Some important tools for verifiable computation: the Sumcheck protocols

Clémence Chevignard

Overview

1. What is a verifiable computation?
2. Arithmetic circuits and R1CS
3. Reed-Solomon codes
4. The univariate Sumcheck
5. The multivariate Sumchecks
6. Conclusion

What is a verifiable computation?

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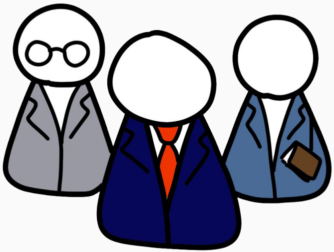
You

programm + input →

A company X



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Big computing capability.

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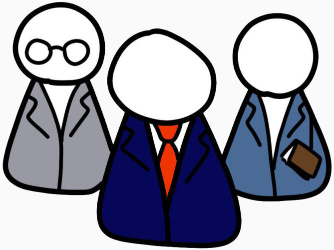

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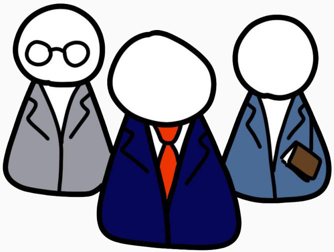

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How to be sure that X's answer is correct?

What is a verifiable computation?

- Solution 1: You do the computation by yourself to check $\to$ not efficient.
- Solution 2: You ask the company X for a proof $\to$ ok, but how?

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Goals of a verifiable computation protocol:

- allowing the company X to give a **proof** that the result is correct.
- ... without spending **more time to craft the proof than to do the computation**.
- You must be able to check the proof **faster than doing the computation**.

Terminology

You $\rightarrow$ the "Verifier"

The company X $\rightarrow$ the "Prover"

Terminology

You $\to$ the "Verifier"

The company X $\to$ the "Prover"

One type of protocols model $\to$ **IOP model:** *Interactive Oracle Proof* [BCS16]:

- allows V and P interactions: they can send each other messages during several rounds.
- allows V to have oracle access to P's messages.
- V can use randomness to make queries to P's oracles.

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The oracle notion is theoretical, but can be implemented with Merkle trees.

Some precisions

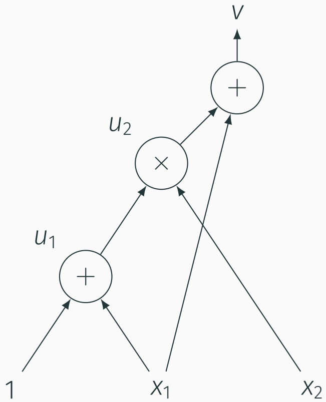
What do we need to be careful about:

- Completeness.
- Linear Prover time.
- Sublinear Verifier time.
- Soundness.

- Linear Proof length, or less.
  proof length = total length of prover's oracles.
- Sublinear Query complexity.
  query complexity = elements read by the Verifier.

Arithmetic circuits and R1CS

What's an **arithmetic circuit** C?



Represents the computation

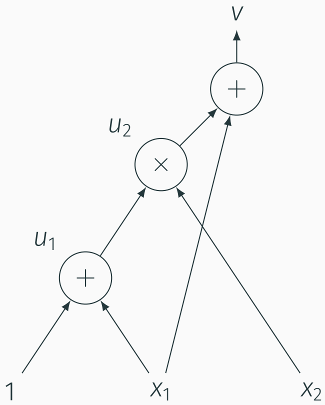
$$v = (1 + x_1) \times x_2 + x_1.$$

Claim: $C(1, x_1, x_2) = v$.

x_1 and x_2 are the inputs of the circuit, v is the output. Every variable belongs to $\mathbb{F}$.

"Length of the computation" $= |(1, x_1, x_2, u_1, u_2, v)|$.

We can build a **Rank 1 Constraint Satisfiability** (A, B, C, x, v) from it.



$\Rightarrow$

$$Az \odot Bz = Cz \text{ with}$$

$$z^T = (1, x_1, x_2, u_1, u_2, v)$$

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where “ $\odot$ ” is a coefficient-wise product.

