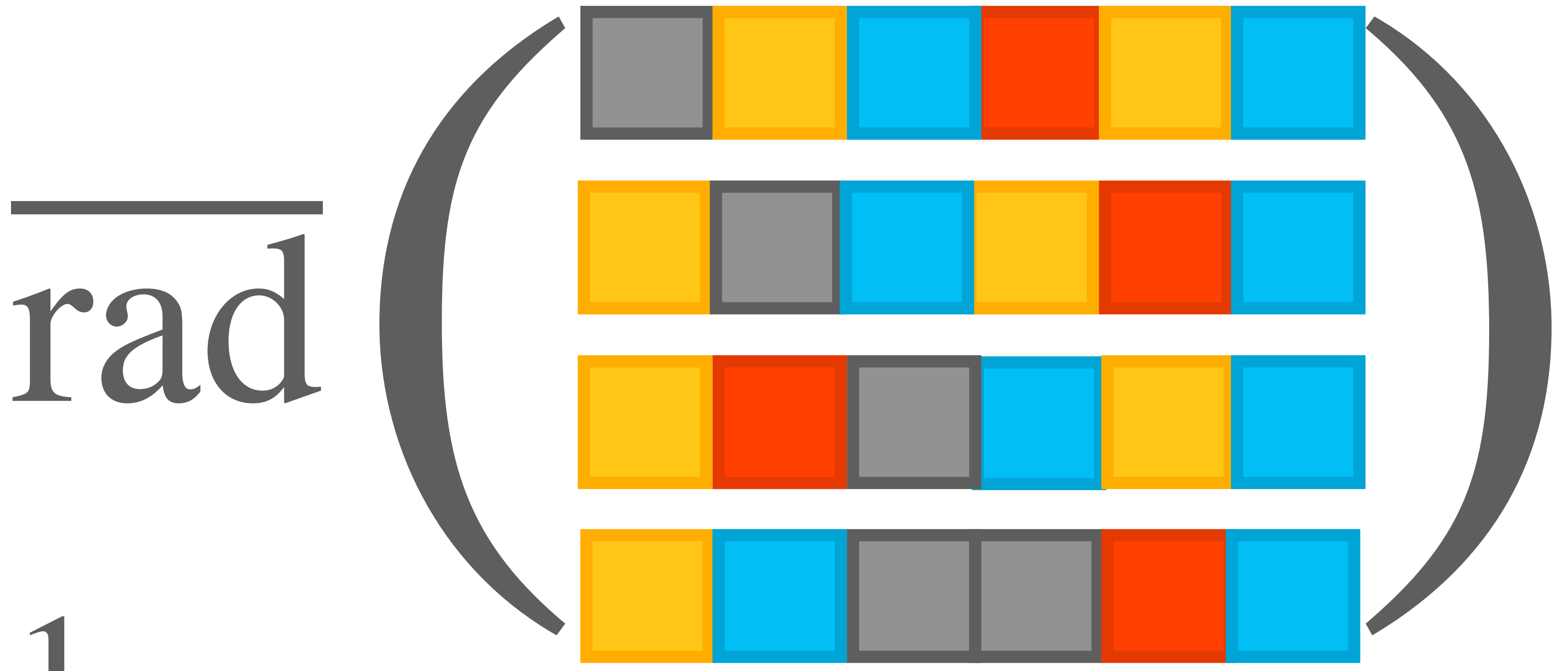


AVERAGE-RADIUS

- Given vectors $\mathbf{x}_1, \ldots, \mathbf{x}_L \in [q]^n$, their *average-radius* is

$$\overline{\mathrm{rad}}(\mathbf{x}_1, \ldots, \mathbf{x}_L) = \min_{\mathbf{y} \in [q]^n} \frac{1}{L} \sum_{j \in [L]} d(\mathbf{x}_i, \mathbf{y}) = \sum_{i=1}^{n} \left(1 - \frac{1}{L} \mathrm{pl}(x_{1,i}, \ldots, x_{L,i}) \right)$$



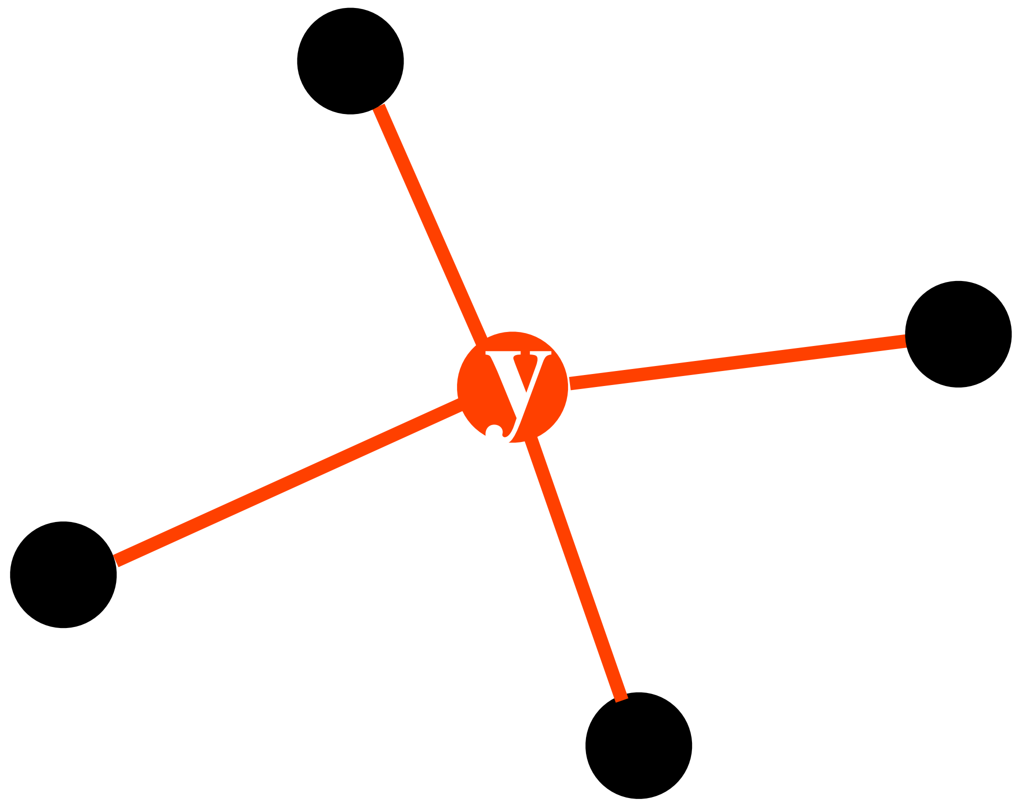


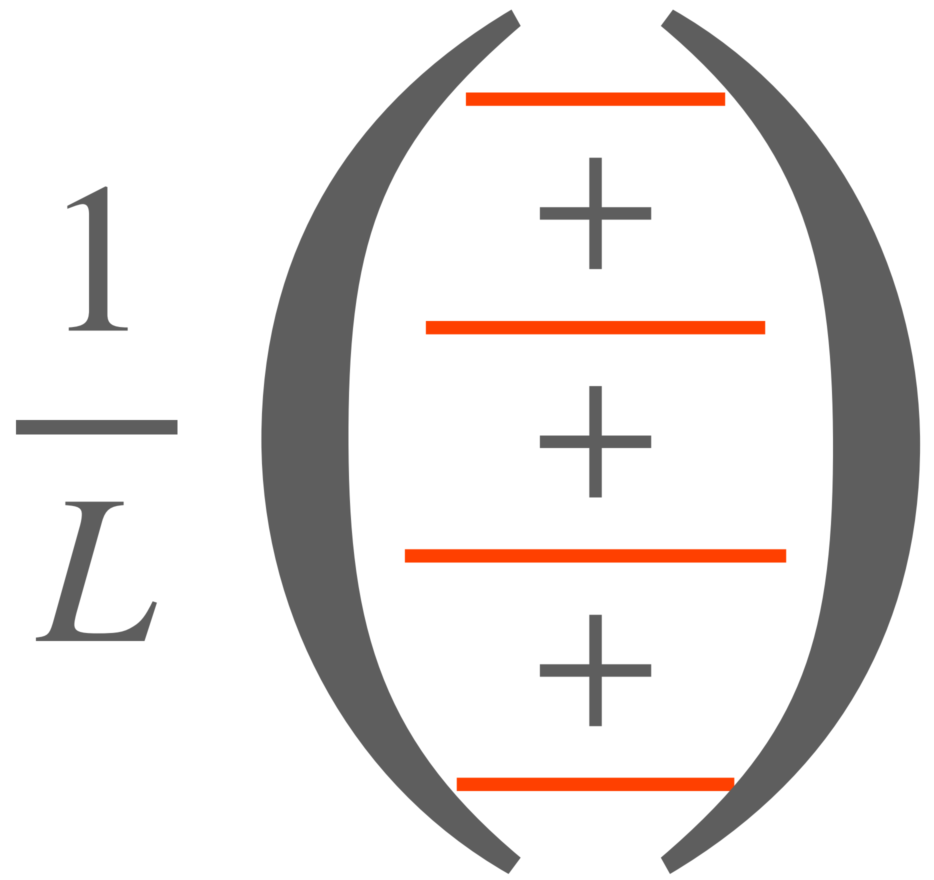
$$1 - \frac{1}{L} \mathrm{pl}(\,\cdot\,)$$

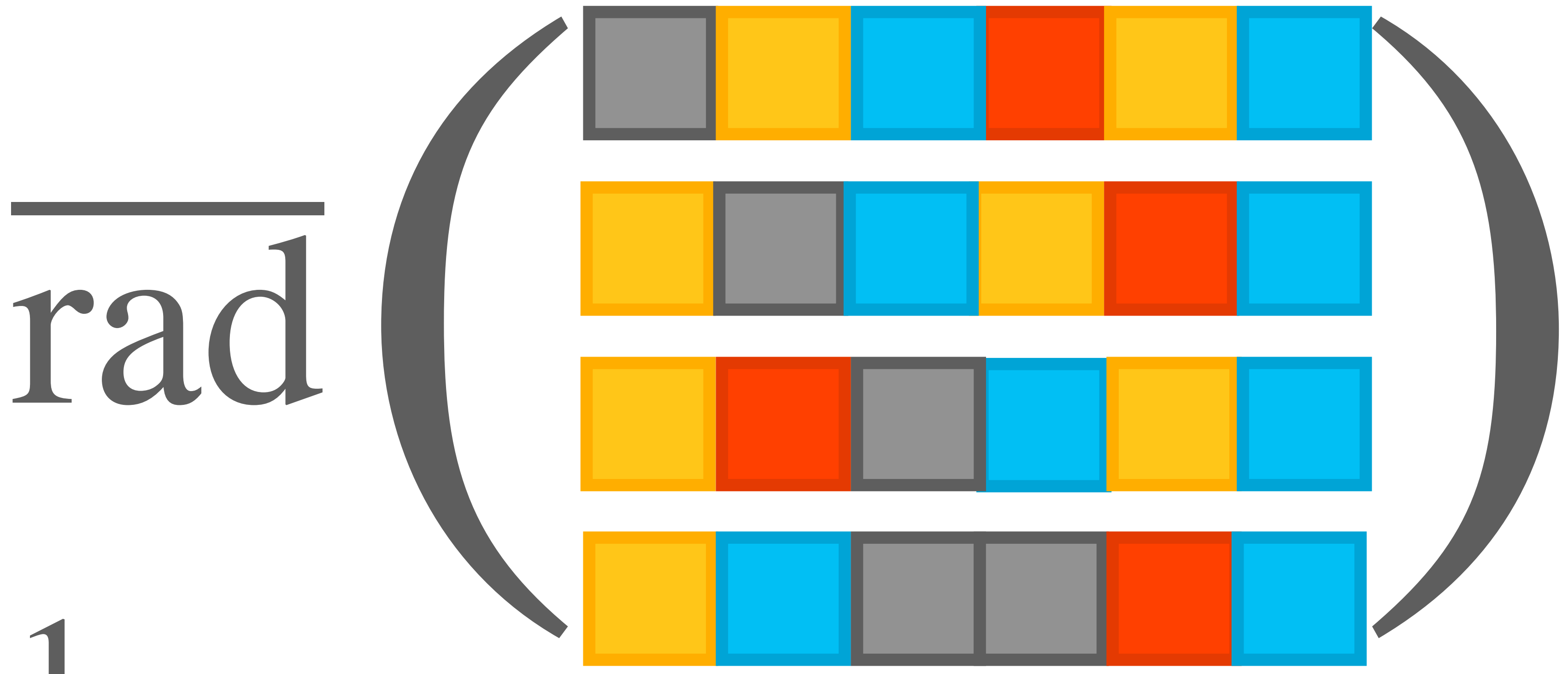
AVERAGE-RADIUS

- Given vectors $\mathbf{x}_1, \ldots, \mathbf{x}_L \in [q]^n$, their *average-radius* is

$$\overline{\mathrm{rad}}(\mathbf{x}_1, \ldots, \mathbf{x}_L) = \min_{\mathbf{y} \in [q]^n} \frac{1}{L} \sum_{j \in [L]} d(\mathbf{x}_i, \mathbf{y}) = \sum_{i=1}^{n} \left(1 - \frac{1}{L} \mathrm{pl}(x_{1,i}, \ldots, x_{L,i}) \right)$$





