Quantum resource estimation

Gustavo Banegas[1]

March 16, 2021

[1]INRIA & LIX - École polytechnique, France
gustavo@cryptme.in

Outline

Cryptoapocalypse

Algorithms for Quantum Computation: Discrete Logarithms and Factoring

Peter W. Shor
AT&T Bell Labs
Room 2D-149
600 Mountain Ave.
Murray Hill, NJ 07974, USA

Abstract

A computer is generally considered to be a universal computational device; i.e., it is believed able to simulate any physical computational device with a cost in computation time of at most a polynomial factor. It is not clear whether this is still true when quantum mechanics is taken into consideration. Several researchers, starting with David Deutsch, have developed models for quantum mechanical computers and have investigated their computational properties. This paper gives Las Vegas algorithms for finding discrete logarithms and factoring integers on a quantum computer that take a number of steps which is polynomial in the input size, e.g., the number of digits of the integer to be factored. These two problems are generally considered hard on a classical computer and have been used as the basis of several proposed cryptosystems. (We thus give the first examples of quantum cryptanalysis.)

[1, 2]. Although he did not ask whether quantum mechanics conferred extra power to computation, he did show that a Turing machine could be simulated by the reversible unitary evolution of a quantum process, which is a necessary prerequisite for quantum computation. Deutsch [9, 10] was the first to give an explicit model of quantum computation. He defined both quantum Turing machines and quantum circuits and investigated some of their properties.

The next part of this paper discusses how quantum computation relates to classical complexity classes. We will thus first give a brief intuitive discussion of complexity classes for those readers who do not have this background. There are generally two resources which limit the ability of computers to solve large problems: time and space (i.e., memory). The field of analysis of algorithms considers the asymptotic demands that algorithms make for these resources as a function of the problem size. Theoretical computer scientists generally classify algorithms as effi-

Cryptoapocalypse

Algorithms for Quantum Computation: Discrete Logarithms and Factoring

Peter W. Shor
AT&T Bell Labs
Room 2D-149
600 Mountain Ave.
Murray Hill, NJ 07974, USA

Abstract

A computer is generally considered to be a universal computational device; i.e., it is believed able to simulate any physical computational device with a cost in computation time of at most a polynomial factor. It is not clear whether this is still true when quantum mechanics is taken into consideration. Several researchers, starting with David Deutsch, have developed models for quantum mechanical computers and have investigated their computational properties. This paper gives Las Vegas algorithms for finding discrete logarithms and factoring integers on a quantum computer that take a number of steps which is polynomial in the input size, e.g., the number of digits of the integer to be factored. These two problems are generally considered hard on a classical computer and have been used as the basis of several proposed cryptosystems. (We thus give the first examples of quantum cryptanalysis.)

[1, 2]. Although he did not ask whether quantum mechanics conferred extra power to computation, he did show that a Turing machine could be simulated by the reversible unitary evolution of a quantum process, which is a necessary prerequisite for quantum computation. Deutsch [9, 10] was the first to give an explicit model of quantum computation. He defined both quantum Turing machines and quantum circuits and investigated some of their properties.

The next part of this paper discusses how quantum computation relates to classical complexity classes. We will thus first give a brief intuitive discussion of complexity classes for those readers who do not have this background. There are generally two resources which limit the ability of computers to solve large problems: time and space (i.e., memory). The field of analysis of algorithms considers the asymptotic demands that algorithms make for these resources as a function of the problem size. Theoretical computer scientists generally classify algorithms as effi-

A fast quantum mechanical algorithm for database search

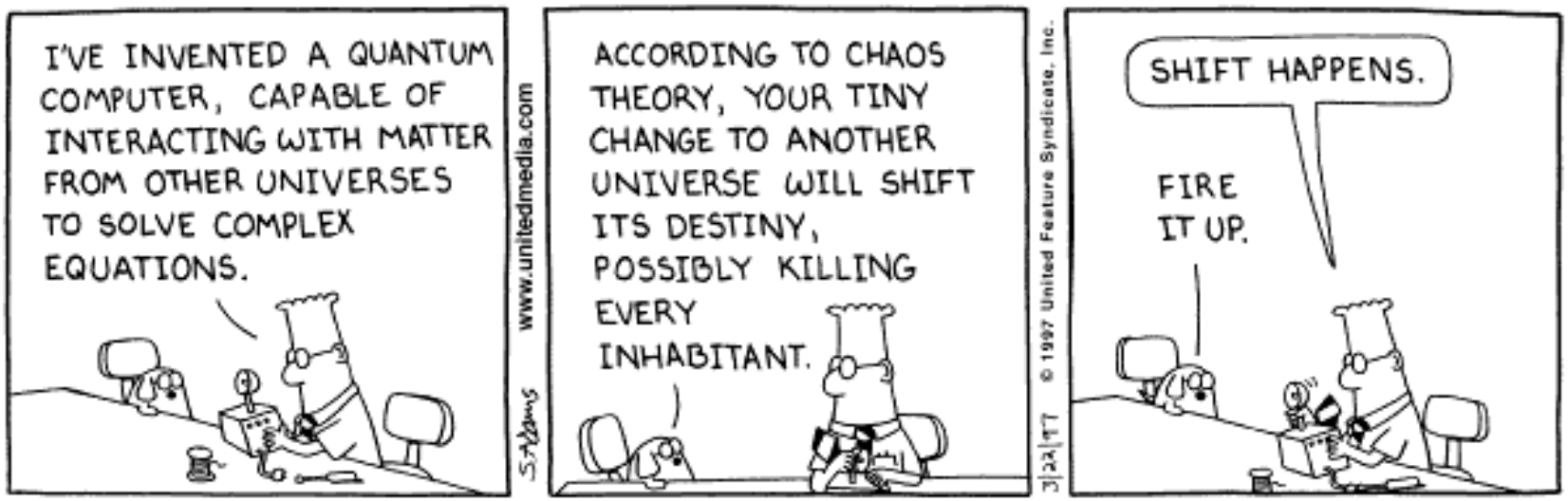
Lov K. Grover
3C-404A, AT&T Bell Labs
600 Mountain Avenue
Murray Hill NJ 07974
lkg@mhcnet.att.com

Summary

An unsorted database contains N records, of which just one satisfies a particular property. The problem is to identify that one record. Any classical algorithm, deterministic or probabilistic, will clearly take $O(N)$ steps since on the average it will have to examine a large fraction of the N records. Quantum mechanical systems can do several operations simultaneously due to their wave like properties. This paper gives an $O(\sqrt{N})$ step quantum mechanical algorithm for identifying that record. It is within a constant factor of the fastest possible quantum mechanical algorithm.

