SQISign: Compact Post-Quantum Signatures from Quaternions and Isogenies

Antonin Leroux, joint work with L. De Feo, D. Kohel, C. Petit and B. Wesolowski

DGA, Ecole Polytechnique, Institut Polytechnique de Paris, Inria Saclay

The State of Post-Quantum Cryptography

Six families still in Round 3 NIST post-quantum competition (Finalists + Alternate Candidates):

Lattices	4 encryption	2 signature
Codes	3 encryption	
Multivariate		2 signature
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Hash-based		1 signature
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Signatures maybe?

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Jao and Soukharev "Isogeny-based quantum-resistant undeniable signatures"

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Yoo et al. "A post-quantum digital signature scheme based on supersingular isogenies"

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De Feo and Galbraith "SeaSign: Compact isogeny signatures from class group actions"

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- [BKV19] CSI-FiSh: Based on CSIDH + precomp. $\Rightarrow$ bad scaling,
  similar to SeaSign with improved efficiency and sizes.

Beullens, Kleinjung, and Vercauteren "CSI-FiSh: Efficient isogeny based signatures through class group computations"

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New security assumption.

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The **dual** isogeny $\hat{\varphi} : F \to E$

$$\hat{\varphi} \circ \varphi = [\deg(\varphi)]_E$$

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All *supersingular* curves have a model over $\mathbb{F}_{p^2}$.

Supersingular Isogeny Diffie Hellman

Key exchange betw. Alice and Bob.

Jao and De Feo "Towards Quantum-Resistant Cryptosystems from Supersingular Elliptic Curve Isogenies"

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Efficient when N_A, N_B are *smooth*.

Jao and De Feo "Towards Quantum-Resistant Cryptosystems from Supersingular Elliptic Curve Isogenies"

Supersingular Isogeny Problem

The underlying *security problem*:

Supersingular ℓ-Isogeny Problem: Given a prime p and two supersingular curves E_1 and E_2 over $\mathbb{F}_{p^2}$, compute an ℓ^e-isogeny $\varphi : E_1 \to E_2$ for $e \in \mathbb{N}^\star$.

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SIDH assumption is *stronger*: additional information required to compute the push-forward maps.

The Deuring Correspondence

Quaternion Algebra, Orders and Ideals

The **Quaternion algebra** $H(a, b)$ is

$$H(a,b) = \mathbb{Q} + i\mathbb{Q} + j\mathbb{Q} + k\mathbb{Q} \text{ with } i^2 = a, j^2 = b$$

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Fractional ideals are $\mathbb{Z}$-lattices of rank 4 inside $H(a,b)$

$$I = \alpha_1\mathbb{Z} + \alpha_2\mathbb{Z} + \alpha_3\mathbb{Z} + \alpha_4\mathbb{Z}$$

The **Reduced norm** $n(I) = \{\mathrm{gcd}(n(\alpha)), \alpha \in I\}$

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The (maximal) **left order**[1] $\mathcal{O}_L(I)$ of an *ideal* is

$$\mathcal{O}_L(I) = \{\alpha \in H(a,b), \alpha I \subset I\}$$

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$\pi : (x, y) \mapsto (x^p, y^p)$ is the Frobenius

$\iota : (x, y) \mapsto (-x, \sqrt{-1}y)$ is the twisting automorphism of E_0.

A new security problem?

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[Koh+14]: *heuristic polynomial* time algorithm `KLPT` for quaternion path problem.

Kohel et al. "On the quaternion ℓ-isogeny path problem"

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[Eis+18]: use `KLPT` to prove *heuristic polynomial* time reduction from supersingular ℓ-isogeny problem to :

Endomorphism Ring Problem: Given a *supersingular elliptic curve E* over $\mathbb{F}_{p^2}$, compute its endomorphism ring.

Eisenträger et al. "Supersingular Isogeny Graphs and Endomorphism Rings: Reductions and Solutions"

Proof of Knowledge of Endomorphism Ring

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First attempt: GPS Signature in 2017.