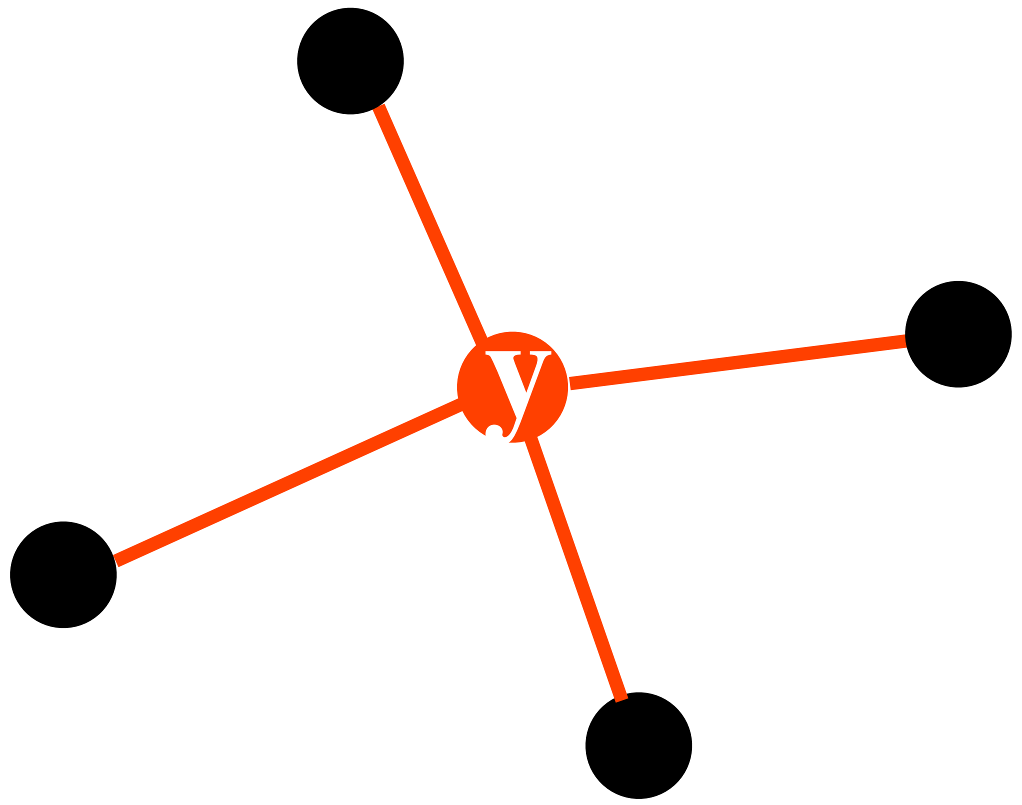


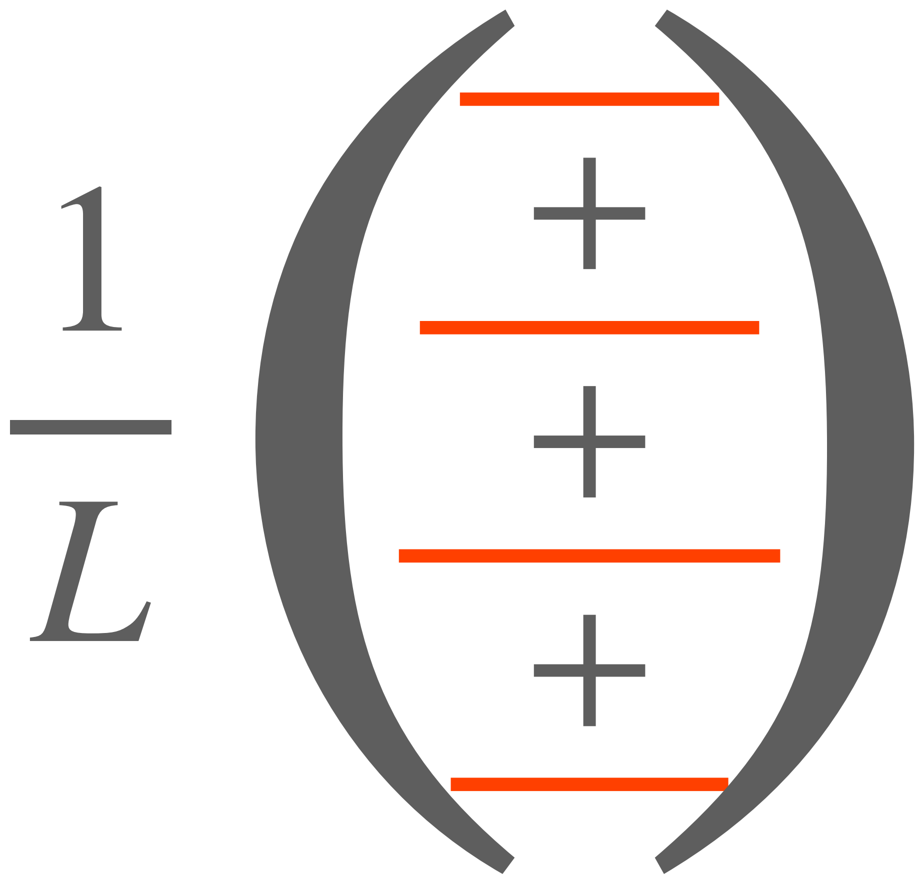
$1 - \frac{1}{L}\mathrm{pl}(\,\cdot\,)$ .25 .75 .5 .75 .5 0

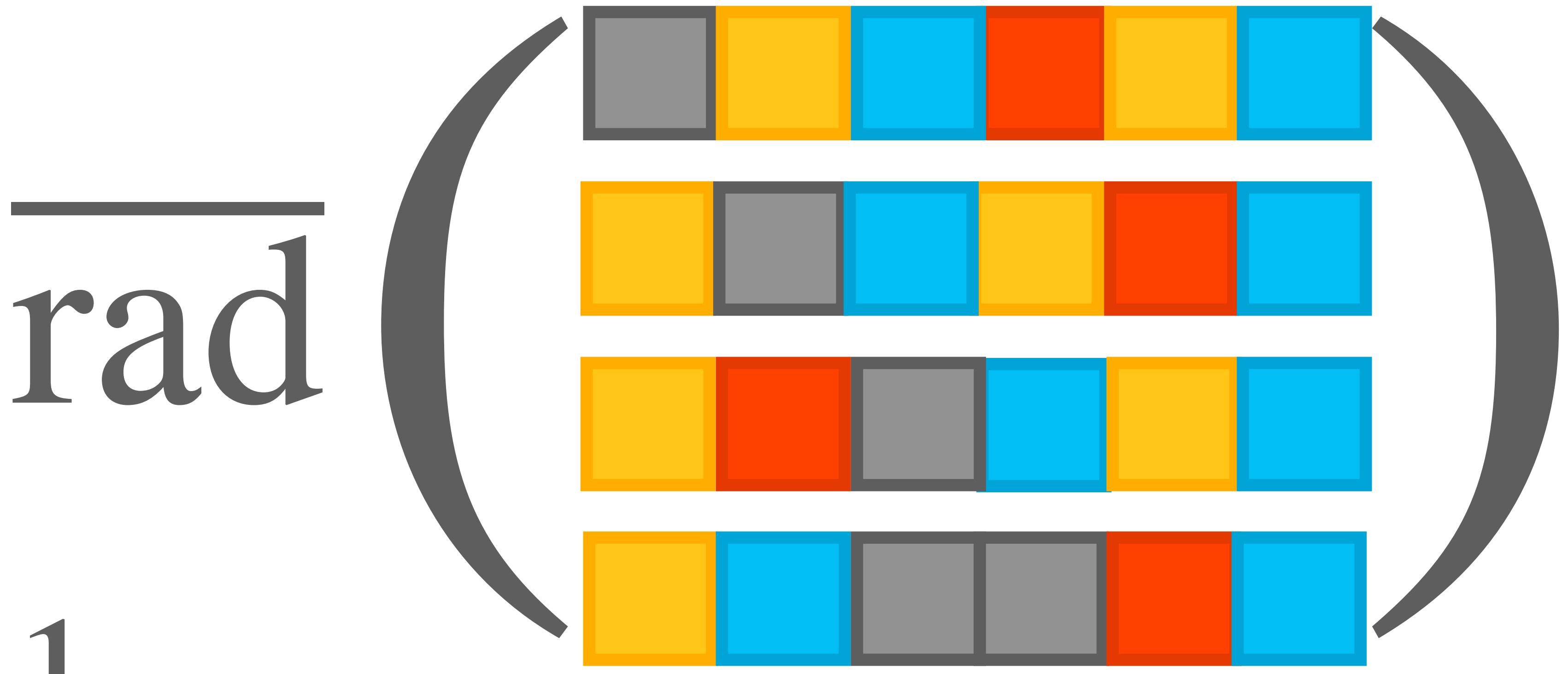
AVERAGE-RADIUS

- Given vectors $\mathbf{x}_1, \ldots, \mathbf{x}_L \in [q]^n$, their *average-radius* is

$$\overline{\text{rad}}(\mathbf{x}_1, \ldots, \mathbf{x}_L) = \min_{\mathbf{y} \in [q]^n} \frac{1}{L} \sum_{j \in [L]} d(\mathbf{x}_i, \mathbf{y}) = \sum_{i=1}^{n} \left(1 - \frac{1}{L} \mathsf{pl}(x_{1,i}, \ldots, x_{L,i}) \right)$$







$$1 - \frac{1}{L}\mathsf{pl}(\,\cdot\,) \quad .25 \; .75 \; .5 \; .75 \; .5 \; 0 = 1.75$$

CENTRAL QUANTITY

- Suppose $\mathscr{C} \subseteq [q]^n$, $|\mathscr{C}| = M$, is $(\rho_* + \epsilon, L)$-list-dec with M "too big"

CENTRAL QUANTITY

"too big" $= O_{q,L,\epsilon}(1)$

(independent of n!)

- Suppose $\mathscr{C} \subseteq [q]^n$, $|\mathscr{C}| = M$, is $(\rho_* + \epsilon, L)$-list-dec with M "too big"

CENTRAL QUANTITY

"too big" $= O_{q,L,\epsilon}(1)$

(independent of n!)

- Suppose $\mathscr{C} \subseteq [q]^n$, $|\mathscr{C}| = M$, is $(\rho_* + \epsilon, L)$-list-dec with M "too big"

$$\Phi = \frac{1}{M^L} \sum_{(\mathbf{x}_1,\ldots,\mathbf{x}_L)\in\mathscr{C}^L} \overline{\mathrm{rad}}(\mathbf{x}_1, \ldots, \mathbf{x}_L)$$

CENTRAL QUANTITY

"too big" $= O_{q,L,\epsilon}(1)$

(independent of n!)

- Suppose $\mathscr{C} \subseteq [q]^n$, $|\mathscr{C}| = M$, is $(\rho_* + \epsilon, L)$-list-dec with M "too big"

$$\Phi = \frac{1}{M^L} \sum_{(\mathbf{x}_1,\ldots,\mathbf{x}_L)\in\mathscr{C}^L} \overline{\mathrm{rad}}(\mathbf{x}_1, \ldots, \mathbf{x}_L)$$

- After "preprocessing"*, may ensure

CENTRAL QUANTITY

"too big" $= O_{q,L,\epsilon}(1)$

(independent of n!)

- Suppose $\mathscr{C} \subseteq [q]^n$, $|\mathscr{C}| = M$, is $(\rho_* + \epsilon, L)$-list-dec with M "too big"

$$\Phi = \frac{1}{M^L} \sum_{(\mathbf{x}_1,\ldots,\mathbf{x}_L)\in\mathscr{C}^L} \overline{\mathrm{rad}}(\mathbf{x}_1, \ldots, \mathbf{x}_L)$$

- After "preprocessing"*, may ensure

$$\Phi \approx \frac{1}{M^L} \sum_{(\mathbf{x}_1,\ldots,\mathbf{x}_L)\in\mathscr{C}^L} \mathrm{rad}(\mathbf{x}_1, \ldots, \mathbf{x}_L) \geq (\rho_* + \epsilon)n$$

CENTRAL QUANTITY

"too big" $= O_{q,L,\epsilon}(1)$

(independent of n!)

- Suppose $\mathscr{C} \subseteq [q]^n$, $|\mathscr{C}| = M$, is $(\rho_* + \epsilon, L)$-list-dec with M "too big"

$$\Phi = \frac{1}{M^L} \sum_{(\mathbf{x}_1,\ldots,\mathbf{x}_L)\in\mathscr{C}^L} \overline{\mathrm{rad}}(\mathbf{x}_1, \ldots, \mathbf{x}_L)$$

- After "preprocessing"*, may ensure

*pass to "very symmetric" subcode of size $\Omega(M)$

$$\Phi \approx \frac{1}{M^L} \sum_{(\mathbf{x}_1,\ldots,\mathbf{x}_L)\in\mathscr{C}^L} \mathrm{rad}(\mathbf{x}_1, \ldots, \mathbf{x}_L) \geq (\rho_* + \epsilon)n$$

CENTRAL QUANTITY

"too big" $= O_{q,L,\epsilon}(1)$

(independent of n!)

- Suppose $\mathscr{C} \subseteq [q]^n$, $|\mathscr{C}| = M$, is $(\rho_* + \epsilon, L)$-list-dec with M "too big"

$$\Phi = \frac{1}{M^L} \sum_{(\mathbf{x}_1,\ldots,\mathbf{x}_L)\in\mathscr{C}^L} \overline{\mathrm{rad}}(\mathbf{x}_1, \ldots, \mathbf{x}_L)$$

Ramsey-theory...

