Union of Balls and α-Complexes

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Laguerre geometry

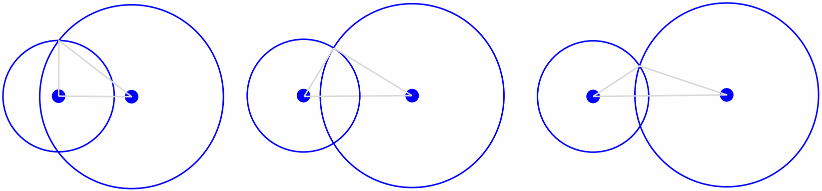
Power distance of two balls or of two weighted points.

ball $b_1(p_1, r_1)$, center p_1 radius $r_1 \longleftrightarrow$ weigthed point $(p_1, r_1^2) \in \mathbb{R}^d$
ball $b_2(p_2, r_2)$, center p_2 radius $r_2 \longleftrightarrow$ weigthed point $(p_2, r_2^2) \in \mathbb{R}^d$

$$\pi(b_1, b_2) = (p_1 - p_2)^2 - r_1^2 - r_2^2$$

Orthogonal balls

b_1, b_2 closer $\iff \pi(b_1, b_2) < 0 \iff (p_1 - p_2)^2 \leq r_1^2 + r_2^2$
b_1, b_2 orthogonal $\iff \pi(b_1, b_2) = 0 \iff (p_1 - p_2)^2 = r_1^2 + r_2^2$
b_1, b_2 further $\iff \pi(b_1, b_2) > 0 \iff (p_1 - p_2)^2 \leq r_1^2 + r_2^2$



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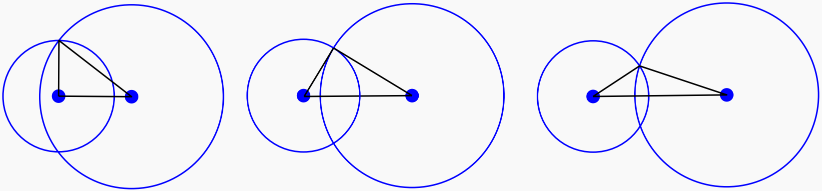
Orthogonal balls

$$\begin{aligned}
b_1, b_2 \text{ closer} \quad &\Longleftrightarrow \quad \pi(b_1, b_2) < 0 \Longleftrightarrow (p_1 - p_2)^2 \leq r_1^2 + r_2^2 \\
b_1, b_2 \text{ orthogonal} \quad &\Longleftrightarrow \quad \pi(b_1, b_2) = 0 \Longleftrightarrow (p_1 - p_2)^2 = r_1^2 + r_2^2 \\
b_1, b_2 \text{ further} \quad &\Longleftrightarrow \quad \pi(b_1, b_2) > 0 \Longleftrightarrow (p_1 - p_2)^2 \leq r_1^2 + r_2^2
\end{aligned}$$

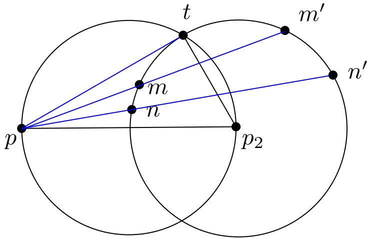


Power distance of a point wrt a ball

If b_1 is reduced to a point p : $\pi(p, b_2) = (p - p_2)^2 - r_2^2$

Normalized equation of bounding sphere :
$p \in \partial b_2 \Longleftrightarrow \pi(p, b_2) = 0$

$$\begin{aligned} p \in \mathrm{int} b_2 &\quad\Longleftrightarrow\quad \pi(p, b) < 0 \\ p \in \partial b_2 &\quad\Longleftrightarrow\quad \pi(p, b) = 0 \\ p \notin b_2 &\quad\Longleftrightarrow\quad \pi(p, b) > 0 \end{aligned}$$

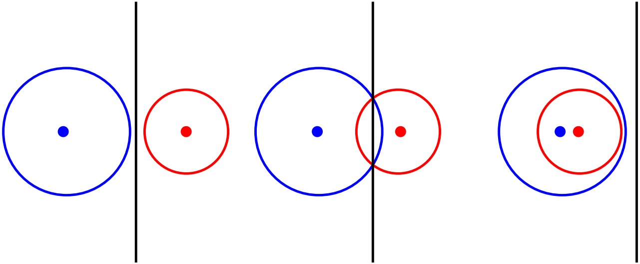


Tangents and secants through p
$\pi(p, b) = pt^2 = \overline{pm} \cdot \overline{pm'} = \overline{pn} \cdot \overline{pn'}$