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$$\begin{tikzcd} E_0 \arrow[d, dashed, "\tau"'] \\ E_A \end{tikzcd}$$

secret key isogeny

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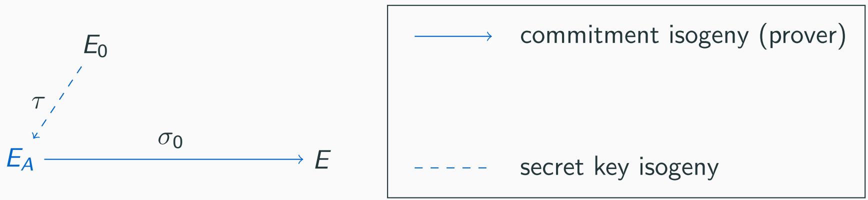

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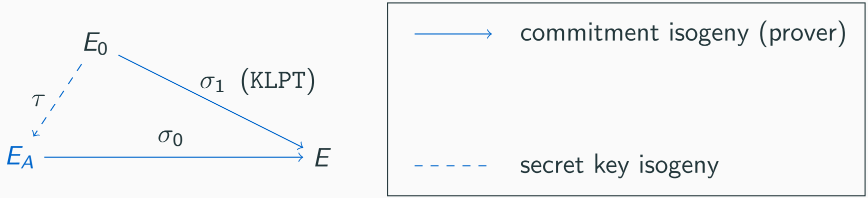

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$b \in \{0, 1\}$

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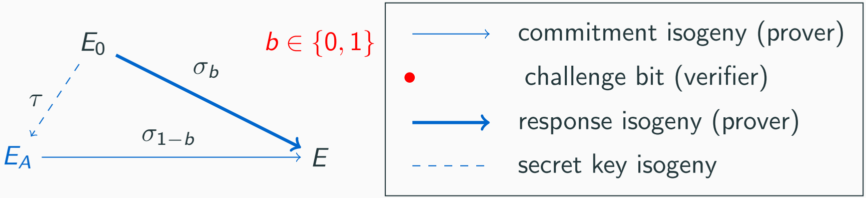

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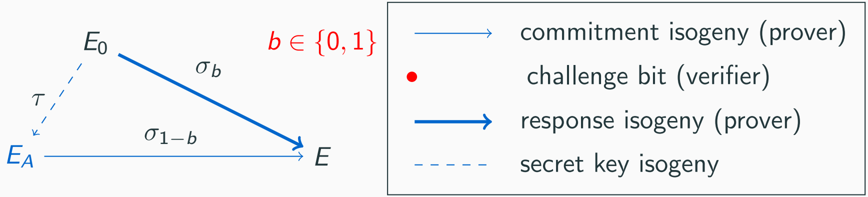

Repeat this λ times to reach 2^{λ}-bits of soundness.

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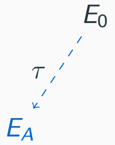

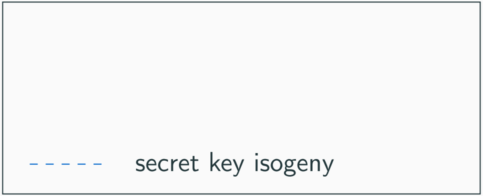

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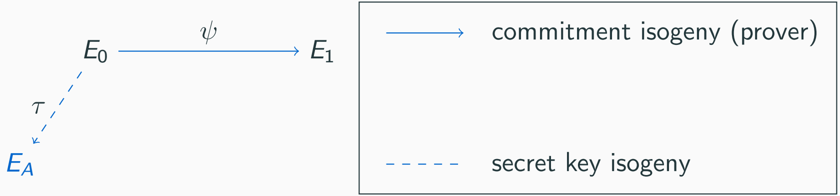

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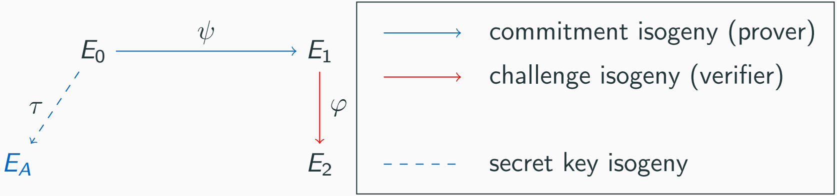

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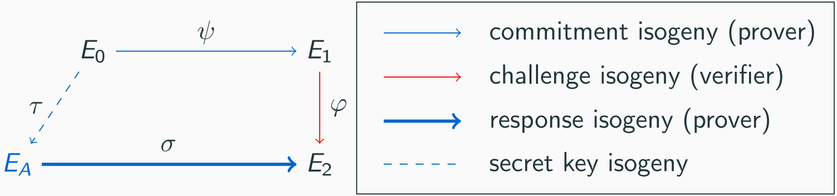

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Probability to cheat without knowledge of $\mathrm{End}(E_A)$: $O(\frac{1}{\deg \varphi})$.