- After "preprocessing"*, may ensure

*pass to "very symmetric" subcode of size $\Omega(M)$

$$\Phi \approx \frac{1}{M^L} \sum_{(\mathbf{x}_1,\ldots,\mathbf{x}_L)\in\mathscr{C}^L} \mathrm{rad}(\mathbf{x}_1, \ldots, \mathbf{x}_L) \geq (\rho_* + \epsilon)n$$

WE NEED AN UPPER BOUND

$$\Phi = \frac{1}{M^L} \sum_{(\mathbf{x}_1,\ldots,\mathbf{x}_L)\in\mathscr{C}^L} \overline{\mathrm{rad}}(\mathbf{x}_1, \ldots, \mathbf{x}_L)$$

WE NEED AN UPPER BOUND

$$\Phi = \frac{1}{M^L} \sum_{(\mathbf{x}_1,\ldots,\mathbf{x}_L)\in\mathscr{C}^L} \overline{\mathrm{rad}}(\mathbf{x}_1, \ldots, \mathbf{x}_L)$$

$$= \cdots = \sum_{k=1}^{n} \sum_{(y_1,\ldots,y_L)\in[q]^L} \left(\prod_{j=1}^{L} P_k(y_j) \right) \left(1 - \frac{1}{L}\mathsf{pl}(y_1, \ldots, y_L) \right)$$

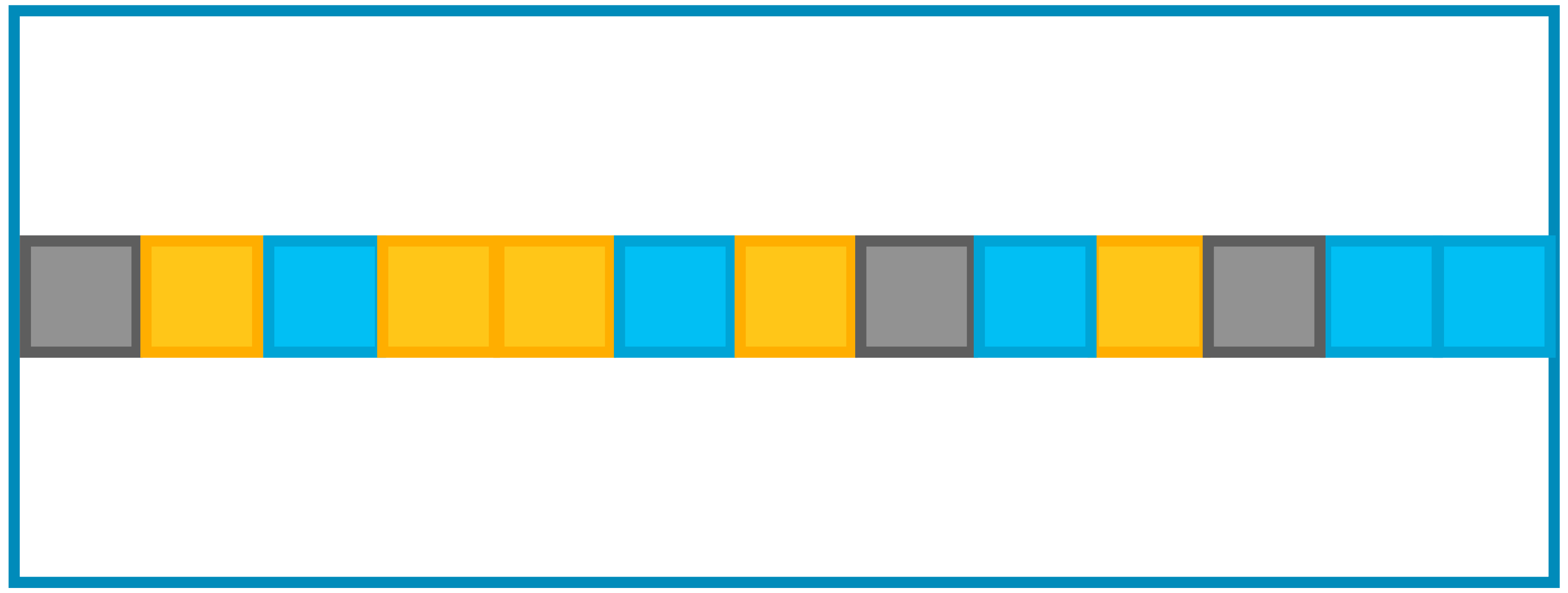
WE NEED AN UPPER BOUND

$$\Phi = \frac{1}{M^L} \sum_{(\mathbf{x}_1,\ldots,\mathbf{x}_L)\in\mathscr{C}^L} \overline{\text{rad}}(\mathbf{x}_1, \ldots, \mathbf{x}_L)$$

$$= \cdots = \sum_{k=1}^{n} \sum_{(y_1,\ldots,y_L)\in[q]^L} \left(\prod_{j=1}^{L} P_k(y_j) \right) \left(1 - \frac{1}{L}\mathsf{pl}(y_1, \ldots, y_L) \right)$$

$$P_k(y) = \frac{1}{M} \sum_{\mathbf{x}\in\mathscr{C}} 1\{x_k = y\}$$

WE NEED AN UPPER BOUND

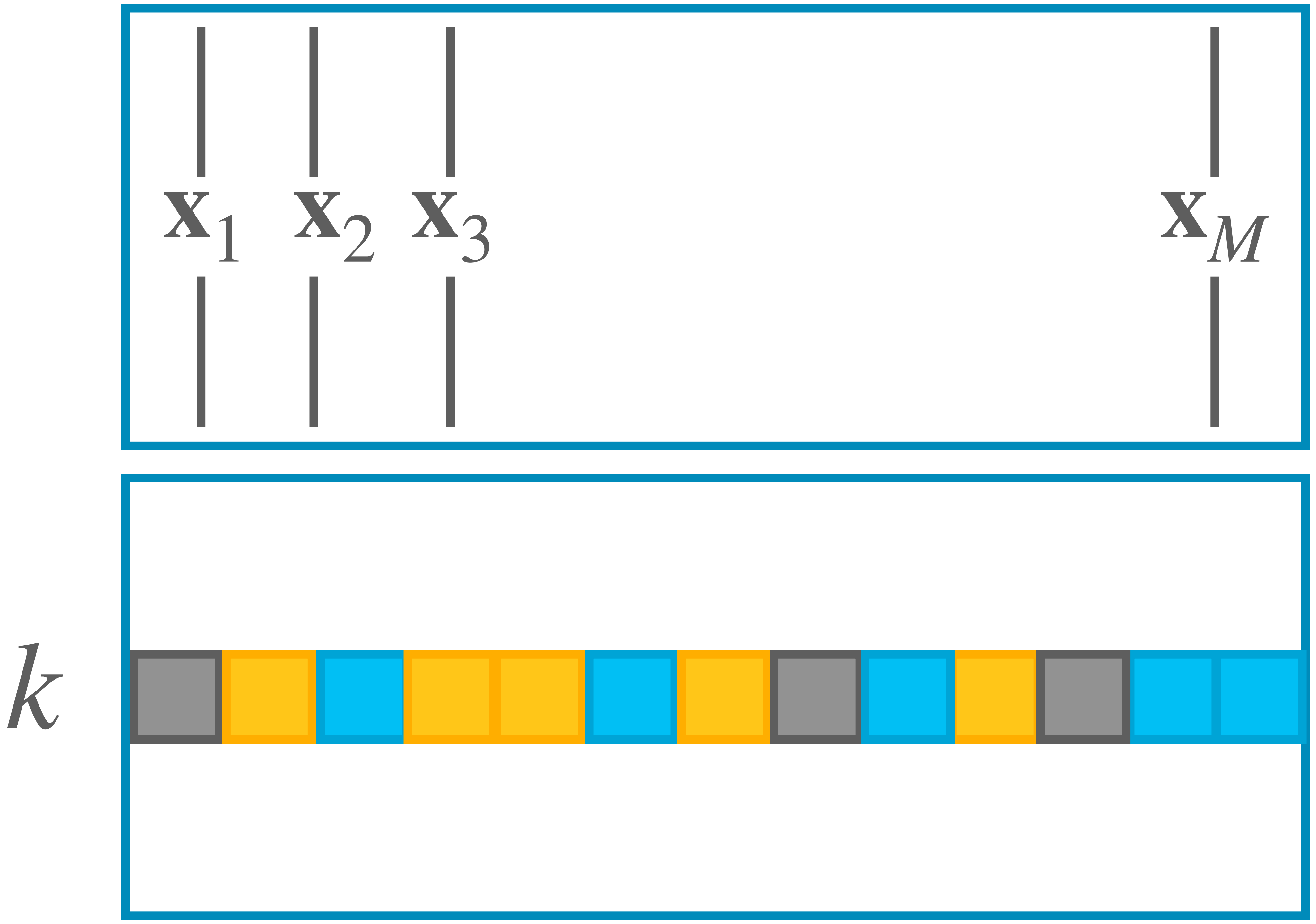


$$\Phi = \frac{1}{M^L} \sum_{(\mathbf{x}_1,\ldots,\mathbf{x}_L)\in\mathscr{C}^L} \overline{\mathrm{rad}}(\mathbf{x}_1, \ldots, \mathbf{x}_L)$$

$$= \cdots = \sum_{k=1}^{n} \sum_{(y_1,\ldots,y_L)\in[q]^L} \left(\prod_{j=1}^{L} P_k(y_j) \right) \left(1 - \frac{1}{L}\mathsf{pl}(y_1, \ldots, y_L) \right)$$

$$P_k(y) = \frac{1}{M} \sum_{\mathbf{x}\in\mathscr{C}} 1\{x_k = y\}$$

WE NEED AN UPPER BOUND

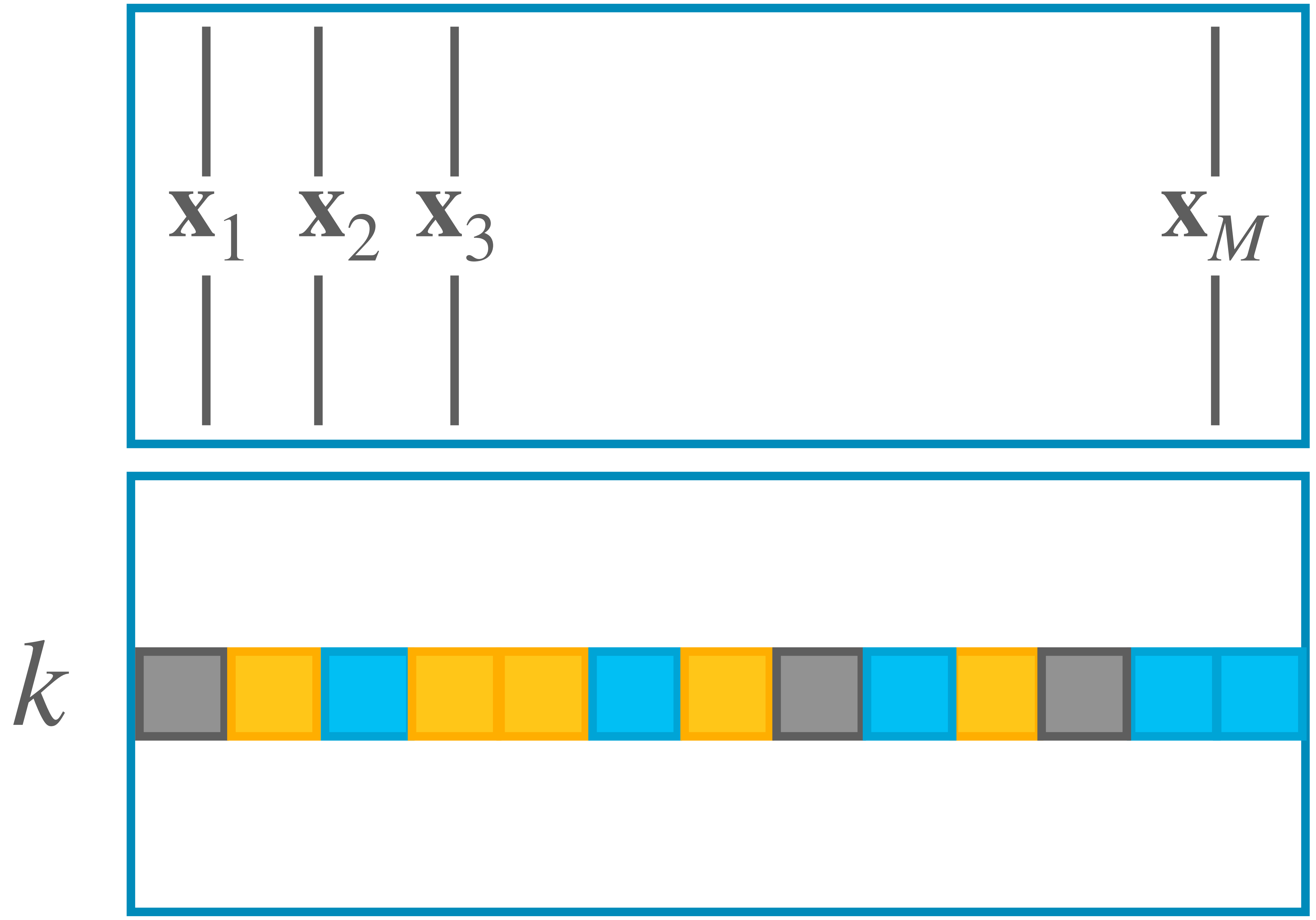


$$\Phi = \frac{1}{M^L} \sum_{(\mathbf{x}_1,\ldots,\mathbf{x}_L)\in\mathscr{C}^L} \overline{\mathrm{rad}}(\mathbf{x}_1, \ldots, \mathbf{x}_L)$$

$$= \cdots = \sum_{k=1}^{n} \sum_{(y_1,\ldots,y_L)\in[q]^L} \left(\prod_{j=1}^{L} P_k(y_j) \right) \left(1 - \tfrac{1}{L}\mathsf{pl}(y_1, \ldots, y_L) \right)$$

$$P_k(y) = \frac{1}{M} \sum_{\mathbf{x}\in\mathscr{C}} 1\{x_k = y\}$$

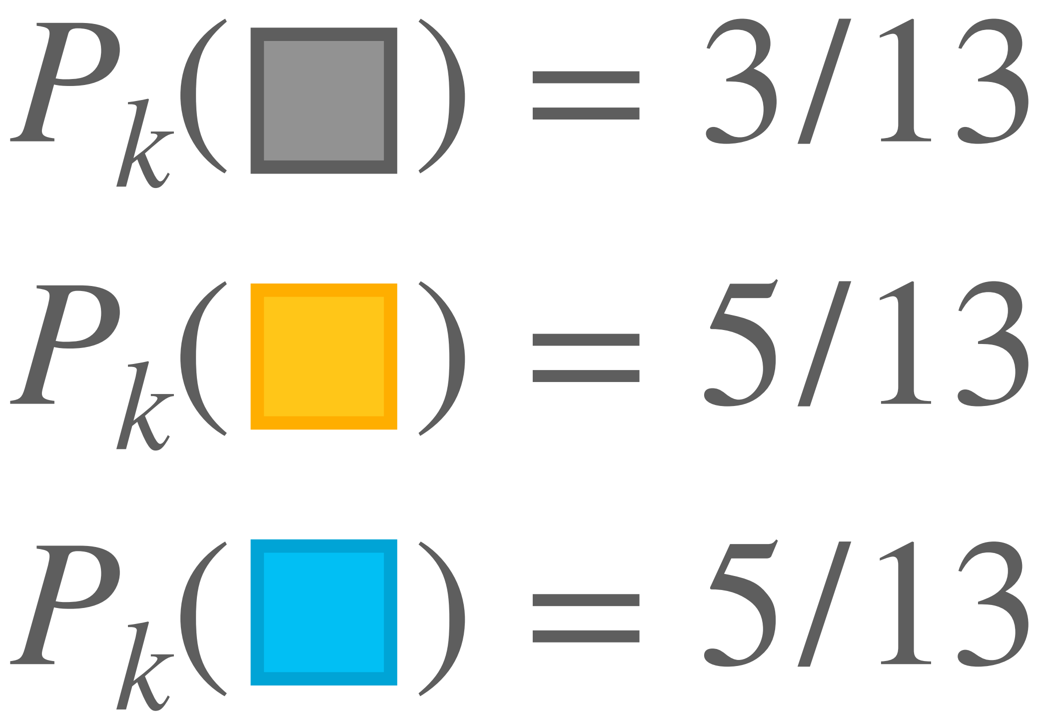
WE NEED AN UPPER BOUND



$$\Phi = \frac{1}{M^L} \sum_{(\mathbf{x}_1,\ldots,\mathbf{x}_L)\in\mathscr{C}^L} \overline{\mathrm{rad}}(\mathbf{x}_1, \ldots, \mathbf{x}_L)$$

$$= \cdots = \sum_{k=1}^{n} \sum_{(y_1,\ldots,y_L)\in[q]^L} \left(\prod_{j=1}^{L} P_k(y_j) \right) \left(1 - \frac{1}{L}\mathsf{pl}(y_1, \ldots, y_L)\right)$$

$$P_k(y) = \frac{1}{M} \sum_{\mathbf{x}\in\mathscr{C}} 1\{x_k = y\}$$



WE NEED AN UPPER BOUND

$$\sum_{(y_1,\ldots,y_L)\in[q]^L} \left(\prod_{j=1}^{L} P_k(y_j)\right)\left(1-\frac{1}{L}\mathsf{pl}(y_1,\ldots,y_L)\right)$$