Radical Hyperplane

The locus of point $\in \mathbb{R}^d$ with same power distance to balls $b_1(p_1, r_1)$ and $b_2(p_2, r_2)$ is a hyperplane of $\mathbb{R}^d$

$$\begin{aligned}\pi(x,b_1) = \pi(x,b_2) \quad &\iff \quad (x-p_1)^2 - r_1^2 = (x-p_2)^2 - r_2^2 \\ &\iff \quad -2p_1x + p_1^2 - r_1^2 = -2p_2x + p_2^2 - r_2^2 \\ &\iff \quad 2(p_2 - p_1)x + (p_1^2 - r_1^2) - (p_2^2 - r_2^2) = 0\end{aligned}$$



A point in h_{12} is the center of a ball orthogonal to b_1 and b_2

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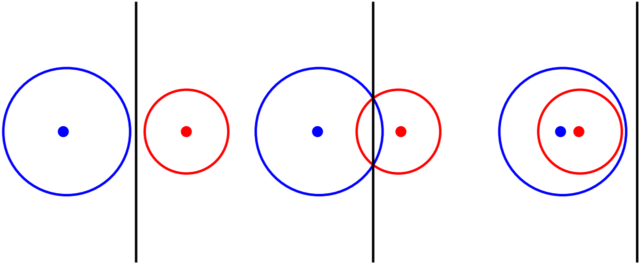

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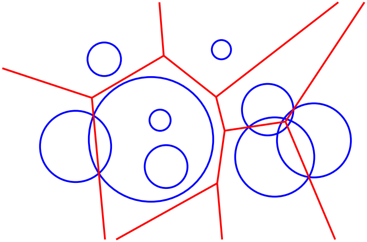
Power Diagrams

also named Laguerre diagrams or weighted Voronoi diagrams



Sites : n balls $B = \{b_i(p_i, r_i), i = 1, \ldots n\}$

Power distance: $\pi(x, b_i) = (x - p_i)^2 - r_i^2$

Power Diagram: $\text{Vor}(B)$
One cell $V(b_i)$ for each site
$V(b_i) = \{x : \pi(x, b_i) \le \pi(x, b_j). \forall j \ne i\}$

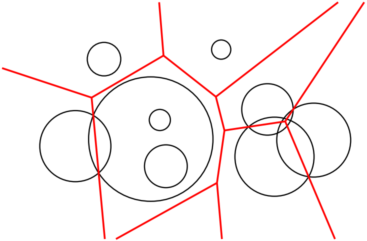
- Each cell is a polytope
- $V(b_i)$ may be empty
- p_i may not belong to $V(b_i)$

Weighted Delaunay triangulations

$B = \{b_i(p_i, r_i)\}$ a set of balls

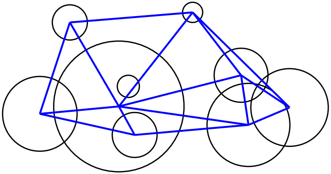
Del(B) = nerve of Vor(B):

$B_\tau = \{b_i(p_i, r_i), i = 0, \dots k\}\} \subset B$
$B_\tau \in \mathrm{Del}(B) \iff \bigcap_{b_i \in B_\tau} V(b_i) \neq \emptyset$

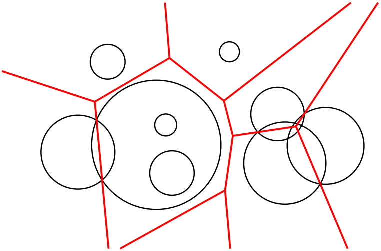


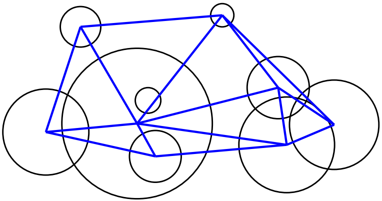
To be proved (next slides):

under a general position condition on B,
$B_\tau \longrightarrow \tau = \mathrm{conv}(\{p_i, i = 0 \dots k\})$
embeds Del(B) as a triangulation in $\mathbb{R}^d$,
called the weighted Delaunay triangulation



Characteristic property of weighted Delaunay complexes





$$\tau \in \mathsf{Del}(B) \iff \bigcap_{b_i \in B_\tau} V(b_i) \neq \emptyset$$

$$\iff \exists\, x \in \mathbb{R}^d \text{ s.t. } \forall b_i, b_j \in B_\tau,\ b_l \in B \setminus B_\tau$$

$$\pi(x, b_i) = \pi(x, b_j) < \pi(x, b_l)$$

$$\iff \exists \text{ ball } b(x, \omega) \text{ s.t. } \forall b_i \in B_\tau,\ b_l \in B \setminus B_\tau$$