Claim that $C(1, x_1, x_2) = v \Leftrightarrow \exists (u_1, u_2, v) / Az \odot Bz = Cz$.

We can build a **Rank 1 Constraint Satisfiability** (A, B, C, x, v) from it.

Why does it work?

Line 1 of A, B, C:

- $(Az)_1 = 1 + x_1$
- $(Bz)_1 = 1$
- $(Cz)_1 = u_1 = 1 \times (1 + x_1)$

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From now on the goal of P is to prove to V that it exists u_1, u_2, v such that $Az \odot Bz = Cz$.

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Still a bit vague, let’s make it more precise.

R1CS

P knows $z = (1, x_1, \ldots, x_n, u_1, \ldots, u_{n'}, v) = (1||x||u||v)$, with $u = (u_1, \ldots, u_{n'})$ supposed to be the outputs of the gates of the circuit.

V knows $(1||x)$ and v.

R1CS [BCRSVW18]

A R1CS instance is specified by $n \times m$ matrices A, B, C over $\mathbb{F}$ and by a vector x and an element v over $\mathbb{F}$.

It is satisfied by a vector u if and only if $Az \odot Bz = Cz$, $z := (1||x||u||v)$.

$\rightarrow$ the whole instance $= (\mathbb{F}, n, m, A, B, C, x, v)$.

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The relation R_{R1CS} is the set of tuples $((\mathbb{F}, n, m, A, B, C, x, v), u)$ such that u satisfies $(\mathbb{F}, n, m, A, B, C, x, v)$.

The Univariate Sumcheck

The “Aurora” [BCRSVW18] article proposes a protocol for R1CS relations

- Define $z_A = Az, z_B = Bz, z_C = Cz$.
- Separately check that $Az = z_A$, $Bz = z_B$, $Cz = z_C \rightarrow$ lincheck.
- Then check that $z_A \odot z_B = z_C \rightarrow$ rowcheck.

A core non-trivial ingredient is to be able to check the statement

$$\sum_{a \in H} \hat{f}(a) = \mu,$$

given $H \subset \mathbb{F}$ with $|H| =$ number of variables, $\hat{f}(X) \in \mathbb{F}[X]$, $\mu \in \mathbb{F}$.

The univariate sumcheck is a protocol that allows to do so.

The Univariate Sumcheck

We need, on input $H \subset \mathbb{F}$, $\hat{f}(X) \in \mathbb{F}[X]$, $\mu \in \mathbb{F}$, to be able to check that

$$\sum_{a \in H} \hat{f}(a) = \mu$$

The **univariate sumcheck** is a protocol that allows to do so.

Why not simply computing the sum?

- $O(|H|)$ evaluations of $\hat{f}(X)$ for the Verifier.
- An evaluation of $\hat{f}(X)$ costs $O(\deg \hat{f}(X))$ operations.

$\rightarrow$ way too long!

Reed-Solomon codes

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Given $L \subset \mathbb{F}$, $0 < d \leqslant |L|$, we denote by $RS[L, d]$ the evaluations over L of all polynomials of $\mathbb{F}[X]$ of degree $< d$.

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Encoding of a vector t into a codeword

Define $H = \{h_1, \ldots, h_d\}, L = \{\ell_1, \ldots, \ell_n\} \subset \mathbb{F}$ such that $|H| \leqslant |L|$, and $t \in \mathbb{F}^{|H|}$:

1. The "low degree extension" $\hat{f}_t(X)$ of t is defined as the **only polynomial of degree** $< |H|$ such that
$$\forall i \in \{1, \ldots, d\}, \hat{f}_t(h_i) = t_i.$$
2. $f_t := \hat{f}_t|_L := (f_t(\ell_1), \ldots, f_t(\ell_n))$ **is the codeword that encodes** t.

The FRI

What are we going to do with RS codewords?