This paper applies quantum computing to a mundane problem in information processing and presents an algorithm that is significantly faster than any classical algorithm can be. The problem is this: there is an unsorted database containing N items out of which just one item satisfies a given condition - that one item has to be retrieved. Once an item is examined, it is possible to tell whether or not it satisfies the condition in one step. However, there does not exist any sorting on the database that would aid its selection. The most efficient classical algorithm for this is to examine the items in the database one by one. If an item satisfies the required condition stop; if it does not, keep track of this item so that it is not examined again. It is easily seen

In other words..



Introduction to Quantum Computing

How a quantum computer works?

- It perform computations based on **probabilities** of an object's state before it is measured;

Introduction to Quantum Computing

How a quantum computer works?

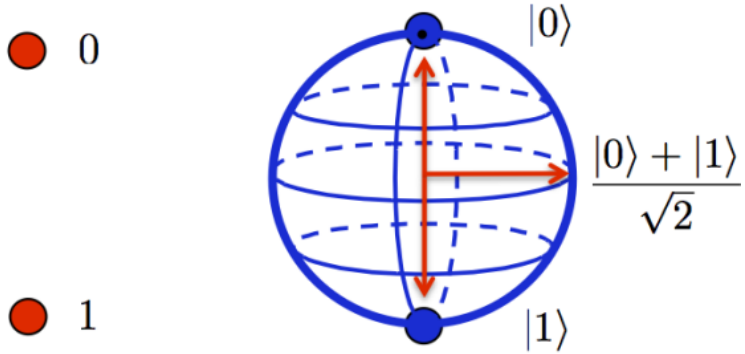
- It perform computations based on **probabilities** of an object's state before it is measured;
- We can change the probabilities of a **state**;

Quantum Computation - qubits

Qubit vs Classical bit

Quantum Computation - qubits

Qubit vs Classical bit



Classical Bit **Qubit**

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\alpha\,|0\rangle + \beta\,|1\rangle\,,$$

$$|\alpha|^2 + |\beta|^2 = 1$$

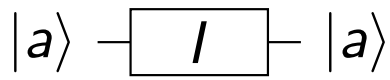
.

Measure quantum state

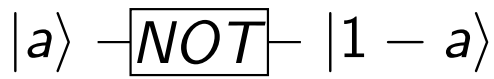
Measuring collapses the state.

Quantum gates

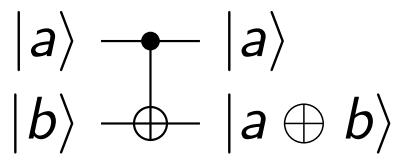
Identity gate:



NOT gate:

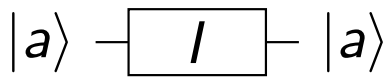


CNOT gate:

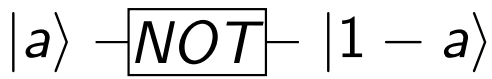


Quantum gates

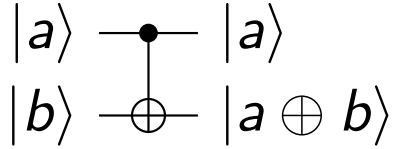
Identity gate:



NOT gate:

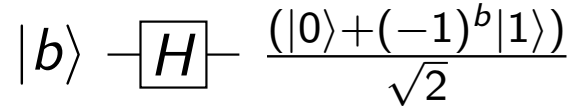


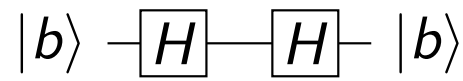
CNOT gate:



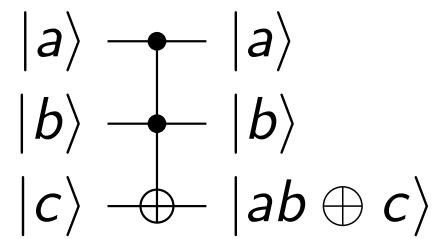
Hadamard Gate:

- $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$





Toffoli gate:



n-Qubit system

Definition

$|\psi\rangle \in \mathbb{C}^2$ such that $|| \, |\psi\rangle \, || = 1$,

$$|\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x \, |x\rangle$$

where

$$\sum_{x \in \{0,1\}^n} |\alpha_x|^2 = 1.$$

Example 2-qubit system

- 4 basis states:
  $|0\rangle \otimes |0\rangle$, $|0\rangle \otimes |1\rangle$,$|1\rangle \otimes |0\rangle$,
  $|1\rangle \otimes |1\rangle$.
- It is common to use just:
  $|0\rangle \, |1\rangle$,$|10\rangle$

Quantum computation and reversibility

Reversibility

Quantum evolution is unitary (or any operation that changes the state needs to be unitary);
Unitary means:

$$UU^\dagger = U^\dagger U = I$$

Quantum computation and reversibility

Reversibility

A unitary transformation taking basis states to basis states must be a permutation.

if $U|x\rangle = |u\rangle$ and $U|y\rangle = |u\rangle$, then $|x\rangle = U^{-1}|u\rangle = |y\rangle$.

Therefore quantum mechanics imposes the constraint that classically it must be reversible computation.

