Asymptotic Performance of G-codes and Uncertainty Principle

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Outline

1. Block codes and G-codes
2. Asymptotic performance of G-codes
3. Uncertainty principle
4. Conclusion and outlook
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Block codes

K finite field of cardinality q.

Basic definitions

- A q-ary **linear code** $\mathcal{C}$ of **length** n is a subspace of K^n.
- If $c = (c_1, \ldots, c_n) \in \mathcal{C}$ (**codeword**), the (Hamming) **weight** of c is

$$\mathrm{wt}(c) = \#\{i \in \{1, \ldots, n\} \mid c_i \neq 0\}.$$

- $\mathcal{C}^\perp = \{v \in K^n \mid \langle v, c \rangle = 0, \text{ for all } c \in \mathcal{C}\}$ (**dual** of $\mathcal{C}$).

Parameters

Parameters: $[n, k, d]_q$.

- $d = \mathrm{d}(\mathcal{C}) = \min_{c \in \mathcal{C}, c \neq 0} \mathrm{wt}(c)$ (**minimum distance**).
- $R = k/n$ (**information rate**).

G-codes

$G \neq \{1_G\}$ **finite group**.

Definition

A G-**code** (or a **group code**) over K is a right ideal in the **group algebra**

$$KG = \left\{ a = \sum_{g \in G} a_g g \;\middle|\; a_g \in K \right\}.$$

Definition

- $G = C_m$ (cyclic group of order m) $\Rightarrow$ **cyclic code**.
- $G = D_{2m}$ (dihedral group of order $2m$) $\Rightarrow$ **dihedral code**.
- $G = C_m \rtimes C_r$ (metacylic group of order rm) $\Rightarrow$ **metacyclic code**.

Remark

If $\#G = n$, fix an ordering $G = \{g_1, \ldots, g_n\}$, then

$$\begin{array}{cccc} \varphi : & KG & \xrightarrow{\sim} & K^n \\ & \sum_{i=1}^{n} a_i g_i & \mapsto & (a_1, \ldots, a_n). \end{array}$$

The isomorphism is not canonical!
Different orderings yield permutation equivalent codes.

Via φ:

G-codes	$\rightsquigarrow$	Linear codes.
Hamming metric in KG	$\leftsquigarrow$	Hamming metric in K^n.
Inner product in KG	$\leftsquigarrow$	Inner product in K^n.
Action of G	$\rightsquigarrow$	Permutation automorphism (regular) subgroup.

Examples and counterexamples

- The self-dual $[24, 12, 8]$ **Golay code** is a S_4-code (Bernhardt, Landrock and Manz - 1990) and a D_{24}-code (McLoughlin and Hurley - 2008).
- The self-dual $[48, 24, 12]$ **extended quadratic residue code** is a D_{48}-code.
- The self-dual $[72, 36, 16]$ code (if it exists!) is not a group code, since $\#\mathrm{PAut}(\mathcal{C}) \leqslant 5$ (B., Willems and many others).
- The $[12, 6, 6]_3$ **Golay code** $\mathcal{G}$ is not a group code, even if $\#\mathrm{PAut}(\mathcal{G}) = 660$.
- The **Reed-Muller codes** $\mathcal{RM}_p(r, m) = J^{m(p-1)-r}$ (p prime), with J Jacobson radical (intersection of maximal ideals) of KG, where G is elementary abelian of rank m (Berman - 1967 and Charpin - 1988).

Principal and checkable codes

Definition (Jitman, Ling, Liu, Xie - 2010)

A G-code $\mathcal{C}$ is **checkable** if $\exists c \in KG$ s.t. $\mathcal{C} = \{v \in KG \mid cv = 0\} = \mathrm{Ann}_r(c)$.

Theorem (B., de la Cruz, Willems - 2019)

For any G-code $\mathcal{C}$,

$$\mathcal{C} \text{ is checkable } \Leftrightarrow \mathcal{C}^{\perp} \text{ is a principal right ideal.}$$

Examples

- Cyclic codes are principal (equivalently checkable).
- If $(m, q) = 1$, all D_{2m}-codes over K are principal (equivalently checkable).
- If l, q are prime, all $C_l \rtimes C_q$-codes over K are principal (equivalently checkable).