1. Compute $\hat{f}_t(X)$ from H and t.
2. "**Check, given a vector f_t, that f_t belongs to** $RS[L, d]$."
   $\rightarrow$ Low degree test FRI [BBHR17]: Fast Reed-Solomon Interactive oracle proof of proximity.

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   $\rightarrow$ **Low degree test FRI [BBHR17]:** Fast Reed-Solomon Interactive oracle proof of proximity.

FRI $=$ **IOPP**, Interactive Oracle Proof of **Proximity**

- Allows interactions, oracle access, randomness ...
- Locality: logarithmic number of query.
- "Proximity" $\rightarrow$ the protocol checks **whether a vector f_t is in** $RS[L, d]$ **(so in** $RS[L, d]$ **with a certain probability) or far from it**.

The FRI, if L is well choosen, has the performance:

- *Prover time* < $6|L|$.
- *Verifier time* $\leqslant 21 \log |L|$.
- *Proof length* $< |L|/3$.
- *Query complexity* $= 2 \log |L|$.

The univariate Sumcheck

The univariate Sumcheck

Sumcheck Relation

The relation R_{SUM} is the set of all pairs $((\mathbb{F}, L, H, d, \mu), f_t)$ where

- $L, H \subset \mathbb{F}$
- $0 < d < |L|$
- $\mu \in \mathbb{F}$
- $f_t \in RS[L, d]$
- $\sum_{a \in H} \hat{f}_t(a) = \mu$.

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We can make an IOP protocol for the Sumcheck relation.

The univariate Sumcheck

A useful result

If H is an additive subgroup of $\mathbb{F}$, given a polynomial $\hat{g}(X)$ such that $\deg \hat{g}(X) \leqslant |H| - 1$ and the coefficient of $X^{|H|-1}$ in $\hat{g}(X)$ is α, we have

$$\sum_{a \in H} \hat{g}(a) = \alpha \sum_{a \in H} a^{|H|-1}.$$

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Setup/Inputs of the sumcheck: P knows f, V has oracle access to f.

Claim: $\sum_{a \in H} \hat{f}(a) = \mu$

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5. V and P computes $\zeta = \sum_{a \in H} a^{|H|-1}$.

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5. V and P computes $\zeta = \sum_{a \in H} a^{|H|-1}$.
6. V and P **run a FRI protocol** with P to check that
$$p := (\zeta\hat{f}(X) - \zeta Z_H(X)\hat{h}(X) - \mu X^{|H|-1})|_L \in RS[L, |H| - 1],$$

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7. **and another** to check that
$$h \in RS[L, \deg \hat{f}(X) - |H| + 1].$$

The univariate Sumcheck

Setup/Inputs of the sumcheck: *P knows f, V has oracle access to f.*

Performance

- *Prover time:*
  - one *IFFT* to get $\widehat{f}(X)$ from f.
  - one "divide-and-conquer" algorithms to get $Z_H(X)$: $O(\log|H|)$.
  - one polynomial divisions to compute $\hat{g}$, $\hat{h}$: $O(M(d))$ $(d = \deg \hat{f}(X))$.
  - two *FFT* to evaluate $\hat{h}(X), \hat{g}(X)$ over L.
  - two FRI: $< 6|L|$.

  so Prover time in $O(M(d)) + 3FFT(\mathbb{F}, L) + 12|L|$.

- *Verifier time:* $O(\log^2|H|)$ (computing ζ) $+42\log|L|)$ (FRI).
- *Query complexity:* $4\log|L|$ related to the low degree test.
- *Proof length:* $2|L|/3$.

Our goal

Sarah, Jade and Daniel made an efficient multivariate FRI recently, for tensor product of Reed-Solomon codes $RS[L_1, d_1] \otimes \ldots \otimes RS[L_n, d_n]$.

Tensor product of RS codes

Given $L_1, \ldots, L_n \subset \mathbb{F}$, $0 < d_1, \ldots, d_n < |L_1|,_d otsc, |L_n|$, we denote by $RS[L_1, d_1] \otimes \ldots \otimes RS[L_n, d_n]$ the evaluations over $L_1 \times \ldots \times L_n$ of all polynomials $\hat{f}(X_1, \ldots, X_n)$ of $\mathbb{F}[X_1, \ldots, X_n]$ such that $\forall i \in \{1, \ldots, n\}$, $\deg_{X_i} \hat{f}(X_1, \ldots, X_n) < d_i$.