WE NEED AN UPPER BOUND

$$\sum_{(y_1,\ldots,y_L)\in[q]^L} \left(\prod_{j=1}^{L} P_k(y_j)\right)\left(1-\frac{1}{L}\mathsf{pl}(y_1,\ldots,y_L)\right)$$

$$= 1 - \frac{1}{L} \mathop{\mathbb{E}}_{(y_1,\ldots,y_L)\sim P_k^{\otimes L}} [\mathsf{pl}(y_1,\ldots,y_L)]$$

WE NEED AN UPPER BOUND

If $P_k = \mathrm{Unif}([q])$, then this equals ρ^*!

$$\sum_{(y_1,\ldots,y_L)\in[q]^L} \left(\prod_{j=1}^{L} P_k(y_j)\right)\left(1-\frac{1}{L}\mathrm{pl}(y_1,\ldots,y_L)\right)$$

$$= 1 - \frac{1}{L}\mathop{\mathbb{E}}_{(y_1,\ldots,y_L)\sim P_k^{\otimes L}}[\mathrm{pl}(y_1,\ldots,y_L)]$$

WE NEED AN UPPER BOUND

If $P_k = \mathrm{Unif}([q])$, then this equals ρ^*!

$$\sum_{(y_1,\ldots,y_L)\in[q]^L}\left(\prod_{j=1}^{L} P_k(y_j)\right)\left(1-\frac{1}{L}\mathrm{pl}(y_1,\ldots,y_L)\right)$$

$$= 1 - \frac{1}{L}\mathop{\mathbb{E}}_{(y_1,\ldots,y_L)\sim P_k^{\otimes L}}[\mathrm{pl}(y_1,\ldots,y_L)]$$

$$\leq 1 - \frac{1}{L}\mathop{\mathbb{E}}_{(y_1,\ldots,y_L)\sim \mathrm{Unif}([q])^{\otimes L}}[\mathrm{pl}(y_1,\ldots,y_L)] = \rho_*$$

Suppose we can prove this inequality

WE NEED AN UPPER BOUND

If $P_k = \text{Unif}([q])$, then this equals ρ^*!

$$\sum_{(y_1,\ldots,y_L)\in[q]^L}\left(\prod_{j=1}^{L} P_k(y_j)\right)\left(1-\frac{1}{L}\mathsf{pl}(y_1,\ldots,y_L)\right)$$

$$(\rho_* + \epsilon)n \le \Phi \le \rho_* n$$

$$= 1 - \frac{1}{L}\mathop{\mathbb{E}}_{(y_1,\ldots,y_L)\sim P_k^{\otimes L}}[\mathsf{pl}(y_1,\ldots,y_L)]$$

Suppose we can prove this inequality

$$\le 1 - \frac{1}{L}\mathop{\mathbb{E}}_{(y_1,\ldots,y_L)\sim \text{Unif}([q])^{\otimes L}}[\mathsf{pl}(y_1,\ldots,y_L)] = \rho_*$$

WE NEED AN UPPER BOUND

If $P_k = \text{Unif}([q])$, then this equals ρ^*!

$$\sum_{(y_1,\ldots,y_L)\in[q]^L} \left(\prod_{j=1}^{L} P_k(y_j)\right)\left(1-\frac{1}{L}\text{pl}(y_1,\ldots,y_L)\right)$$

$(\rho_* + \epsilon)n \leq \Phi \leq \rho_* n$

$$= 1 - \frac{1}{L} \mathop{\mathbb{E}}_{(y_1,\ldots,y_L)\sim P_k^{\otimes L}} [\text{pl}(y_1,\ldots,y_L)]$$

Contradiction!

Suppose we can prove this inequality

$$\leq 1 - \frac{1}{L} \mathop{\mathbb{E}}_{(y_1,\ldots,y_L)\sim \text{Unif}([q])^{\otimes L}} [\text{pl}(y_1,\ldots,y_L)] = \rho_*$$

PROVING THAT INEQUALITY

INEQUALITY TO PROVE

$$1 - \frac{1}{L} \mathop{\mathbb{E}}_{(y_1,\ldots,y_L)\sim P_k^{\otimes L}} [\mathsf{pl}(y_1, \ldots, y_L)] \leq 1 - \frac{1}{L} \mathop{\mathbb{E}}_{(y_1,\ldots,y_L)\sim(\mathrm{Unif}([q])^{\otimes L}} [\mathsf{pl}(y_1, \ldots, y_L)]$$

INEQUALITY TO PROVE

$$1 - \frac{1}{L} \mathop{\mathbb{E}}_{(y_1,\ldots,y_L)\sim P_k^{\otimes L}} [\mathrm{pl}(y_1, \ldots, y_L)] \leq 1 - \frac{1}{L} \mathop{\mathbb{E}}_{(y_1,\ldots,y_L)\sim(\mathrm{Unif}([q]))^{\otimes L}} [\mathrm{pl}(y_1, \ldots, y_L)]$$

- Define function on prob. dists. $P \in \Delta([q]) = \left\{ Q \in \mathbb{R}^q_{\geq 0} : \sum_i Q_i = 1 \right\}$

$$f_{q,L}(P) := \mathop{\mathbb{E}}_{(X_1,\ldots,X_L)\sim P^{\otimes L}} [\mathrm{pl}(X_1, \ldots, X_L)]$$

INEQUALITY TO PROVE

$$1 - \frac{1}{L} \mathop{\mathbb{E}}_{(y_1,\ldots,y_L)\sim P_k^{\otimes L}} [\mathrm{pl}(y_1,\ldots,y_L)] \leq 1 - \frac{1}{L} \mathop{\mathbb{E}}_{(y_1,\ldots,y_L)\sim(\mathrm{Unif}([q]))^{\otimes L}} [\mathrm{pl}(y_1,\ldots,y_L)]$$

- Define function on prob. dists. $P \in \Delta([q]) = \left\{ Q \in \mathbb{R}^q_{\geq 0} : \sum_i Q_i = 1 \right\}$

$$f_{q,L}(P) := \mathop{\mathbb{E}}_{(X_1,\ldots,X_L)\sim P^{\otimes L}} [\mathrm{pl}(X_1,\ldots,X_L)]$$

- Want to show: f is minimized on uniform distribution

SCHUR CONVEXITY

SCHUR CONVEXITY

- Well-studied class of functions minimized on uniform distribution: Schur-convex functions

SCHUR CONVEXITY

- Well-studied class of functions minimized on uniform distribution: Schur-convex functions
- Functions f such that $f(\mathbf{x}) \leq f(\mathbf{y})$ whenever $\mathbf{x} \preceq \mathbf{y}$, where $\preceq$ is majorization ordering

SCHUR CONVEXITY

- Well-studied class of functions minimized on uniform distribution: Schur-convex functions
- Functions f such that $f(\mathbf{x}) \leq f(\mathbf{y})$ whenever $\mathbf{x} \preceq \mathbf{y}$, where $\preceq$ is majorization ordering
- $\mathbf{x}, \mathbf{y} \in \Delta([n])$ satisfy $\mathbf{x} \preceq \mathbf{y}$ if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow} \quad \forall k \in [n]$$

where $\mathbf{x}^{\downarrow}$ is $\mathbf{x}$ sorted in descending order

SCHUR CONVEXITY

- Well-studied class of functions minimized on uniform distribution: Schur-convex functions
- Functions f such that $f(\mathbf{x}) \leq f(\mathbf{y})$ whenever $\mathbf{x} \preceq \mathbf{y}$, where $\preceq$ is majorization ordering
- $\mathbf{x}, \mathbf{y} \in \Delta([n])$ satisfy $\mathbf{x} \preceq \mathbf{y}$ if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow} \quad \forall k \in [n]$$

where $\mathbf{x}^{\downarrow}$ is $\mathbf{x}$ sorted in descending order

$\mathbf{x}$			
	1/3	1/3	1/3

$\mathbf{y}$			
	1/5	1/2	3/10

SCHUR CONVEXITY

- Well-studied class of functions minimized on uniform distribution: Schur-convex functions
- Functions f such that $f(\mathbf{x}) \leq f(\mathbf{y})$ whenever $\mathbf{x} \preceq \mathbf{y}$, where $\preceq$ is majorization ordering
- $\mathbf{x}, \mathbf{y} \in \Delta([n])$ satisfy $\mathbf{x} \preceq \mathbf{y}$ if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow} \quad \forall k \in [n]$$

where $\mathbf{x}^{\downarrow}$ is $\mathbf{x}$ sorted in descending order

$\mathbf{x}$	1/3	1/3	1/3
$\mathbf{x}^{\downarrow}$	1/3	1/3	1/3

$\mathbf{y}$	1/5	1/2	3/10
$\mathbf{y}^{\downarrow}$	1/2	3/10	1/5

SCHUR CONVEXITY

- Well-studied class of functions minimized on uniform distribution: Schur-convex functions
- Functions f such that $f(\mathbf{x}) \leq f(\mathbf{y})$ whenever $\mathbf{x} \preceq \mathbf{y}$, where $\preceq$ is majorization ordering
- $\mathbf{x}, \mathbf{y} \in \Delta([n])$ satisfy $\mathbf{x} \preceq \mathbf{y}$ if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow} \quad \forall k \in [n]$$

where $\mathbf{x}^{\downarrow}$ is $\mathbf{x}$ sorted in descending order

$\mathbf{x}$	1/3	1/3	1/3
$\mathbf{x}^{\downarrow}$	1/3	1/3	1/3
	1/3	2/3	1

$\mathbf{y}$	1/5	1/2	3/10
$\mathbf{y}^{\downarrow}$	1/2	3/10	1/5
	1/2	8/10	1

SCHUR CONVEXITY

- Well-studied class of functions minimized on uniform distribution: Schur-convex functions
- Functions f such that $f(\mathbf{x}) \leq f(\mathbf{y})$ whenever $\mathbf{x} \preceq \mathbf{y}$, where $\preceq$ is majorization ordering
- $\mathbf{x}, \mathbf{y} \in \Delta([n])$ satisfy $\mathbf{x} \preceq \mathbf{y}$ if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow} \quad \forall k \in [n]$$

where $\mathbf{x}^{\downarrow}$ is $\mathbf{x}$ sorted in descending order

$\mathbf{x}$	1/3	1/3	1/3
$\mathbf{x}^{\downarrow}$	1/3	1/3	1/3
	1/3	2/3	1

$\mathbf{y}$	1/5	1/2	3/10
$\mathbf{y}^{\downarrow}$	1/2	3/10	1/5
	1/2	8/10	1

$$\Longrightarrow \quad \mathbf{x} \preceq \mathbf{y}$$

SCHUR CONVEXITY

$(1/q, \ldots, 1/q)$ is majorized by every other probability vector

- Well-studied class of functions minimized on uniform distribution: Schur-convex functions
- Functions f such that $f(\mathbf{x}) \leq f(\mathbf{y})$ whenever $\mathbf{x} \preceq \mathbf{y}$, where $\preceq$ is majorization ordering
- $\mathbf{x}, \mathbf{y} \in \Delta([n])$ satisfy $\mathbf{x} \preceq \mathbf{y}$ if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow} \quad \forall k \in [n]$$

where $\mathbf{x}^{\downarrow}$ is $\mathbf{x}$ sorted in descending order

$\mathbf{x}$			
	1/3	1/3	1/3

$\mathbf{y}$			
	1/5	1/2	3/10

$\mathbf{x}^{\downarrow}$			
	1/3	1/3	1/3
	1/3	2/3	1

$\mathbf{y}^{\downarrow}$			
	1/2	3/10	1/5
	1/2	8/10	1

$$\implies \mathbf{x} \preceq \mathbf{y}$$

SCHUR-OSTROWSKI CRITERION

SCHUR-OSTROWSKI CRITERION

- f is Schur-convex iff $(x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \geq 0 \quad \forall i \neq j$

SCHUR-OSTROWSKI CRITERION

- f is Schur-convex iff $(x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \geq 0 \quad \forall i \neq j$