Asymptotic performance of G-codes

Definition

A family of codes $\mathcal{F}$ is called **asymptotically good** if it exists an infinite set $\{\mathcal{C}_n\}_{n\in\mathcal{I}} \subseteq \mathcal{F}$ of $[n, k_n, d_n]_q$ codes such that

$$R = \liminf_{n\to\infty} k_n/n > 0 \quad (\textbf{asymptotic rate}),$$

$$\delta = \liminf_{n\to\infty} d_n/n > 0 \quad (\textbf{asymptotic relative minimum distance}).$$

Open problem (Assmus, Mattson, Turyn - 1966)

Is the family of cyclic codes asymptotically good?

Theorem (Lin, Weldon - 1967)

Long BCH codes are bad.

Theorem (Berman - 1967)

Cyclic codes are bad if only finitely many primes are involved in the lengths of the codes.

Theorem (Babai, Shpilka, Stefankovic - 2005)

- There are no good cyclic LDPC (low density parity check) codes.
- There are no good cyclic locally testable codes.

Open problem (Assmus, Mattson, Turyn - 1966)

Is the family of cyclic codes asymptotically good? Maybe not!

Theorem (Bazzi, Mitter - 2006)

Binary dihedral codes are asymptotically good.

Theorem (B., Willems - 2020)

$C_p \rtimes C_q$-codes over K are asymptotically good.

Corollary

Principal (equivalently checkable) codes are asymptotically good.

Theorem (B., Moree, Solé - 2020)

Assuming Artin's conjecture for primitive roots in arithmetic progression (true under GRH), metacyclic codes are aymptotically good.

Open problem (Assmus, Mattson, Turyn - 1966)

Is the family of cyclic codes asymptotically good? Maybe yes!

The Uncertainty Principle

G **finite abelian group** and $f : G \to \mathbb{C}$.

Definition

The **dual group** of G is

$$\hat{G} = \{\text{homomorphisms } \chi : G \to \mathbb{S}^1\} \cong G$$

where $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

Definition

The **Fourier transform** of f is $\hat{f} : \hat{G} \to \mathbb{C}$ defined by

$$\hat{f}(\chi) = \frac{1}{\#G} \sum_{g \in G} f(g)\overline{\chi(g)}$$

$\mathrm{supp}(f) = \{g \in G \mid f(g) \neq 0\}$.

Theorem (Donoho, Stark - 1989)

Every $f : G \to \mathbb{C}$, $f \neq 0$, satisfies

$$\#\mathrm{supp}(f) \cdot \#\mathrm{supp}(\hat{f}) \geqslant \#G.$$

(Uncertainty Principle)

Stronger version for $G = C_p$, observed first by Meshulam.

Theorem (Goldstein, Guralnick, Isaac / Tao - 2005)

Every $f : C_p \to \mathbb{C}$, $f \neq 0$, satisfies

$$\#\mathrm{supp}(f) + \#\mathrm{supp}(\hat{f}) \geqslant p + 1.$$

(Uncertainty Principle for simple cyclic group)

- $f : G \to \mathbb{C} \longleftrightarrow \sum_{g\in G} f(g)g \in \mathbb{C}G$
- $\mathbb{C}C_p = \mathbb{C}[x]/(x^p - 1)$ and

$$f = a_0 + a_1 x + \ldots + a_{p-1}x^{p-1}$$

- $\hat{C}_p \cong \mu_p(\mathbb{C}) = \{\zeta \in \mathbb{C} \mid \zeta^p = 1\}$ by $\chi \mapsto \chi(1)$ and

$$\hat{f}(\zeta) = \frac{1}{p}(a_0 + a_1\zeta^{-1} + \ldots + a_{p-1}\zeta^{-(p-1)})$$

- Let $\mathcal{I}_f = (f)$ in $\mathbb{C}[x]/(x^p - 1)$, with $f | x^p - 1$. Then

$$\dim \mathcal{I}_f = p - \deg(f) = p - \#zeros(f) = \#\mathrm{supp}(\hat{f}).$$

Theorem (Uncertainty Principle reformulated)

Every $f \in \mathbb{C}[x]/(x^p - 1)$, $f \neq 0$, satisfies

$$\mathrm{wt}(f) + \dim \mathcal{I}_f \geqslant p + 1.$$

Corollary (Evra, Kowalski, Lubotzky - 2017)

Cyclic codes over $\mathbb{C}$ are asymptotically good.