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$\rightarrow$ Let's use it to build a multivariate Sumcheck.

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. . . **Actually, this already exists.**

The first multivariate sumcheck is from Carsten Lund et al [LFKN90] and was related to the SAT and UNSAT problems.

In fact, it has many applications.

Multivariate Sumcheck

IP protocol: Interactive protocol, with V reading all the messages it receives.

Inputs: P knows $\hat{p}(X_1, \ldots, X_n)$, V has oracle access to $\hat{p}(X_1, \ldots, X_n)$ and its degree.

Claim: $\sum_{a_1,\ldots,a_n \in H} \hat{p}(a_1, \ldots, a_n) = \alpha$.

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Protocol

Prover		Verifier
$\hat{p}_1(X) := \sum_{a_2,\ldots,a_n\in H}\hat{p}(X,a_2,\ldots,a_n)$	$\xrightarrow{\hat{p}_1(X)}$	$\sum_{a_1\in H}\hat{p}_1(a_1)\stackrel{?}{=}\alpha$
	$\xleftarrow{w_1}$	$w_1\stackrel{\$}{\leftarrow}\mathbb{F}$
$\hat{p}_2(X) := \sum_{a_3,\ldots,a_n\in H}\hat{p}(w_1,X,a_3,\ldots,a_n)$	$\xrightarrow{\hat{p}_2(X)}$	$\sum_{a_2\in H}\hat{p}_2(a_2)\stackrel{?}{=}\hat{p}_1(w_1)$
$\vdots$	$\vdots$	$\vdots$
$\hat{p}_n(X) := \hat{p}(w_1,\ldots,w_{n-1},X)$	$\xrightarrow{\hat{p}_n(X)}$	$\sum_{a_n\in H}\hat{p}_n(a_n)\stackrel{?}{=}\hat{p}_{n-1}(w_{n-1})$
		$w_n\stackrel{\$}{\leftarrow}\mathbb{F}$
		$\hat{p}(w_1,w_2,\ldots,w_n)\stackrel{?}{=}\hat{p}_n(w_n)$

Multivariate Sumcheck

Performance

- *Prover time:* $|H|^n$.
- *Verifier time:* $n|H| \deg_{ind} \hat{p}(X_1, \ldots, X_n)$.
- *Communication cost:* $n \deg_{ind} \hat{p}(X_1, \ldots, X_n)$.

Multivariate Sumcheck - an improvement

Original Multivariate Sumcheck $\leftarrow$ [LFKN90].

Ben-Sasson et al [BCGRS17] proposed an alternative algorithm, **using a univariate sumcheck**, **Reed-Solomon codes**, and a big abstract theorem [MIE09], to have better computing time.

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[BCGRS17]

- *Prover times:* $npoly(\log|\mathbb{F}|) + nO(|L|^2 + |H|\log(|L|^2 + |H|)) + n|L|^n$.
- *Verifier times:* $n \times poly(\log|\mathbb{F}| + \log(|L|^2 + |H|)) + O(n)$.
- *Proof length:* $O(n(|L|^2 + |H|)\log(|L|^2 + |H|))$.
- *Query complexity:* $O(n)$.

[LFKN90]]

- *Prover time:* $|H|^n$.
- *Verifier time:* $n|H| \deg_{ind} \hat{p}(X_1, \ldots, X_n)$.
- *Communication cost:* $n \deg_{ind} \hat{p}(X_1, \ldots, X_n)$.

Much better!

Can we do better ?