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Soundness: Given *two* valid transcripts for *two* different challenges for the *same* commitment, some knowledge is revealed on the secret key.

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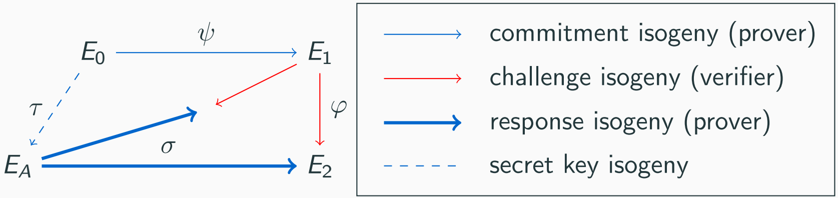

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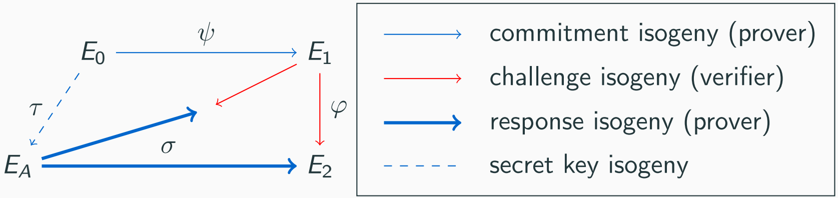

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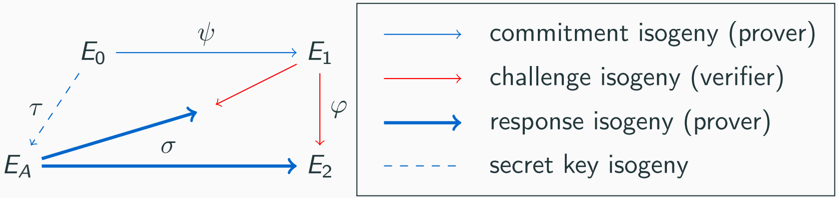

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[Eis+18]: prove *heuristic polynomial* reduction to the **Endomorphism Ring Problem**.

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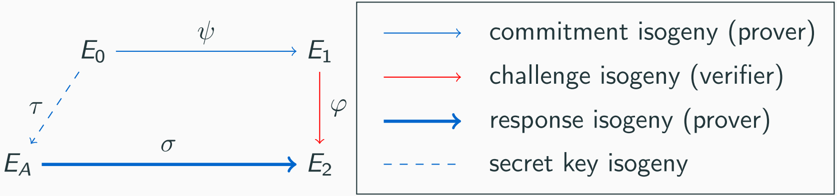

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We propose a new `SigningKLPT` algorithm.

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$\mathcal{P}_{\mathsf{deg}(\tau)}$ can be computed from $\deg(\tau)$ only and has exponential size.

SQISign in Practice

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- Take D powersmooth $\rightarrow E[D]$ in $\sim$ small extension ([GPS17]).
- Take $D = \ell^f$ and split σ in smaller isogenies of degree ℓ^e and apply `IdealToIsogeny` for each (**SQISign**).

New Pb: for generic E of known $\mathrm{End}(E)$, hard to evaluate $\mathrm{End}(E)$...

Choice of Parameters

In summary, for efficient translation: accessible $\ell^e T$-torsion for e as big as possible and smooth $T \wedge \ell = 1$ with $T^2 \sim p^3$ (constraint from `KLPT`).

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We found a 256 bits prime p with $e = 33$ and 2^{13}-smooth integer of 395 bits:

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Bottleneck of the signature: T-isogeny computations $O(1000/33)$.

What now?

Future implementation improvements

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- Various tradeoffs to explore.
- The size of `KLPT` solutions: huge impact on almost every aspect of the scheme. Current best is $O(p^3)$, going to $O(p^{5/2})$ could allow to cut in two the signing time (the best possible is $O(p)$)

Conclusion and Important Problems

We introduced the most compact post-quantum signatures but efficiency is still order of magnitudes below the competitors. The underlying sigma protocol has zero-knowledge property relying on a new assumption.

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- Find new algorithms for effective Deuring Correspondence.

Questions?

https://eprint.iacr.org/2020/1240