Proof

Let ζ_p is a primitive p-th root of unity and

$$f = \prod_{i=1}^{\frac{p-1}{2}} (x - \zeta_p^i).$$

Then $\dim \mathcal{I}_f = p - \deg(f) = \frac{p+1}{2}$ and for $h \in \mathcal{I}_f$, $h \neq 0$,

$$\mathrm{wt}(h) \geqslant p + 1 - \dim \mathcal{I}_h \geqslant p + 1 - \dim \mathcal{I}_f = \frac{p+1}{2}.$$

So $\mathcal{I}_f$ is a $[p, \frac{p+1}{2}, \frac{p+1}{2}]_{\mathbb{C}}$ cyclic code.

Special cases of Reed-Solomon codes over $\mathbb{C}$.

Uncertainty Principle over finite fields

What about finite fields?

Definition

$$\mu(K,n) = \min\{\mathrm{d}(\mathcal{I}_f) + \dim \mathcal{I}_f \mid f \in K[x]/(x^n - 1)\}.$$

- $\mu(\mathbb{C}, p) = p + 1$ for all prime p.
- $\mu(K,n) \leqslant n+1$ (Singleton bound).
- $\mu(K,p) = p+1$ if q is primitive modulo p, i.e. $\mathrm{ord}_p(q) = p - 1$.

Definition (Evra, Kowalski, Lubotzky - 2017)

K satisfies the **(strong) Uncertainty Principle** if for all prime p

$$\mu(K,p) = p+1.$$

Theorem (B., Solé - 2020)

Assume MDS conjecture. If q is not primitive modulo p and $p > q + 2$, then

$$\mu(K, p) < p + 1.$$

Proof

- q is not primitive modulo $p \Rightarrow$ it exists $f | x^p - 1$ such that

$$1 < \deg(f) < p - 1, \text{ i.e. } 1 < \dim \mathcal{I}_f < p - 1.$$

- By contradiction,

$$\mathrm{d}(\mathcal{I}_f) + \dim \mathcal{I}_f \geqslant \mu(K, p) \geqslant p + 1$$

  $\Rightarrow \mathcal{I}_f$ is MDS of length p, non-trivial.
- MDS conjecture $\Rightarrow p \leqslant q + 2$.

Something similar is true without MDS conjecture (e.g. nontrivial MDS codes have length at most $2q - 2$). So, **the (strong) UP is not true for any K.**

Definition (Weak Uncertainty Principle)

Let $0 < \varepsilon < \lambda \leqslant 1$. K satisfies the (ε, λ)-**Uncertainty Principle** if there exists an infinite set of primes $\mathcal{P}$ such that for all $p \in \mathcal{P}$,

- $\mu(K, p) > \lambda p$
- $\mathrm{ord}_p(q) < \varepsilon p$.

Theorem (Evra, Kowalski, Lubotzky - 2017)

If K satisfies the (ε, λ)-Uncertainty Principle, then cyclic codes over K are asymptotically good.

Idea:

- $\mu(K, p) > \lambda p \Rightarrow$ we can find ideals with large distance.
- $\mathrm{ord}_p(q) < \varepsilon p \Rightarrow$ we can find ideals with large dimension.

Proposition (B., Solé - 2020)

If K satisfies the (ε, λ)-Uncertainty Principle, then $\lambda < \frac{q-1}{q}$.