Computing functions

Using classical computers

Example to compute $H(H(H(H(x)))) = H^4(x)$:

Computing functions

Using classical computers

Example to compute $H(H(H(H(x)))) = H^4(x)$:

$Time$	$R0$	$R1$	$R2$

Computing functions

Using classical computers

Example to compute $H(H(H(H(x)))) = H^4(x)$:

Time	$R0$	$R1$	$R2$
time 0:	x	0	0

Computing functions

Using classical computers

Example to compute $H(H(H(H(x)))) = H^4(x)$:

Time	$R0$	$R1$	$R2$
time 0:	x	0	0
time 1:	x	0	$H(x)$

Computing functions

Using classical computers

Example to compute $H(H(H(H(x)))) = H^4(x)$:

Time	$R0$	$R1$	$R2$
time 0:	x	0	0
time 1:	x	0	$H(x)$
time 2:	x	0	$H^2(x)$

Computing functions

Using classical computers

Example to compute $H(H(H(H(x)))) = H^4(x)$:

Time	$R0$	$R1$	$R2$
time 0:	x	0	0
time 1:	x	0	$H(x)$
time 2:	x	0	$H^2(x)$
time 3:	x	0	$H^3(x)$

Computing functions

Using classical computers

Example to compute $H(H(H(H(x)))) = H^4(x)$:

Time	$R0$	$R1$	$R2$
time 0:	x	0	0
time 1:	x	0	$H(x)$
time 2:	x	0	$H^2(x)$
time 3:	x	0	$H^3(x)$
time 4:	x	$H^4(x)$	$H^3(x)$

Computing functions

Using classical computers

Example to compute $H(H(H(H(x)))) = H^4(x)$:

Time	$R0$	$R1$	$R2$
time 0:	x	0	0
time 1:	x	0	$H(x)$
time 2:	x	0	$H^2(x)$
time 3:	x	0	$H^3(x)$
time 4:	x	$H^4(x)$	$H^3(x)$

Computing functions

Make it reversible with Bennett–Tompa's method

Example to compute $H^4(x)$:

Computing functions

Make it reversible with Bennett–Tompa's method

Example to compute $H^4(x)$:

Time	*Register*0	$R1$	$R2$	$R3$	$R4$

Computing functions

Make it reversible with Bennett–Tompa's method

Example to compute $H^4(x)$:

Time	*Register*0	$R1$	$R2$	$R3$	$R4$
time 0:	x	0	0	0	0

Computing functions

Make it reversible with Bennett–Tompa's method

Example to compute $H^4(x)$:

Time	*Register*0	$R1$	$R2$	$R3$	$R4$
time 0:	x	0	0	0	0
time 1:	x	0	$H(x)$	0	0

Computing functions

Make it reversible with Bennett–Tompa's method

Example to compute $H^4(x)$:

Time	*Register*0	$R1$	$R2$	$R3$	$R4$
time 0:	x	0	0	0	0
time 1:	x	0	$H(x)$	0	0
time 2:	x	0	$H(x)$	$H^2(x)$	0

Computing functions

Make it reversible with Bennett–Tompa's method

Example to compute $H^4(x)$:

Time	*Register*0	$R1$	$R2$	$R3$	$R4$
time 0:	x	0	0	0	0
time 1:	x	0	$H(x)$	0	0
time 2:	x	0	$H(x)$	$H^2(x)$	0
time 3:	x	0	$H(x)$	$H^2(x)$	$H^3(x)$

Computing functions

Make it reversible with Bennett–Tompa's method

Example to compute $H^4(x)$:

Time	*Register*0	$R1$	$R2$	$R3$	$R4$
time 0:	x	0	0	0	0
time 1:	x	0	$H(x)$	0	0
time 2:	x	0	$H(x)$	$H^2(x)$	0
time 3:	x	0	$H(x)$	$H^2(x)$	$H^3(x)$
time 4:	x	$H^4(x)$	$H(x)$	$H^2(x)$	$H^3(x)$

Computing functions

Make it reversible with Bennett–Tompa's method

Example to compute $H^4(x)$:

Time	*Register*0	$R1$	$R2$	$R3$	$R4$
time 0:	x	0	0	0	0
time 1:	x	0	$H(x)$	0	0
time 2:	x	0	$H(x)$	$H^2(x)$	0
time 3:	x	0	$H(x)$	$H^2(x)$	$H^3(x)$
time 4:	x	$H^4(x)$	$H(x)$	$H^2(x)$	$H^3(x)$
time 5:	x	$H^4(x)$	$H(x)$	$H^2(x)$	0

Computing functions

Make it reversible with Bennett–Tompa's method

Example to compute $H^4(x)$:

Time	*Register*0	$R1$	$R2$	$R3$	$R4$
time 0:	x	0	0	0	0
time 1:	x	0	$H(x)$	0	0
time 2:	x	0	$H(x)$	$H^2(x)$	0
time 3:	x	0	$H(x)$	$H^2(x)$	$H^3(x)$
time 4:	x	$H^4(x)$	$H(x)$	$H^2(x)$	$H^3(x)$
time 5:	x	$H^4(x)$	$H(x)$	$H^2(x)$	0
time 6:	x	$H^4(x)$	$H(x)$	0	0

Computing functions

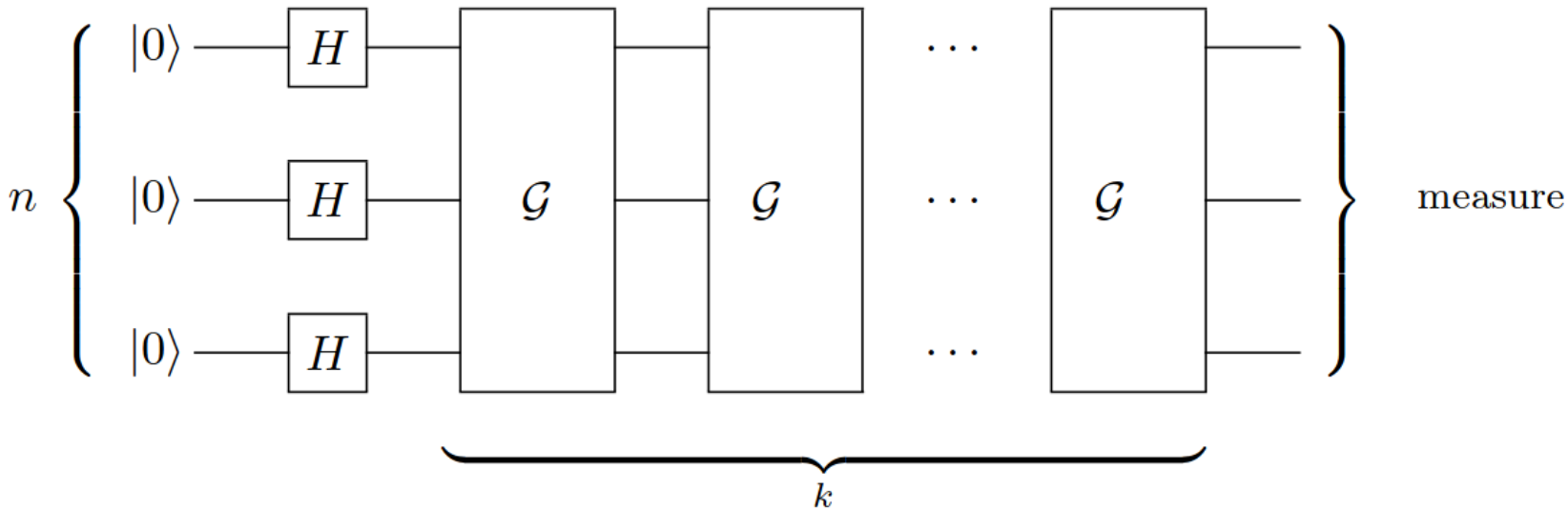
Make it reversible with Bennett–Tompa's method