- Can verify this holds for

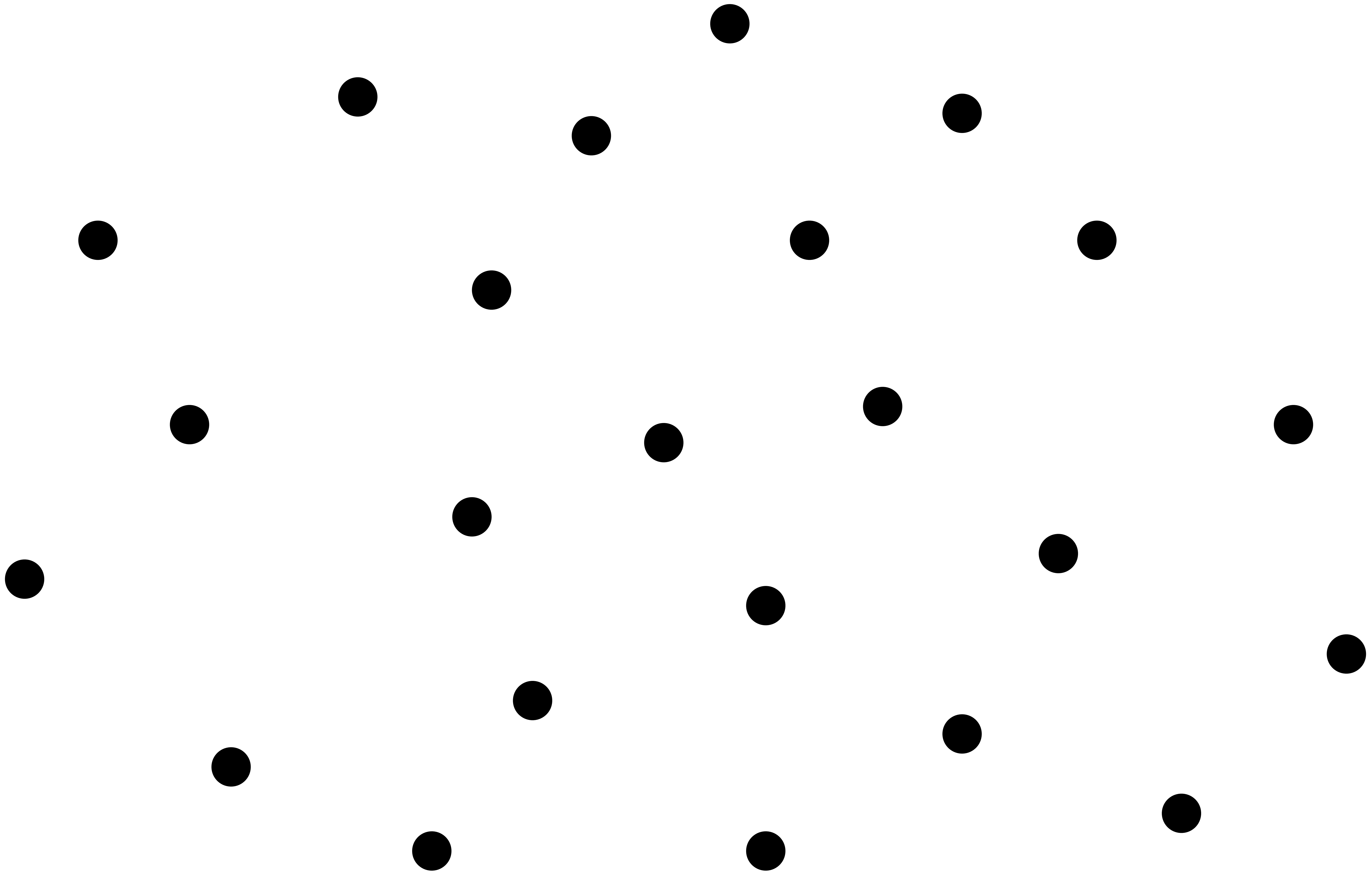
$$f_{q,L}(P) = \mathop{\mathbb{E}}_{(X_1,\ldots,X_L)\sim P^{\otimes L}}[\mathrm{pl}(X_1, \ldots, X_L)] = \sum_{(a_0,\ldots,a_{q-1})} \binom{L}{a_0, \ldots, a_{q-1}} \max\{a_0, \ldots, a_{q-1}\} \prod_{i\in[q]} P_i^{a_i}$$

where sum runs over all $(a_0, \ldots, a_{q-1}) \in \mathbb{N}^q_{\geq 0}$ s.t. $\sum_i a_i = L$

ELIAS-BASSALYGO-LIKE BOUND

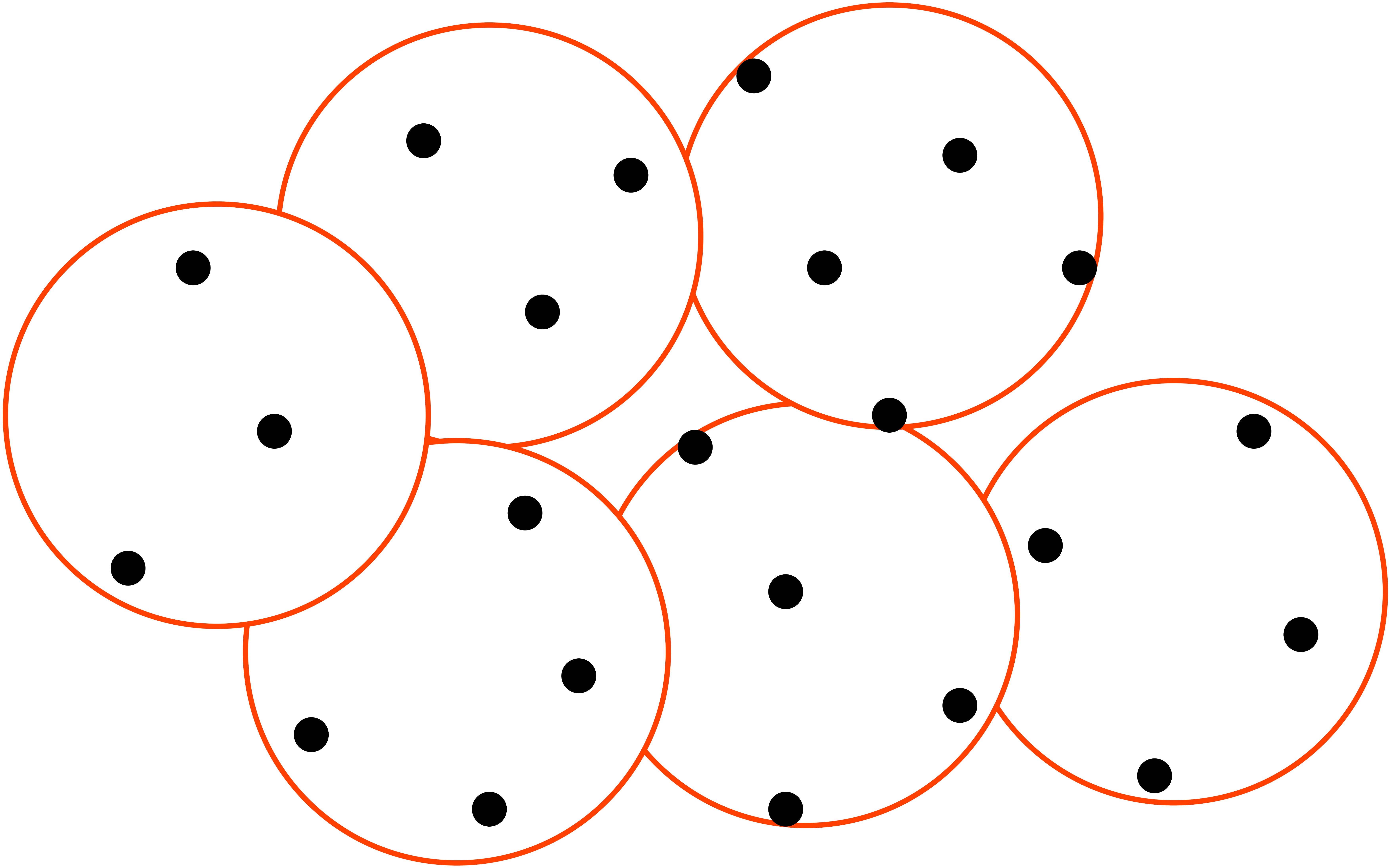
RATE-DECODING RADIUS TRADEOFFS

[Bassalygo'65,Elias]



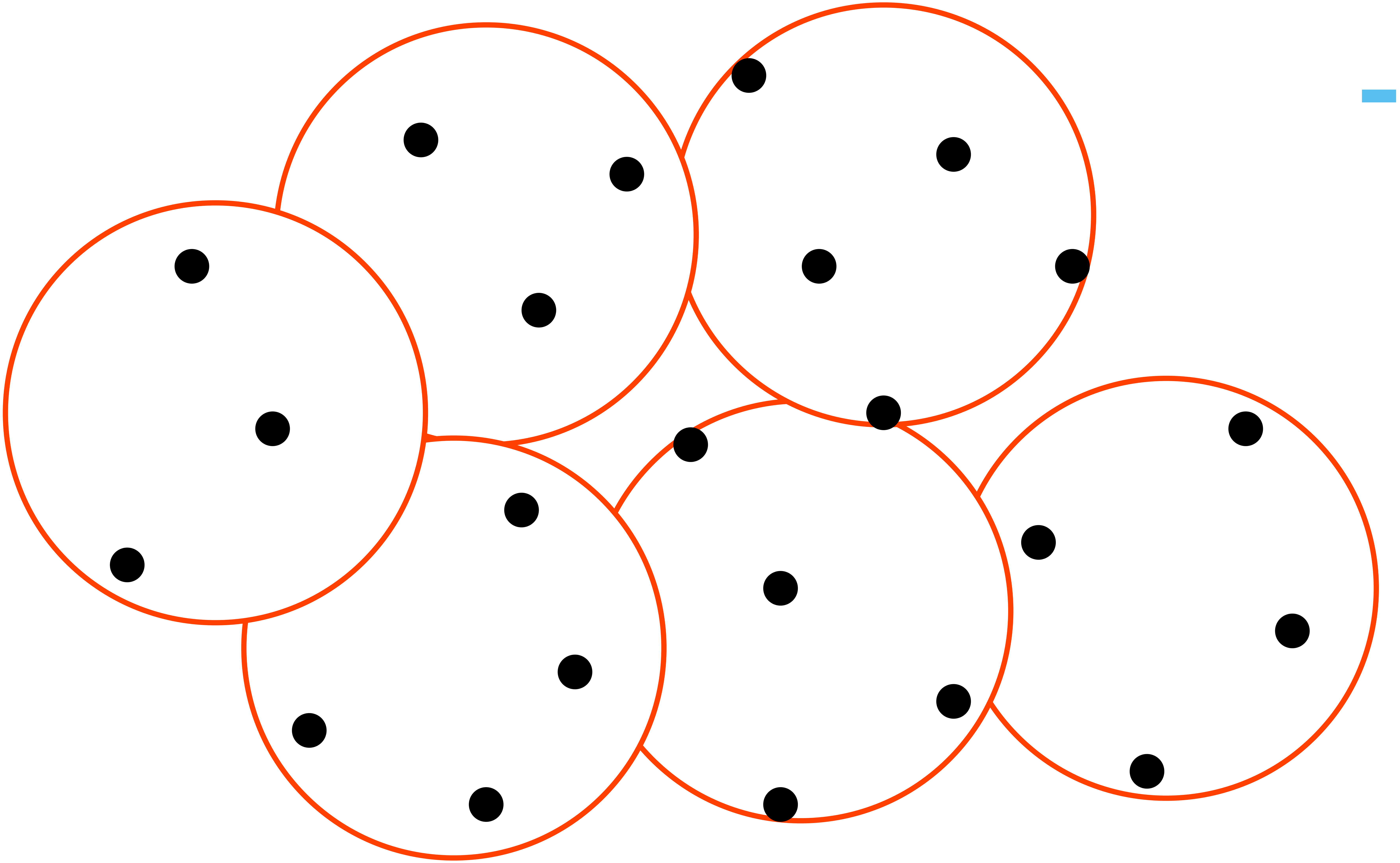
RATE-DECODING RADIUS TRADEOFFS

[Bassalygo'65,Elias]



RATE-DECODING RADIUS TRADEOFFS

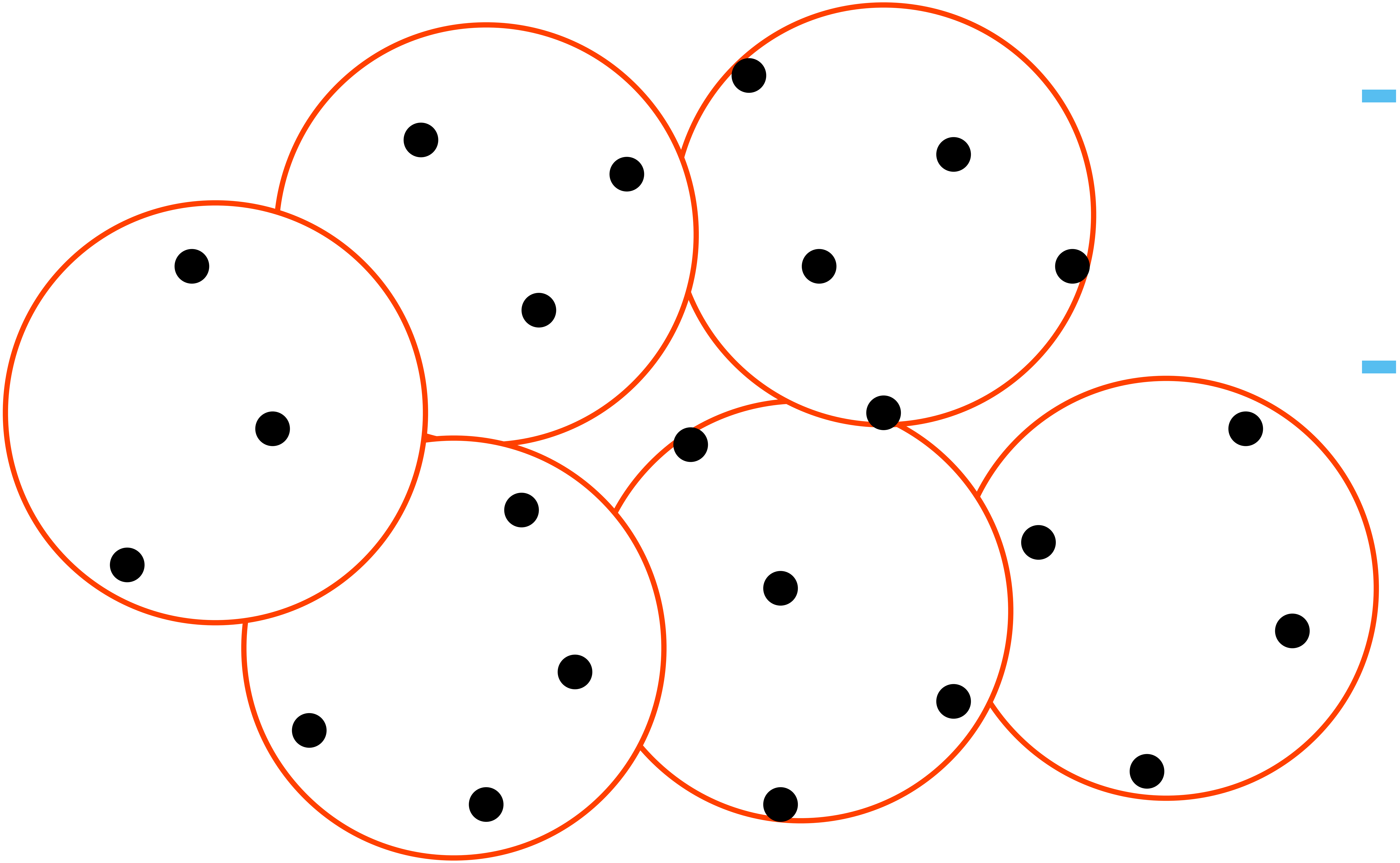
[Bassalygo'65,Elias]