Proof

- There exists a sequence of cyclic codes of length $p \in \mathcal{P}$, asymptotic rate R and asymptotic relative distance δ.
- $p\delta + pR \geqslant \mu(K, p) > \lambda p$.
- $\lambda < \min\{\delta + \alpha_q(\delta)\}$, where $\alpha_q(\delta)$ is the largest possible rate of a code of relative distance δ.
- Asymptotic Plotkin bound $\Rightarrow \min\{\delta + \alpha_q(\delta)\} = \frac{q-1}{q}$.

Does it exist any K satisfying the Weak Uncertainty Principle for some ε, λ?

Naive Uncertainty Principle

Analogue of Fourier transform for finite fields:

Definition

Let ζ_n be a primitive n-th root of unity in $\overline{K}$. For

$$f : C_n \to K \longleftrightarrow f \in K[x]/(x^n - 1)$$

the **Mattson-Solomon polynomial** is

$$\hat{f} = (f(\zeta_n), f(\zeta_n^2), \ldots, f(\zeta_n^n)) \longleftrightarrow \hat{f} \in K[x]/(x^n - 1).$$

Generalization of Donoho-Stark :

Proposition (B., Solé - 2020)

For $f \neq 0$,

$$\mathrm{wt}(f) \cdot \mathrm{wt}(\hat{f}) \geqslant n.$$

(Naive Uncertainty Principle)

Reminder: BCH bound

If among the zeros of f there exists m consecutive powers of ζ_n, then

$$\mathrm{d}(\mathcal{I}_f) \geqslant m + 1.$$

Proof (of Naive Uncertainty Principle)

- Let $\mathrm{wt}(f) = w$.
- By BCH bound, $\hat{f}$ cannot have w consecutive zeros.
- If w divides n, in each interval

$$[1, \ldots, w], \ldots, [n - w + 1, \ldots, n]$$

  there is a nonzero of $\hat{f}$. So $\mathrm{wt}(\hat{f}) \geqslant n/w$.
- Similarly otherwise.

Theorem (B., Solé - 2020)

For every real number $0 < \alpha < 1/2$, there are sequences of cyclic codes of asymptotic rate R with minimum distance $\Omega(n^\alpha)$.

Proof

- $n = q^p - 1$, with p prime.
- $x^n - 1 = \prod_{a \neq 0}(x - a) \prod_{i=1}^{s} f_i$, with f_i irreducible of degree p.
- $g_I = \prod_{i \in I} f_i$, with $\#I = \lfloor s(1 - R) \rfloor$.
- $\mathcal{I}_{g_I} = (g_I)$ has asymptotic rate R.
- Calculate $\Lambda_n \geqslant \#\{\text{codes containing a codewords of weight at most } n^\alpha\}$ (using naive UP).
- Prove that asymptotically $\Lambda_n \cdot \#B_0(n^\alpha) \leqslant \#\{\text{possible } g_I\}$.

Ramsey Theory

Definition

Let $b \neq 0$. An **arithmetic progression of length** m **in** $\mathbb{Z}/n\mathbb{Z}$ is

$$\{a + kb \mid k \in \{0, \dots, m-1\}\}.$$

definition

The **Szemeredi function** $r_m(n)$ is the largest size of a subset of $\mathbb{Z}/n\mathbb{Z}$ not containing an arithmetic progression of length m.

By BCH bound, if $\mathrm{wt}(f) = m$, then

$$\mathrm{wt}(\hat{f}) = n - \#zeros(f) \geqslant n - r_m(n)$$

(proved by Quader, Russell, Sundaram - 2019, without BCH bound). So

$$\mu(K, n) \geqslant \min\{m + n - r_m(n) \mid 0 \leqslant m \leqslant n\}.$$

Conclusion and outlook

Conclusion

- We presented arguments **for and against** the existence of asymptotically good families of cyclic codes.
- We presented different versions of the **Uncertainty Principle over finite fields** and the relation with the problem above.
- One of these is sufficient to prove the existence of infinite families of **"almost good" cyclic codes** of any asymptotic rate.

Outlook

- Develop the **approach with arithmetic progressions** in order to prove the Weak Uncertainty Principle for some finite field.
- Generalize all these results to **abelian codes** or to G-codes.

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Thank you very much for the attention!