Example to compute $H^4(x)$:

Time	*Register*0	$R1$	$R2$	$R3$	$R4$
time 0:	x	0	0	0	0
time 1:	x	0	$H(x)$	0	0
time 2:	x	0	$H(x)$	$H^2(x)$	0
time 3:	x	0	$H(x)$	$H^2(x)$	$H^3(x)$
time 4:	x	$H^4(x)$	$H(x)$	$H^2(x)$	$H^3(x)$
time 5:	x	$H^4(x)$	$H(x)$	$H^2(x)$	0
time 6:	x	$H^4(x)$	$H(x)$	0	0
time 7:	x	$H^4(x)$	0	0	0

Grover's Algorithm

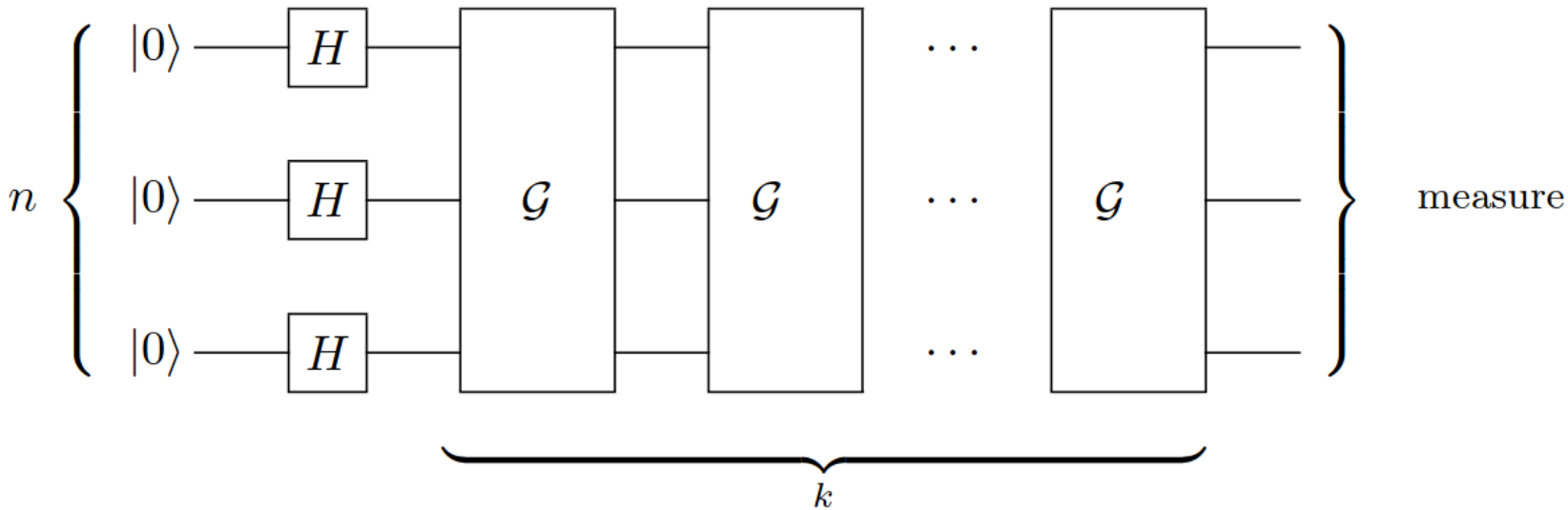
Grover's algorithm in a nutshell



- Originally described as search of an element in an unoreded database.

Grover's Algorithm

Grover's algorithm in a nutshell



- Originally described as search of an element in an unoreded database.
- Needs $O(\sqrt{N})$ queries in database of size $N = 2^n$ elements.

Grover's Algorithm

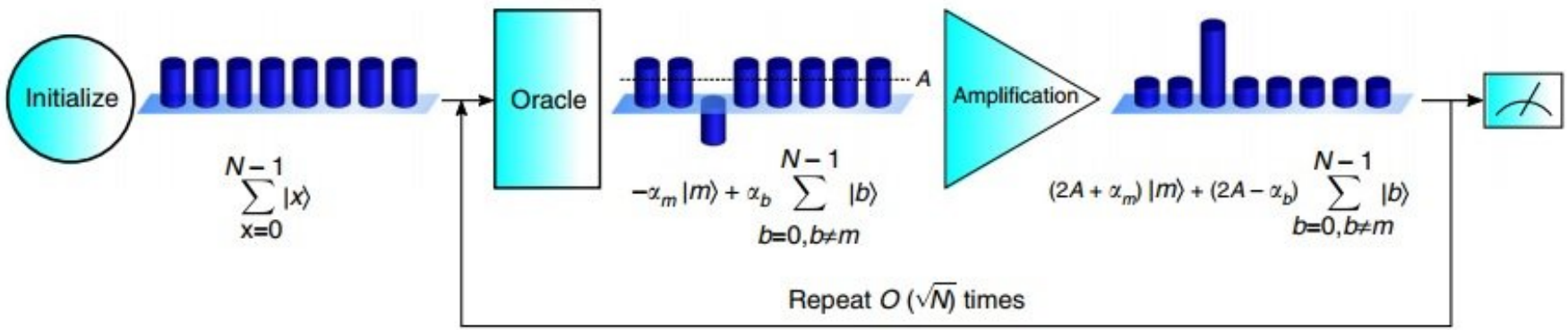
Grover's algorithm in a nutshell

Grover(f, t):

1. Start with $|\phi_0\rangle = |1^n\rangle$
2. Apply $\mathbf{H}^{\otimes n}$
3. Repeat $O\left(\sqrt{2^n}\right)$ times
4. Query to oracle $\mathcal{O}_f$
5. Amplification;
6. Return $x = |\phi\rangle$ with $f(x) = 1$.