- Use Plotkin bound on each Hamming ball subcode

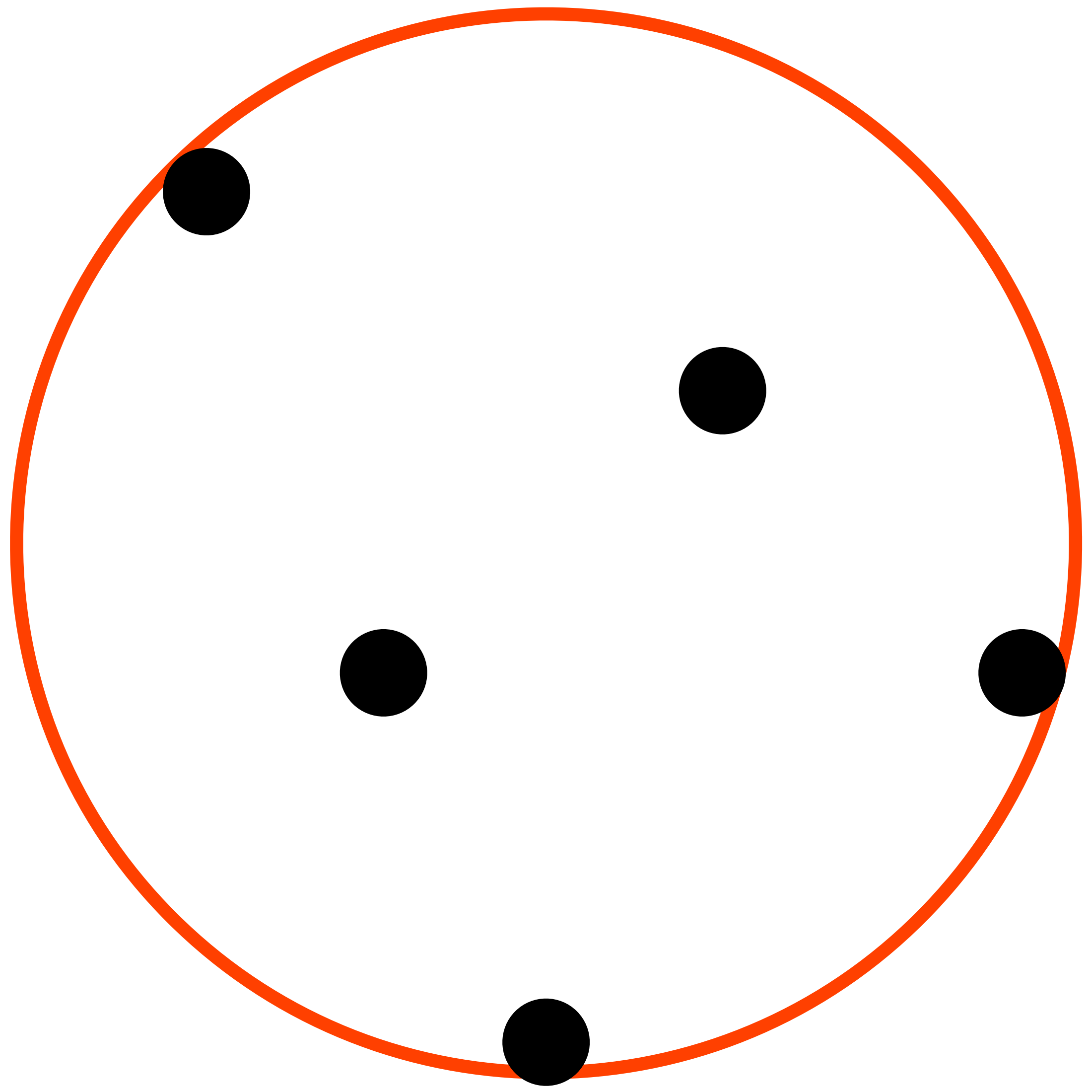
RATE-DECODING RADIUS TRADEOFFS

[Bassalygo'65,Elias]

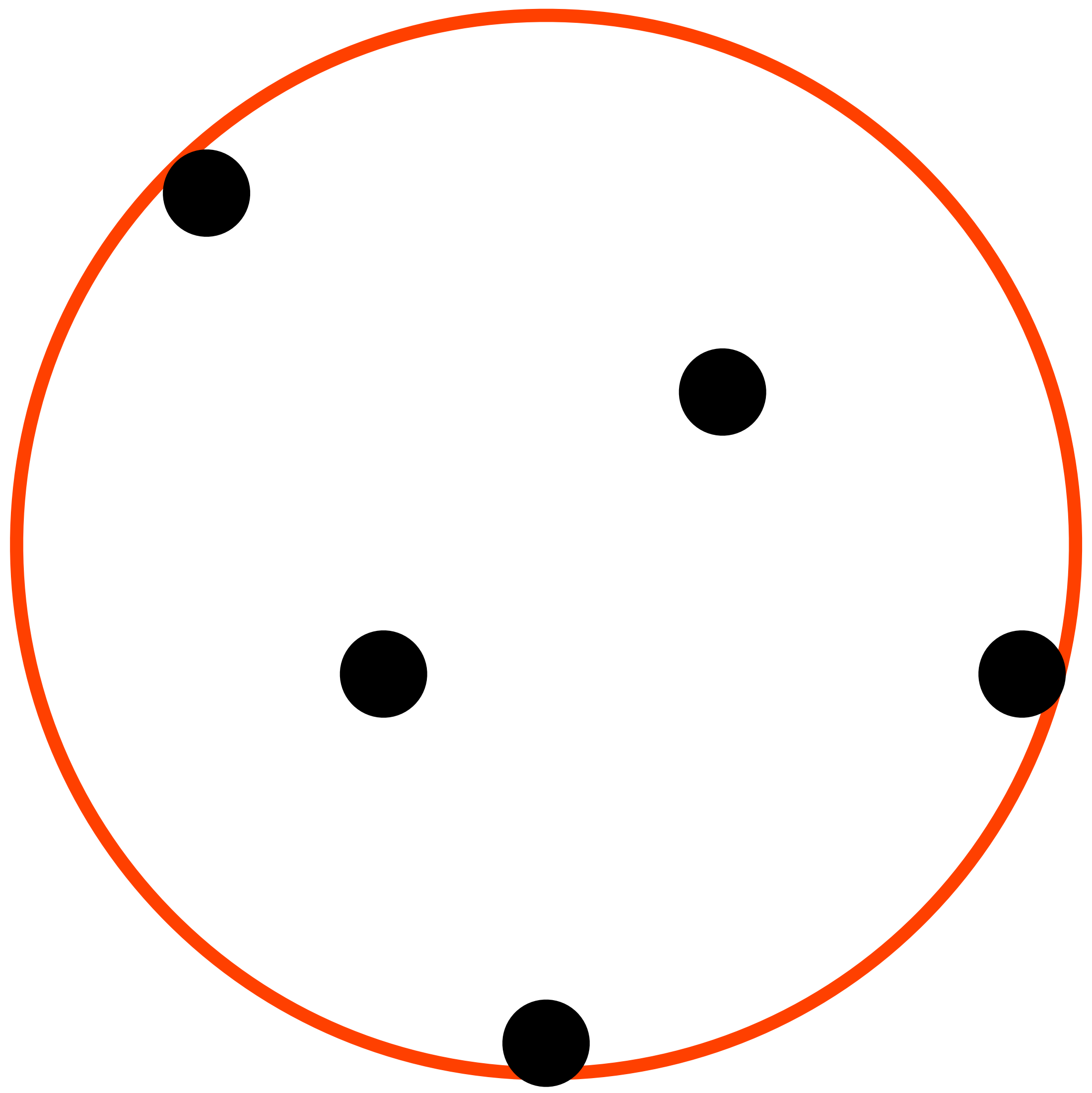


- Use Plotkin bound on each Hamming ball subcode
- Use fact that you don't need *too many* Hamming balls to cover $[q]^n$

NEED TO USE WEIGHT CONSTRAINT!

- Each subcode  is not only list-decodable, but is (equivalent to) code **with bounded weight**

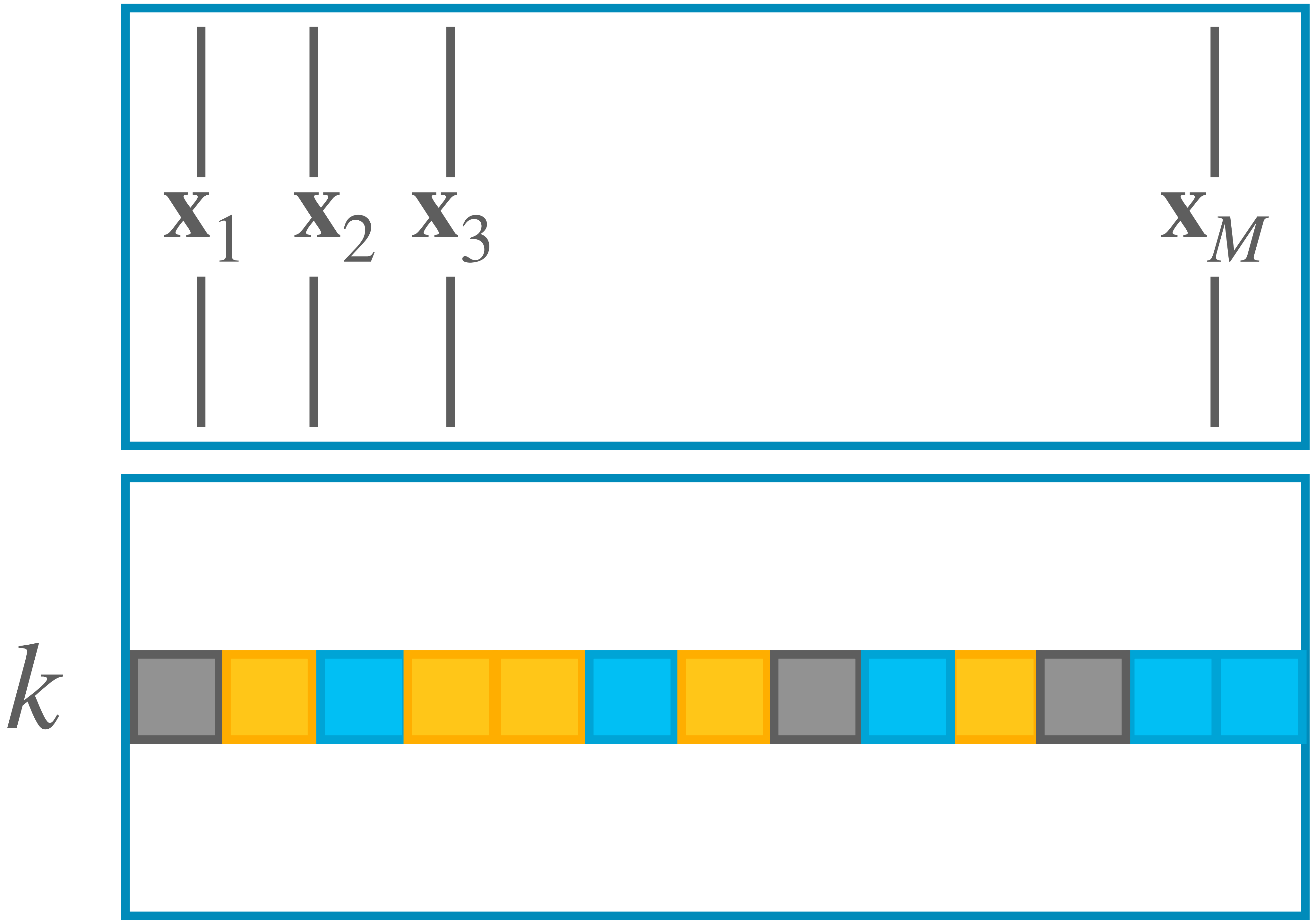
NEED TO USE WEIGHT CONSTRAINT!