My multivariate Sumcheck

Let's focus on the bivariate case:

Another useful result

If H is an additive subgroup of $\mathbb{F}$, given a polynomial $\hat{f}(X, Y)$ such that $\deg_{X,Y} \hat{f} \leqslant |H| - 1$ and α is the coefficient of $X^{|H|-1}Y^{|H|-1}$ in $\hat{f}$, we have

$$\sum_{a_1, a_2 \in H} \hat{f}(a_1, a_2) = \alpha \sum_{a_1, a_2 \in H} a_1^{|H|-1} a_2^{|H|-1}.$$

Inputs: P knows $f = \hat{f}|_{L \times L}$, V has oracle access to $f = \hat{f}|_{L \times L}$. **Claim:** $\sum_{a_1, a_2 \in H} \hat{f}(a_1, a_2) = \mu$.

My multivariate Sumcheck

Protocol

1. P computes $\hat{f}(X, Y)$.

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6. V computes $\zeta = \left(\sum_{a \in H} a^{|H|-1}\right)^2$ and accepts if and only if
$$p \in RS[L, |H|] \otimes RS[L, |H| - 1]$$
where
$$\widehat{p} := \zeta(\widehat{f} - Y^{|H|-1}\widehat{g}_2 - \mu X^{|H|-1}Y^{|H|-1} - Z_H(X)\widehat{q}_1 - Z_H(Y)\widehat{q}_2).$$

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7. V and P also runs low-degree tests to check the degrees of $\hat{g}_2(X, Y), \hat{q}_1(X, Y)$ and $\hat{q}_2(X, Y)$.

Our multivariate Sumcheck

Performance

- *Prover time:*
  - one 2*DIFFT* to get $\widehat{f}(X, Y)$.
  - one "divide-and-conquer" algorithms to get$Z_H(X)$ and $Z_H(Y) \rightarrow O(\log|H|)$
  - four polynomial divisions to compute $\widehat{g}_2(X, Y)$, β, $\widehat{q}_1(X, Y)$, $\widehat{q}_2(X, Y) \rightarrow O(M(d) \times d)$.
  - three 2*DFFT* to evaluate those polynomials over L^2.
  - four 2*DFRI*: $O(|L|^2)$.

  so Prover time in $O(\log|H| + M(d)d) + 4FFT(\mathbb{F}, L^2) + O(|L|^2)$.
- *Verifier time:* $O(\log^2|H|) + O(\log|H|)$, related to the 2*DFRI*.
- *Query complexity:* $O(4\log|H|)$, related to the 2*DFRI*.
- *Proof length:* $O(|L|^2)$.

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If we have n variables

- *Prover time:* $O(n)FFT(\mathbb{F}, L^n) + O(\log|H|) + O(nM(d)d^{n-1}) + O(|L|^n)$.
- *Verifier time:* $O(n\log^2|H|) + O(n\log|H|)$.
- *Query complexity:* $O(n\log|H|)$.
- *Proof length:* $O(n|L|^n)$.

Comparison

	Sumcheck from [BCGRS17]	Our multivariate Sumcheck
Prover time	$\|L\|^n + npoly(\log\|\mathbb{F}\|) + n\tilde{O}(\|L\|^2 + \|H\|) + n\|L\|^n$	$O(n)FFT(\mathbb{F}, L^n) + O(\log\|H\|) + O(nM(d)d^{n-1}) + O(\|L\|^n)$
Verifier time	$poly(n + \|L\|) + npoly(\log\|\mathbb{F}\| + \log(\|L\|^2 + \|H\|)) + O(n)$	$O(n\log^2\|H\|) + O(n\log\|H\|)$
Proof length	$O(\|L\|^n\log(q) + n\tilde{O}(\|L\|^2 + \|H\|))$	$O(n\|L\|^n)$
Query complexity	$O(n)$	$O(n\log\|H\|)$

Conclusion

The univariate Sumcheck is well known and used, and it's efficiency is mostly due to the FRI protocol.

Since Sarah, Jade and Daniel made a multivariate version of the FRI, we made a multivariate version of the sumcheck that uses the FRI.

- it should have better performance in practice. Sumcheck from [3] Our multivariate Sumcheck
- it could be used within specific arithmetization with multivariate polynomials.

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Thank you for listening!

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