Grover's Algorithm

Grover's algorithm in a nutshell



Quantum AES

AES in quantum gates

- All the operations can only be build using quantum gates;

Quantum AES

AES in quantum gates

- All the operations can only be build using quantum gates;
- It needs to be reversible;

Quantum AES

AES in quantum gates

- All the operations can only be build using quantum gates;
- It needs to be reversible;
- Lower depth and low amount of qubits.

Quantum AES

AES in quantum gates

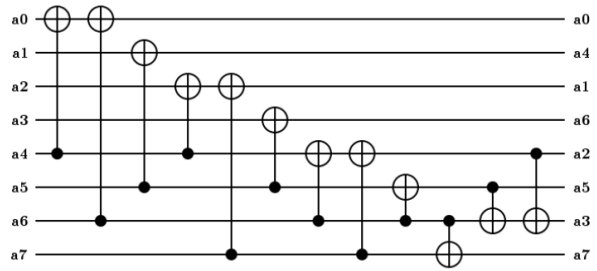
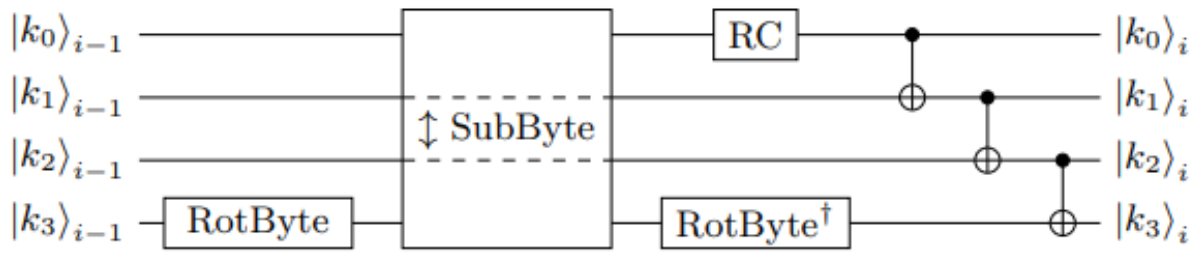


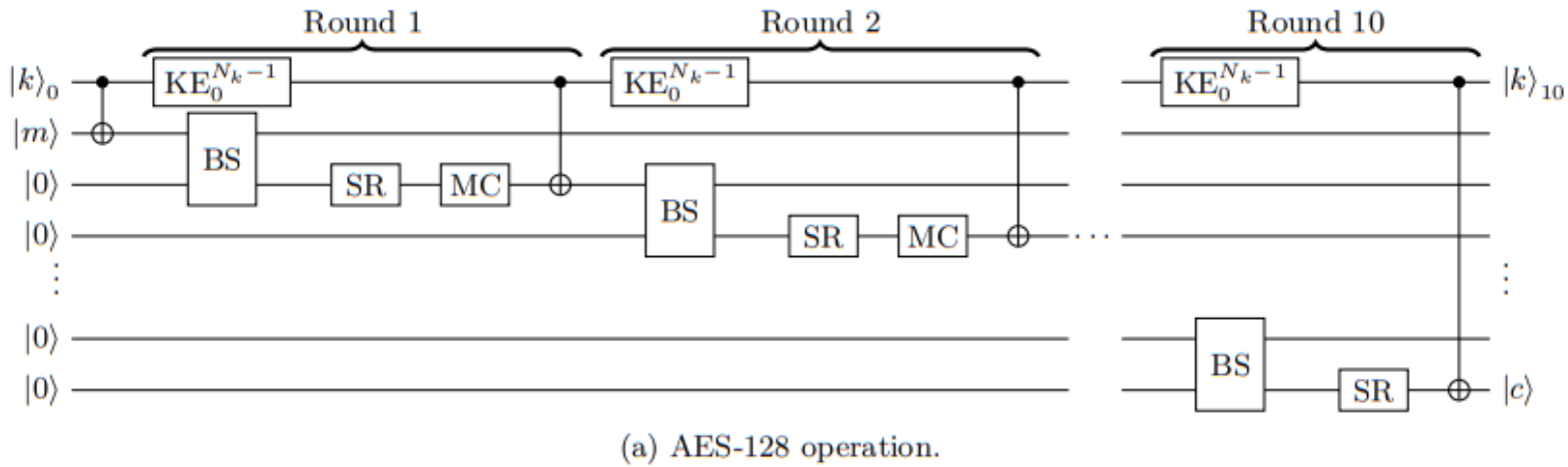
Figure: Squaring in $\mathsf{F}_2[x]/x^8 + x^4 + x^3 + x + 1$



(a) AES-128 in-place key expansion step producing the i-th round key.

Quantum AES

AES in quantum gates



(a) AES-128 operation.

Figure: Complete AES-128.

Grover's algorithm for breaking AES

Quantum Resources

Table: Number of gates for running Grover's algorithm on AES-128

	Cliff +CNOT	T gates
GLRS[2]	$1.55 \cdot 2^{86}$	$1.19 \cdot 2^{86}$
LPS[3]	$1.46 \cdot 2^{82}$	$1.47 \cdot 2^{81}$
JNRV[4]	$1.13 \cdot 2^{82}$	$1.32 \cdot 2^{78}$

[2] Markus Grassl, Brandon Langenberg, Martin Roetteler, and Rainer Steinwandt. Applying Grovers algorithm to AES: quantum resource estimates. In PQCRYPTO 16. Springer, 2016.

[3] Brandon Langenberg, Hai Pham, and Rainer Steinwandt. Reducing the cost of implementing AES as a quantum circuit. Cryptology ePrint Archive, Report 2019/854, 2019.

[4] Samuel Jaques, Michael Naehrig, Martin Roetteler and Fernando Virdia. Implementing Grover oracles for quantum key search on AES and LowMC. In EUROCRYPT 2020, 2020.