- Each subcode (equivalent to)  is not only list-decodable, but is code with bounded weight

Want: Plotkin bound for list-decodable $\mathscr{C}$ with weight constraint

$$\mathrm{wt}(\mathbf{x}) \leq w \quad \forall \mathbf{x} \in \mathscr{C}$$

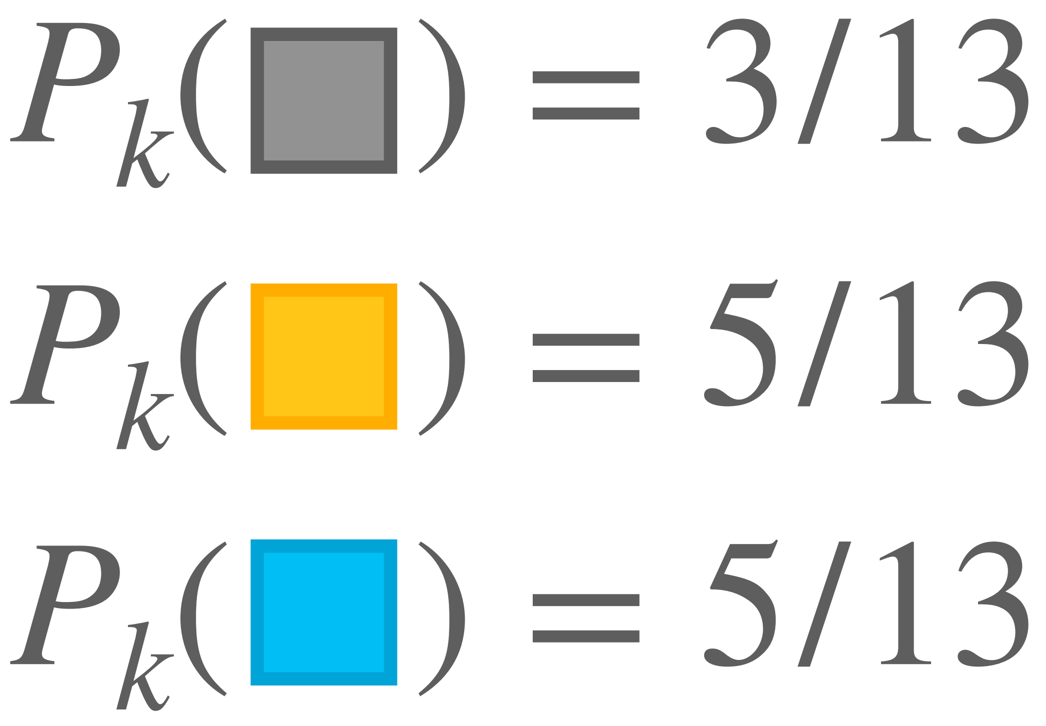
LET'S RETURN HERE



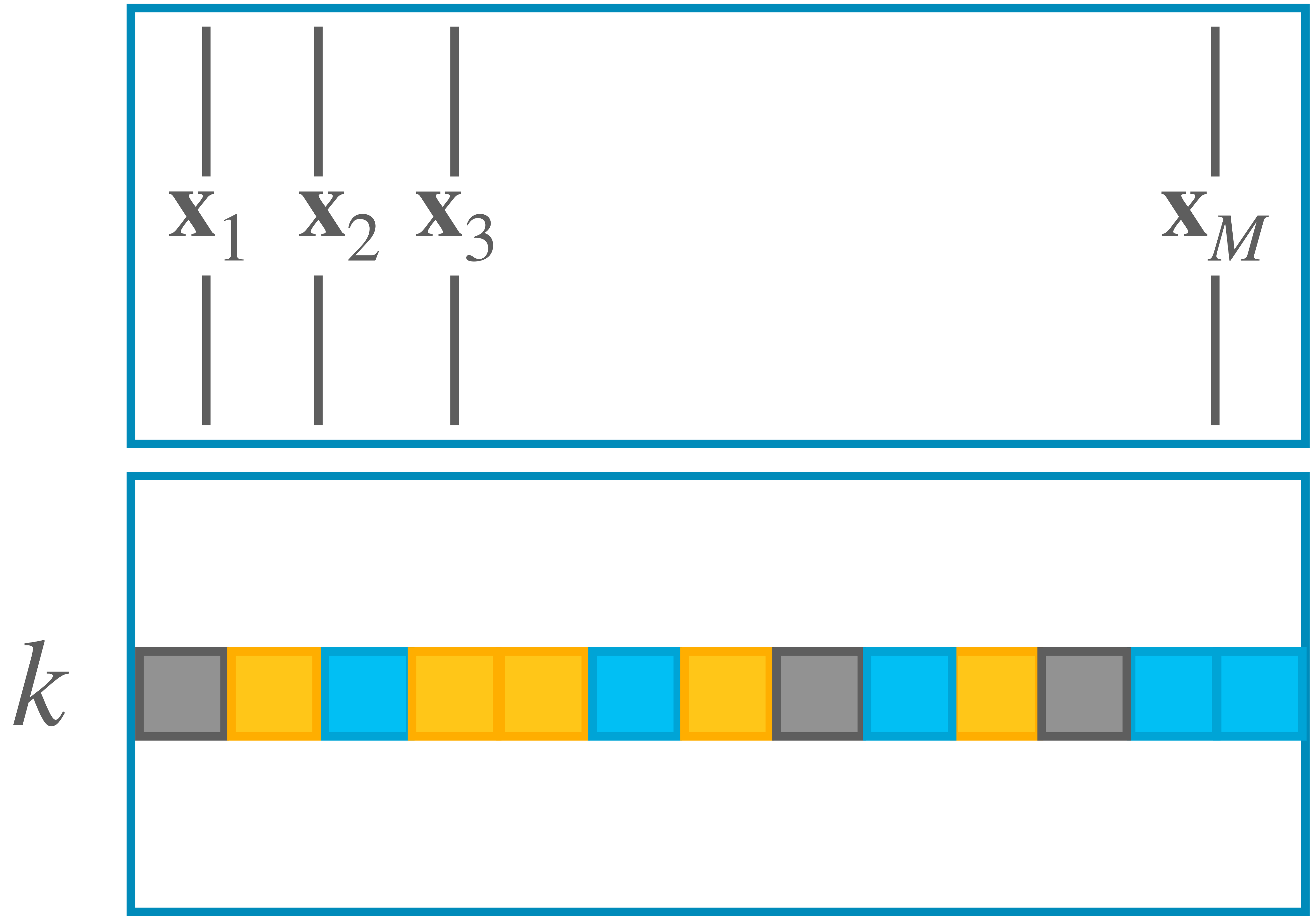
$$\Phi = \frac{1}{M^L} \sum_{(\mathbf{x}_1,\ldots,\mathbf{x}_L)\in\mathscr{C}^L} \overline{\mathrm{rad}}(\mathbf{x}_1, \ldots, \mathbf{x}_L)$$

$$= \cdots = \sum_{k=1}^{n} \sum_{(y_1,\ldots,y_L)\in[q]^L} \left(\prod_{j=1}^{L} P_k(y_j) \right) \left(1 - \frac{1}{L}\mathsf{pl}(y_1, \ldots, y_L)\right)$$

$$P_k(y) = \frac{1}{M} \sum_{\mathbf{x}\in\mathscr{C}} 1\{x_k = y\}$$



LET'S RETURN HERE



$$\Phi = \frac{1}{M^L} \sum_{(\mathbf{x}_1,\ldots,\mathbf{x}_L)\in\mathscr{C}^L} \overline{\mathrm{rad}}(\mathbf{x}_1, \ldots, \mathbf{x}_L)$$

$$= \sum_{k=1}^{n} \left(1 - \frac{1}{L} f_{q,L}(P_k)\right)$$

$$P_k(y) = \frac{1}{M} \sum_{\mathbf{x}\in\mathscr{C}} 1\{x_k = y\}$$

$$P_k(\square) = 3/13$$

$$P_k(\square) = 5/13$$

$$P_k(\square) = 5/13$$

USING WEIGHT CONSTRAINT

$$\Phi = \sum_{k=1}^{n} \left(1 - \frac{1}{L} f_{q,L}(P_k) \right)$$

USING WEIGHT CONSTRAINT

$$\Phi = \sum_{k=1}^{n} \left(1 - \frac{1}{L} f_{q,L}(P_k)\right)$$

- Let w_k be fractional weight of k-th column, i.e., $1 - P_k(0)$

USING WEIGHT CONSTRAINT

$$\Phi = \sum_{k=1}^{n} \left(1 - \frac{1}{L} f_{q,L}(P_k)\right)$$

- Let w_k be fractional weight of k-th column, i.e., $1 - P_k(0)$
- Define $P_u = (1 - u, u/(q-1), \ldots, u/(q-1))$; then $P_{w_k} \preceq P_k$

USING WEIGHT CONSTRAINT

$$\Phi = \sum_{k=1}^{n} \left(1 - \frac{1}{L} f_{q,L}(P_k)\right) \leq \sum_{k=1}^{n} \left(1 - \frac{1}{L} f_{q,L}\left(P_{w_k}\right)\right)$$

- Let w_k be fractional weight of k-th column, i.e., $1 - P_k(0)$
- Define $P_u = (1 - u, u/(q-1), \ldots, u/(q-1))$; then $P_{w_k} \preceq P_k$

USING WEIGHT CONSTRAINT

$$\Phi = \sum_{k=1}^{n} \left(1 - \frac{1}{L} f_{q,L}(P_k)\right) \leq \sum_{k=1}^{n} \left(1 - \frac{1}{L} f_{q,L}\left(P_{w_k}\right)\right)$$

- Let w_k be fractional weight of k-th column, i.e., $1 - P_k(0)$
- Define $P_u = (1 - u, u/(q-1), \ldots, u/(q-1))$; then $P_{w_k} \preceq P_k$
- Consider univariate function $g(u) := f_{q,L}(P_u)$, suppose its convex

USING WEIGHT CONSTRAINT

$$\Phi = \sum_{k=1}^{n}\left(1-\frac{1}{L}f_{q,L}(P_k)\right) \leq \sum_{k=1}^{n}\left(1-\frac{1}{L}f_{q,L}\left(P_{w_k}\right)\right)$$

$$\leq n\left(1-\frac{1}{L}g_{q,L}\left(\frac{1}{n}\sum_{k=1}^{n}w_k\right)\right)$$

- Let w_k be fractional weight of k-th column, i.e., $1 - P_k(0)$
- Define $P_u = (1 - u, u/(q-1), \ldots, u/(q-1))$; then $P_{w_k} \preceq P_k$
- Consider univariate function $g(u) := f_{q,L}(P_u)$, suppose its convex

USING WEIGHT CONSTRAINT

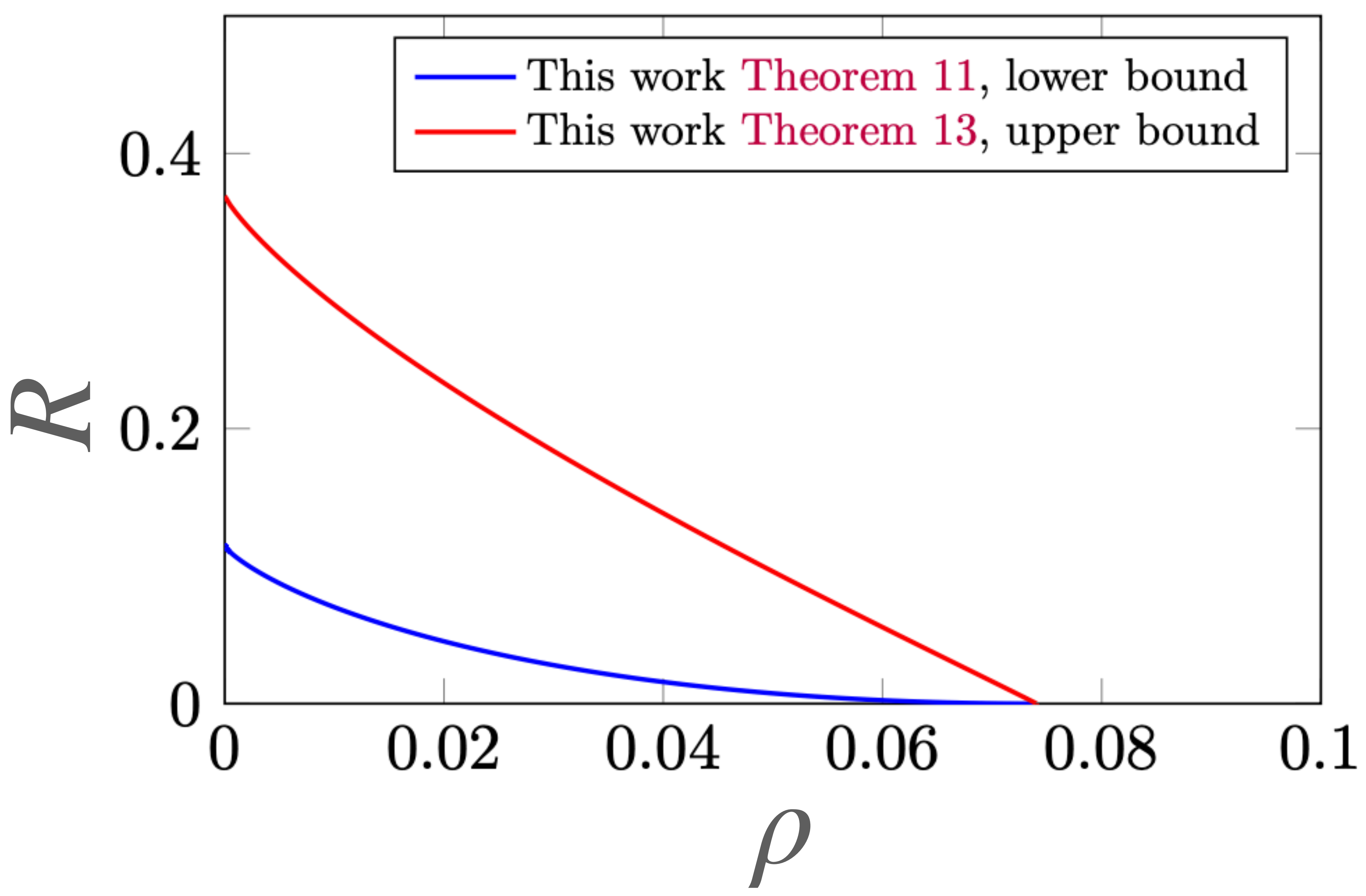
$$\Phi = \sum_{k=1}^{n}\left(1-\frac{1}{L}f_{q,L}(P_k)\right) \leq \sum_{k=1}^{n}\left(1-\frac{1}{L}f_{q,L}\left(P_{w_k}\right)\right)$$

$$\leq n\left(1-\frac{1}{L}g_{q,L}\left(\frac{1}{n}\sum_{k=1}^{n}w_k\right)\right) \leq n\left(1-\frac{1}{L}g_{q,L}(w)\right) =: n\rho_*(q,L,w)$$