Introduction to Binary ECC

Basic overview

- Binary elliptic curves are elliptic curves defined over a binary field $\mathbb{F}_{2^n}$;

Introduction to Binary ECC

Basic overview

- Binary elliptic curves are elliptic curves defined over a binary field $\mathbb{F}_{2^n}$;
- We use polynomial representation and the operations are in $\mathbb{F}_2$ since $\mathbb{F}_{2^n} \cong \mathbb{F}_2[z]/(m(z))$, where $m(z)$ is an irreducible polynomial of degree n;

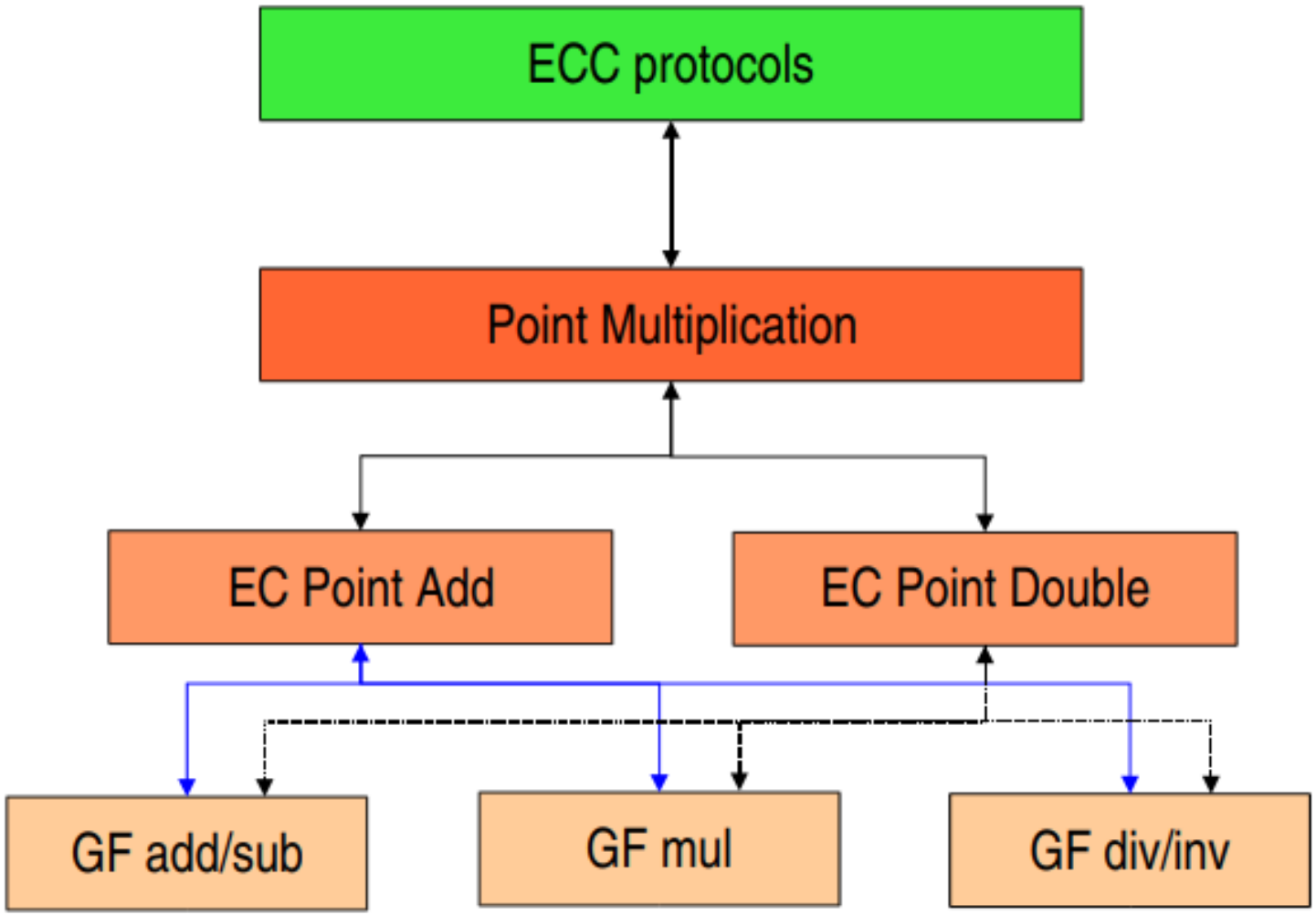
Introduction to Binary ECC

Basic overview

- Binary elliptic curves are elliptic curves defined over a binary field $\mathbb{F}_{2^n}$;
- We use polynomial representation and the operations are in $\mathbb{F}_2$ since $\mathbb{F}_{2^n} \cong \mathbb{F}_2[z]/(m(z))$, where $m(z)$ is an irreducible polynomial of degree n;
- All computations are done mod $m(z)$.

Introduction to Binary ECC

Basic overview of operations



Introduction to Binary ECC

Hardness of ECC

- Alice and Bob agrees in the same point P over a curve;

Introduction to Binary ECC

Hardness of ECC

- Alice and Bob agrees in the same point P over a curve;
- Alice selects a secret integer α and Bob selects an integer β;

Hardness of ECC

- Alice and Bob agrees in the same point P over a curve;
- Alice selects a secret integer α and Bob selects an integer β;
- Then, they calculate and tell each other $P_\alpha = [\alpha]P$ and $P_\beta = [\beta]P$;

Hardness of ECC

- Alice and Bob agrees in the same point P over a curve;
- Alice selects a secret integer α and Bob selects an integer β;
- Then, they calculate and tell each other $P_\alpha = [\alpha]P$ and $P_\beta = [\beta]P$;
- Finally, they calculate their shared point $P_{\alpha\beta} = [\alpha \cdot \beta]P = [\alpha]P_\beta = [\beta]P_\alpha$.

Shor's algorithm

In summary Shor's algorithm has two parts:

- A reduction of the factoring problem to the problem of **order-finding**, which can be done on a classical computer;

Shor's algorithm

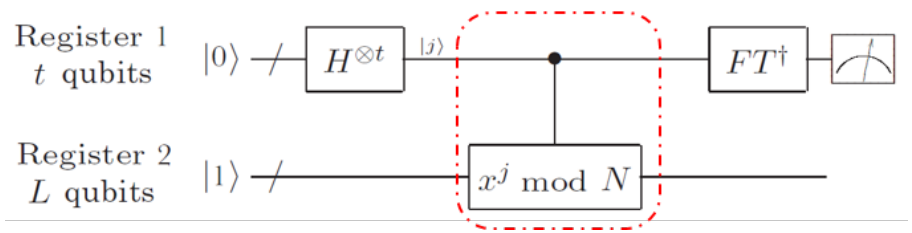
In summary Shor's algorithm has two parts:

- A reduction of the factoring problem to the problem of **order-finding**, which can be done on a classical computer;
- A quantum algorithm to solve the **order-finding** problem.

Shor's algorithm

A toy example can be when we have $N = 15$. Let's see how Shor's algorithm works:

1. Select an arbitrary number, such as $a = 2\ (< 15)$
2. $gcd(a, N) = gcd(2, 15) = 1$
3. Find the period of function $f(x) = a^x \mod N$, which satisfies $f(x + r) = f(x)$;
4. Get $r = 4$ through the circuit below;
5. $gcd(a^{\frac{r}{2}} + 1, N) = gcd(5, 15) = 3$;
6. $gcd(a^{\frac{r}{2}} - 1, N) = gcd(3, 15) = 5$;
7. For $N = 15$, the two decomposed prime numbers are 3 and 5.



Ressource Estimation

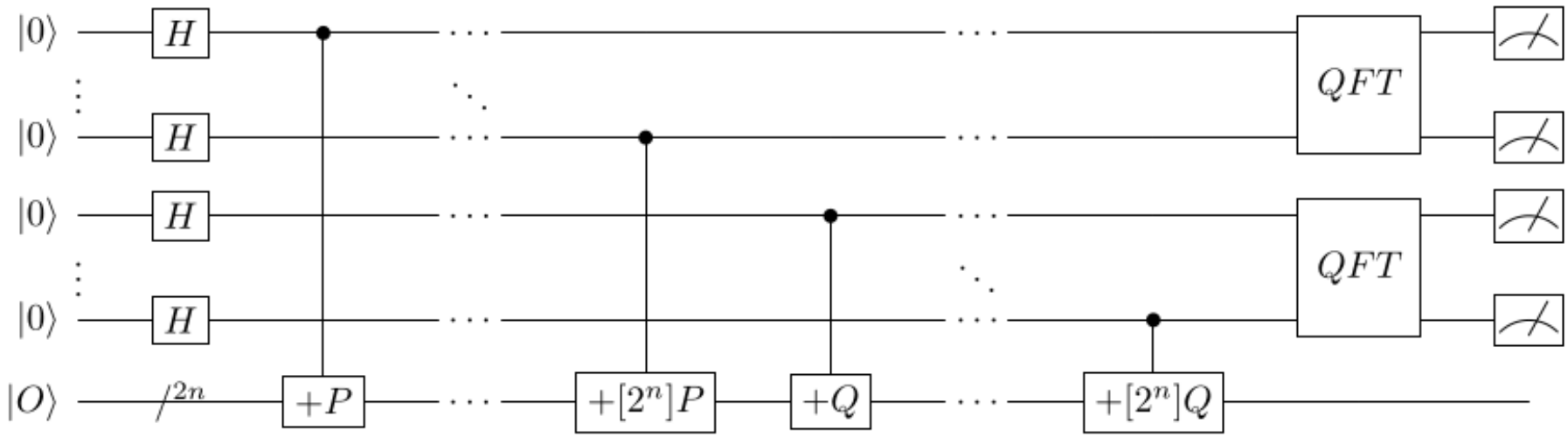
Break RSA (Integer Factoring)

From Gidney & Ekerå(2019)[5] uses "$3n + 0.002nlg(n)$ logical qubits, $0.3n^3 + 0.0005n^3lg(n)$ Toffolis, and $500n^2 + n^2lg(n)$ measurement depth to factor n-bit RSA integers"

RSA Bits	Qubits	Toffoli + T Gates (billions)
1024	3092	0.4
2048	6189	2.7
3072	9287	9.9

[5] Craig Gidney and Martin Ekerå. How to factor 2048 bit RSA integers in 8 hours using 20 million noisy qubits. arXiv preprint quant-ph/1904.09749, 2019. https://arxiv.org/abs/1905.09749

Shor's circuit for finding elliptic curve discrete logarithm



Shor's circuit for finding elliptic curve discrete logarithm

- Implementation (Quantumly) of Inversion using GCD and FLT (Fermat's little theorem);

[6] Iggy van Hoof. Space-efficient quantum multiplication of polynomials for binaryfinite fields with sub-quadratic Toffoli gate count.Quantum Information & Computation, pages 721–735, 2020.https://arxiv.org/abs/1910.02849.

Shor's circuit for finding elliptic curve discrete logarithm

- Implementation (Quantumly) of Inversion using GCD and FLT (Fermat's little theorem);
  - We use for multiplication Karatsuba from Iggy's paper[6];

[6] Iggy van Hoof. Space-efficient quantum multiplication of polynomials for binaryfinite fields with sub-quadratic Toffoli gate count.Quantum Information & Computation, pages 721–735, 2020.https://arxiv.org/abs/1910.02849.

Shor's circuit for finding elliptic curve discrete logarithm

- Implementation (Quantumly) of Inversion using GCD and FLT (Fermat's little theorem);
  - We use for multiplication Karatsuba from Iggy's paper[6];
  - The GCD-based inversion performed better in number of qubits and gates.

[6] Iggy van Hoof. Space-efficient quantum multiplication of polynomials for binaryfinite fields with sub-quadratic Toffoli gate count.Quantum Information & Computation, pages 721–735, 2020.https://arxiv.org/abs/1910.02849.

Shor's circuit for finding elliptic curve discrete logarithm

- Implementation (Quantumly) of Inversion using GCD and FLT (Fermat's little theorem);
  - We use for multiplication Karatsuba from Iggy's paper[6];
  - The GCD-based inversion performed better in number of qubits and gates.
- Implementation of quantum Point addition and Point "doubling";

[6] Iggy van Hoof. Space-efficient quantum multiplication of polynomials for binaryfinite fields with sub-quadratic Toffoli gate count.Quantum Information & Computation, pages 721–735, 2020.https://arxiv.org/abs/1910.02849.