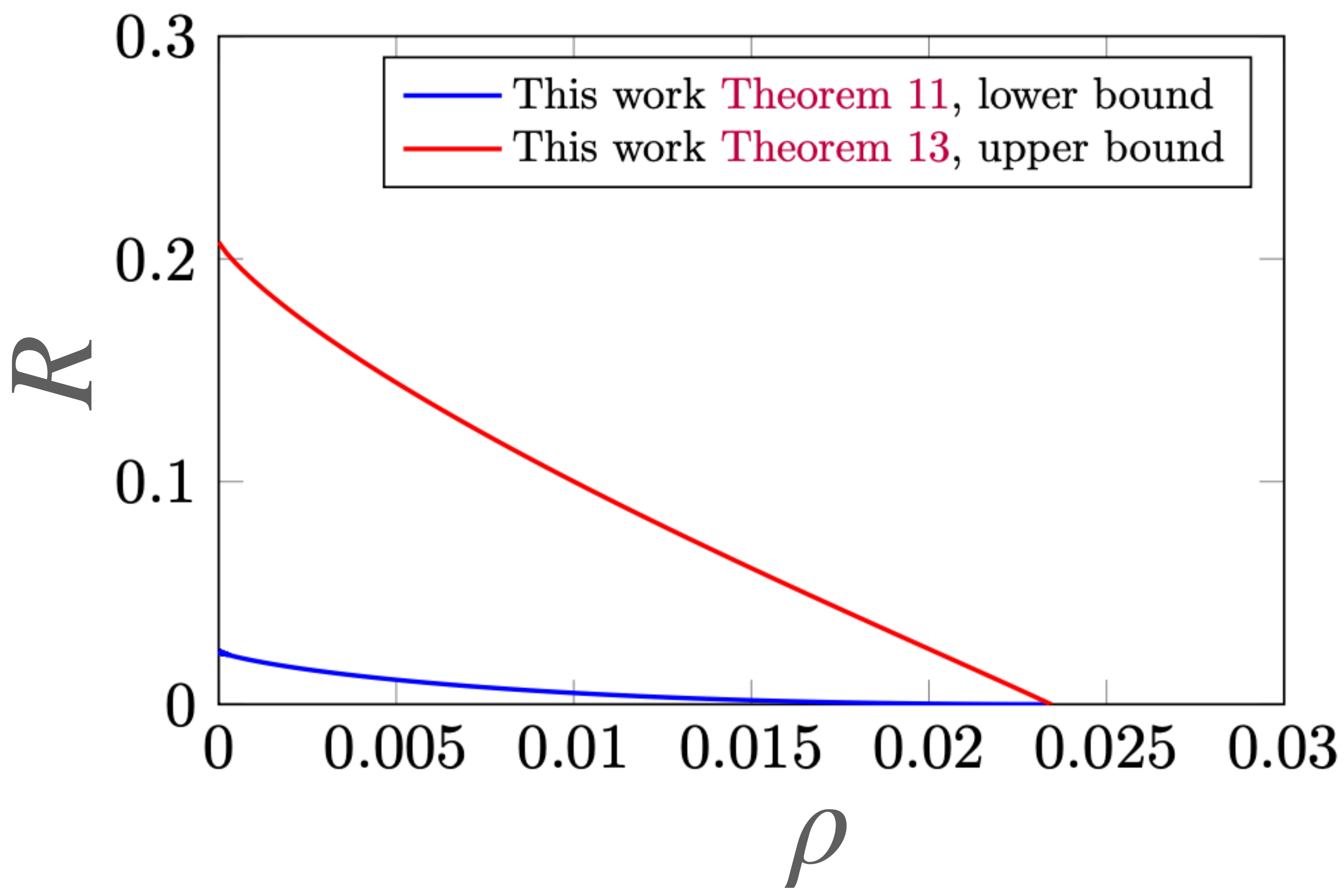
- Let w_k be fractional weight of k-th column, i.e., $1 - P_k(0)$
- Define $P_u = (1-u, u/(q-1), \ldots, u/(q-1))$; then $P_{w_k} \preceq P_k$
- Consider univariate function $g(u) := f_{q,L}(P_u)$, suppose its convex

SOME PLOTS

$q = 3, \ L = 3$



$q = 3, \ L = 4$



CONCLUSION

RECAP

Analogous result for list-recovery

From this: derive new upper bounds on rate of (ρ, L)-list-decodable codes

- For all $q \geq 2$ and $L \geq 2$, we prove the equality

$$\rho_*(q, L) = 1 - \frac{1}{L} \mathop{\mathbb{E}}_{(X_1,\ldots,X_L)\sim \mathrm{Unif}([q])^L} [\mathrm{pl}(X_1, \ldots, X_L)]$$

- Two parts:

<u>Possibility Result</u>: $\exists$ positive rate $(\rho_* - \epsilon, L)$-list-dec. codes (standard random code argument)

<u>Impossibility Result</u>: If $\mathscr{C} \subseteq [q]^n$ is $(\rho_* + \epsilon, L)$-list-dec., then $|\mathscr{C}| \leq O_\epsilon(1)$

RECAP

Analogous result for list-recovery

From this: derive new upper bounds on rate of (ρ, L)-list-decodable codes

- For all $q \geq 2$ and $L \geq 2$, we prove the equality

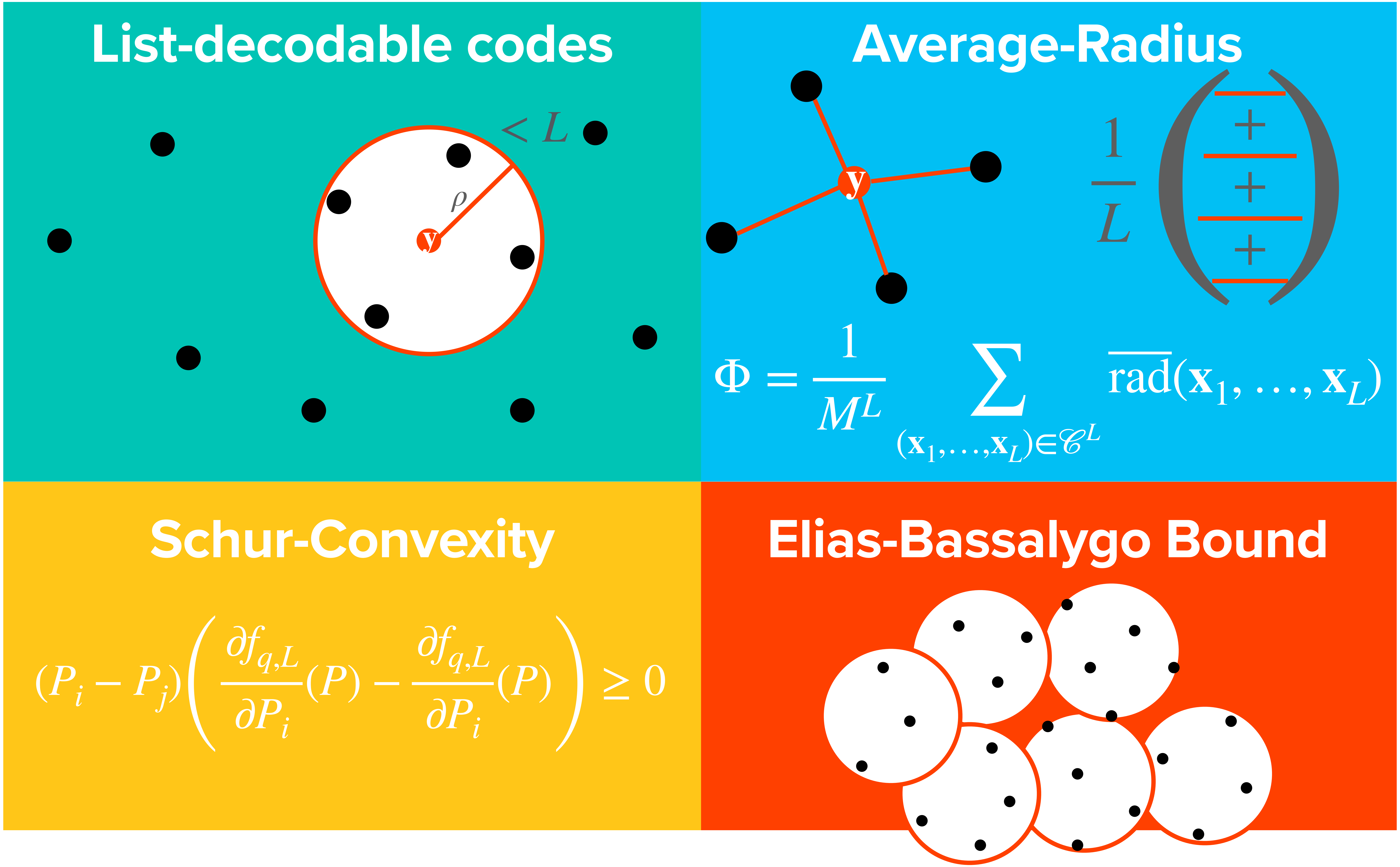
$$\rho_*(q, L) = 1 - \frac{1}{L} \mathop{\mathbb{E}}_{(X_1,\ldots,X_L)\sim\text{Unif(}}$$

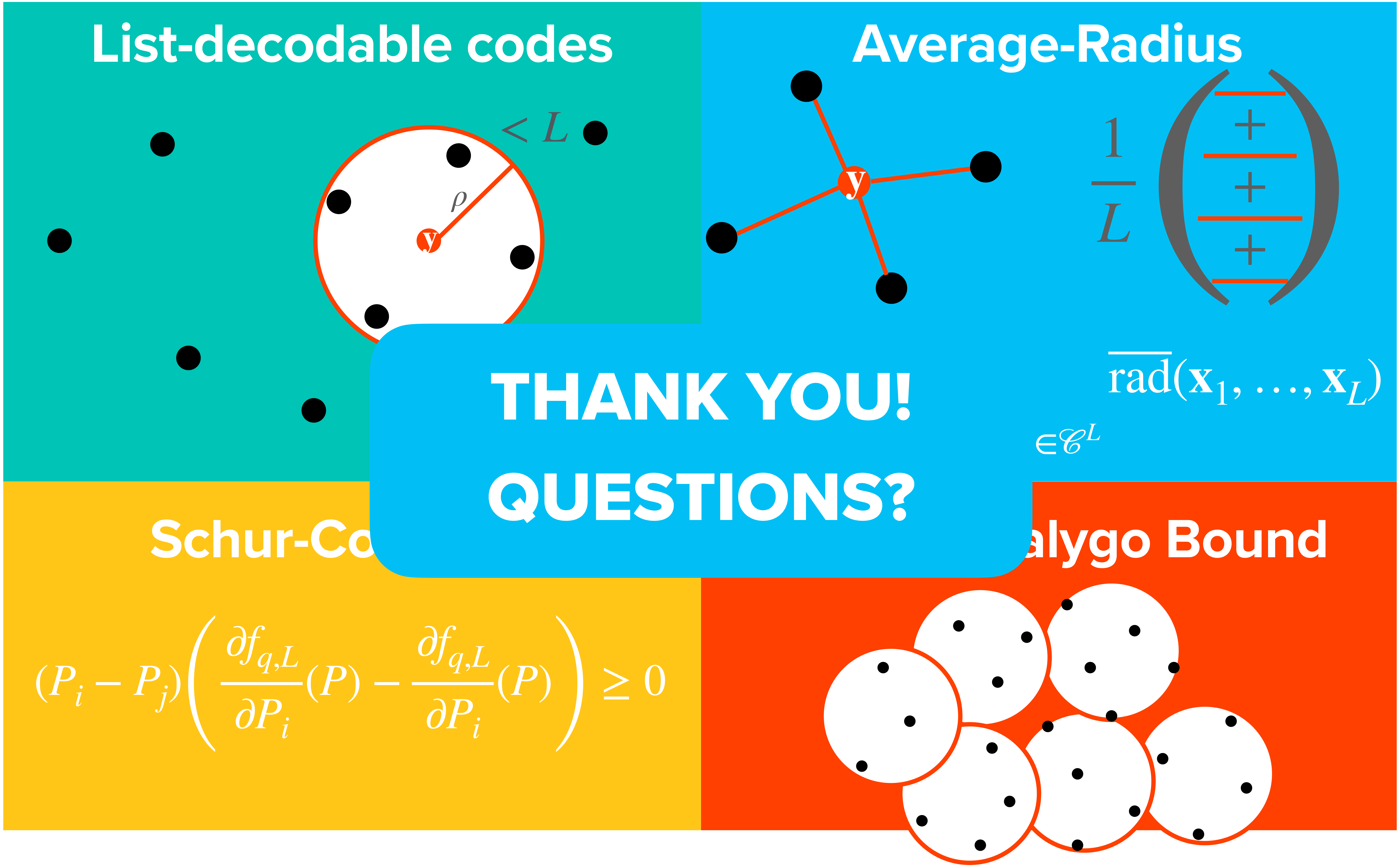
Question: explicit bound à la [Alon-Bukh-Polyanskiy'18]?

- Two parts:

<u>Possibility Result</u>: ∃ positive rate $(\rho_* - \epsilon, L)$-list-dec. codes (standard random code argument)

<u>Impossibility Result</u>: If $\mathscr{C} \subseteq [q]^n$ is $(\rho_* + \epsilon, L)$-list-dec., then $|\mathscr{C}| \leq O_\epsilon(1)$


List-decodable codes
$< L$
ρ
$\mathbf{y}$
Average-Radius
$\mathbf{y}$
$\frac{1}{L}$
+
+
+
$\Phi = \frac{1}{M^L} \sum_{(\mathbf{x}_1,\ldots,\mathbf{x}_L)\in\mathscr{C}^L} \overline{\mathrm{rad}}(\mathbf{x}_1, \ldots, \mathbf{x}_L)$
Schur-Convexity
$(P_i - P_j)\left(\frac{\partial f_{q,L}}{\partial P_i}(P) - \frac{\partial f_{q,L}}{\partial P_i}(P)\right) \geq 0$
Elias-Bassalygo Bound


List-decodable codes
$< L$
ρ
y
Average-Radius
y
$\frac{1}{L}$
$\overline{\text{rad}}(\mathbf{x}_1, \ldots, \mathbf{x}_L)$
$\in \mathscr{C}^L$
THANK YOU!
QUESTIONS?
Schur-Co
alygo Bound
$(P_i - P_j)\left(\frac{\partial f_{q,L}}{\partial P_i}(P) - \frac{\partial f_{q,L}}{\partial P_i}(P)\right) \geq 0$