Shor's circuit for finding elliptic curve discrete logarithm

- Implementation (Quantumly) of Inversion using GCD and FLT (Fermat's little theorem);
  - We use for multiplication Karatsuba from Iggy's paper[6];
  - The GCD-based inversion performed better in number of qubits and gates.
- Implementation of quantum Point addition and Point "doubling";
- Present the a quantum version of "window" addition;
- Q# implementation of Karatsuba and other functions.

[6] Iggy van Hoof. Space-efficient quantum multiplication of polynomials for binaryfinite fields with sub-quadratic Toffoli gate count.Quantum Information & Computation, pages 721–735, 2020.https://arxiv.org/abs/1910.02849.

Ressource Estimation

Break Binary ECC (DLP)

From Banegas, Bernstein, von Hoof and Lange(2021)[7] we have that for breaking binary ECC we have $7n + \lfloor \log(n) \rfloor + 9$ qubits, $48n^3 + 8n^{\log(3)+1} + 352n^2 \log(n) + 512n^2 + O(n^{\log(3)})$ Toffoli gates and $O(n^3)$ CNOT gates (More details in the presentation at CHES2021).

		Single step			Total
n	qubits	TOF gates	CNOT gates	depth upper bound	TOF gates
163	1,157	893,585	827,379	1,262,035	293,095,880
233	1,647	1,669,299	1,614,947	2,405,889	781,231,932
283	1,998	2,427,369	2,358,734	3,503,510	1,378,745,592
571	4,015	8,987,401	9,080,190	13,237,682	10,281,586,744

[7] Banegas, G., Bernstein, D. J., van Hoof, I., Lange, T. Concrete quantum cryptanalysis of binary elliptic curves. IACR Transactions on Cryptographic Hardware and Embedded Systems, 2021(1)

Future works

In a not so distant future:

- Improve the analysis of CNOT gates;
- Improve the depth of the algorithms;
- Improve small circuits such as incrementer.

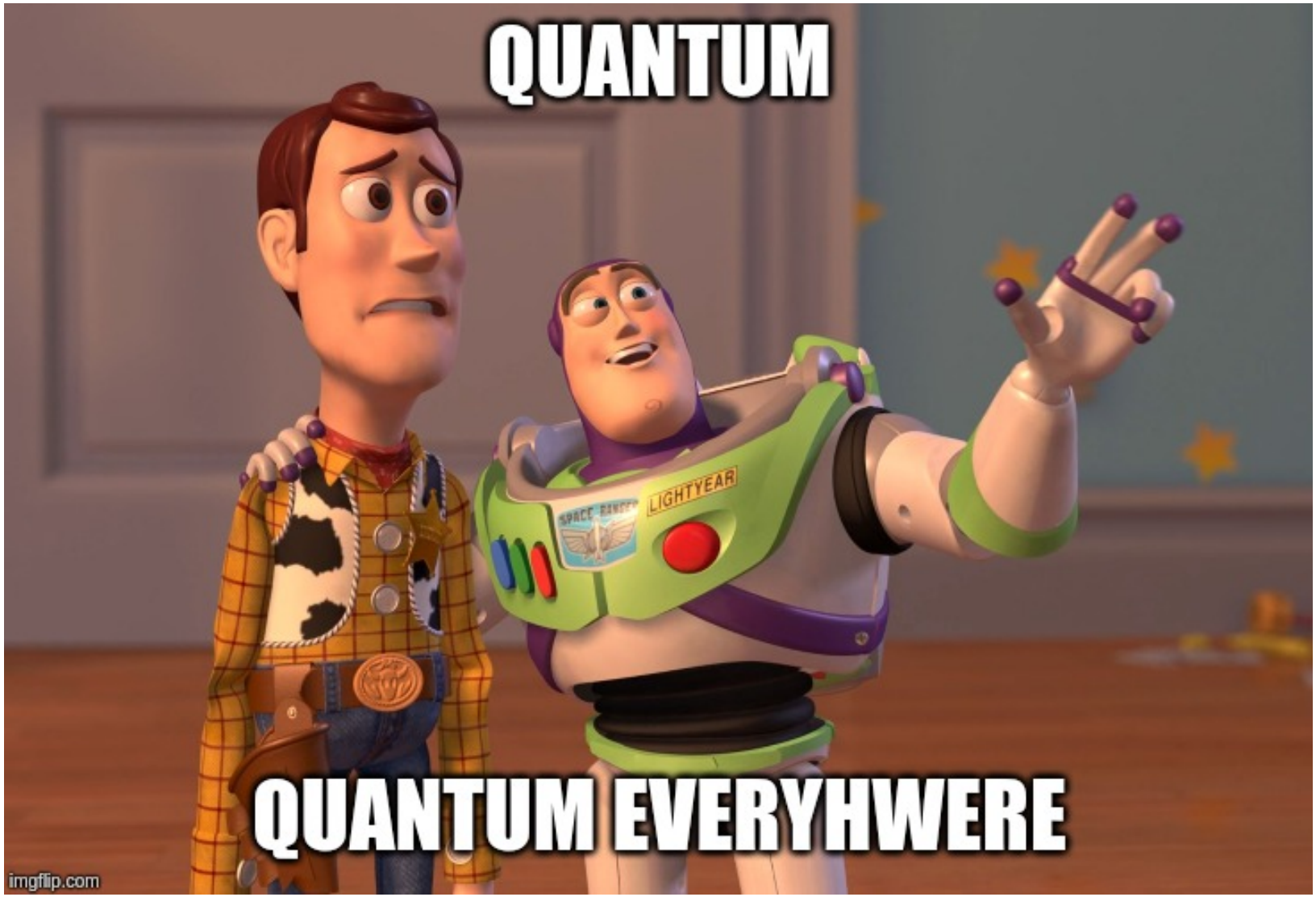
Future works

In a not so distant future:

- Improve the analysis of CNOT gates;
- Improve the depth of the algorithms;
- Improve small circuits such as incrementer.

In a more distante future:

- Quantum resource estimation for McEliece (Grover);
- Improve quantum attacks to LowMC (Picnic).


QUANTUM
LIGHTYEAR
QUANTUM EVERYHWERE
imgflip.com

Other Quantum algorithms

- Simon's Algorithm (QFT);
- Ambaini's Algorithm (Element disticness);
- Claw finding Algorithm;
- Kuperberg's Algorithm (dihedral hidden subgroup problem);

Questions

Thank you for your attention.
Questions?
gustavo@cryptme.in