$$0 = \pi(b, b_i) < \pi(b, b_l)$$

The space of spheres

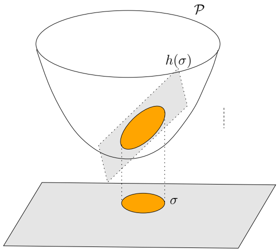
$b(p, r)$ ball of $\mathbb{R}^d$

$\rightarrow$ point $\phi(b) \in \mathbb{R}^{d+1}$

$\phi(b) = (p, s = p^2 - r^2)$

$\rightarrow$ polar hyperplane $h_b = \phi(b)^* \in \mathbb{R}^{d+1}$

$$\mathcal{P} = \{\hat{x} \in \mathbb{R}^{d+1} : x_{d+1} = x^2\}$$
$$h_b = \{\hat{x} \in \mathbb{R}^{d+1} : x_{d+1} = 2p \cdot x - s\}$$



- Balls will null radius are mapped onto $\mathcal{P}$

  h_p is tangent to $\mathcal{P}$ at $\phi(p)$.

- The vertical projection of $h_b \cap \mathcal{P}$ onto $x_{d+1} = 0$ is ∂b

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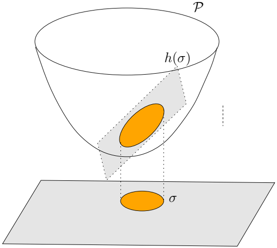

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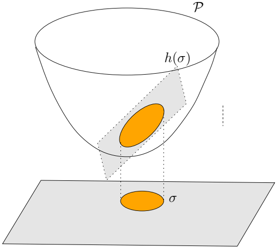

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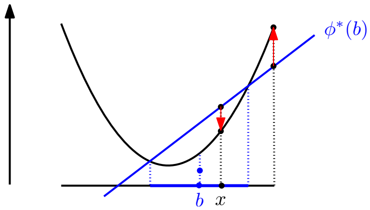

- The vertical distance between $\hat{x} = (x, x^2)$ and h_b is equal to

$$x^2 - 2p \cdot x + s = \pi(x, b)$$

- The faces of the power diagram of B are the vertical projections onto $x_{d+1} = 0$ of the faces of the polytope $\mathcal{V}(B) = \bigcap_i h_b^+$ of $\mathbb{R}^{d+1}$

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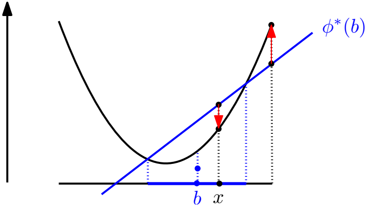

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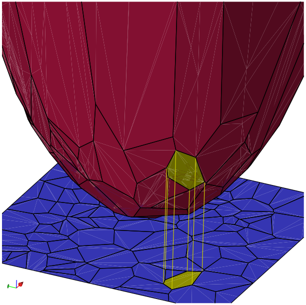
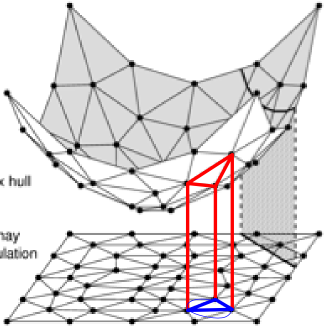
Power diagrams, weighted Delaunay triangulations and polytopes



$$\mathcal{V}(B) = \cap_i \, \phi(b_i)^{*+}$$



$$\mathcal{D}(B) = \mathsf{conv}^-(\hat{P})$$

Proof of the theorem

$$B_\tau \subset B, |B_\tau| = d+1,\ \tau = \textsf{conv}(\{p_i, b_i(p_i, r_i) \in B_\tau\}),$$
$$\phi(\tau) = \textsf{conv}(\{\phi(b_i), b_i \in B_\tau\})$$

$\exists\ b(p,r)$ s.t. $h_b = \phi(b)^* = \textsf{aff}(\{\phi(b_i), b_i \in B_\tau\})$

$$\begin{aligned}
\phi(\tau) \in \mathcal{D}(B) \quad &= \quad \textsf{conv}^-(\{\phi(b_i)\}) \\
&\iff \quad \forall b_i \in B_\tau, \phi(b_i) \in h_b \quad \forall b_j \notin B_\tau, \phi(b_j) \in h_b^{*+} \\
&\iff \quad \forall b_i \in B_\tau, \pi(b, b_i) = 0 \quad \forall b_j \notin B_\tau, \pi(b, b_j) > 0 \\
&\iff \quad p \in \bigcap_{b_i \in B_\tau} V(b_i) \\
&\iff \quad \tau \in \textsf{Del}(B)
\end{aligned}$$

Delaunay's theorem extended

$B = \{b_1, b_2 \ldots b_n\}$ is said to be in general position wrt spheres if

$\nexists\ x \in \mathbb{R}^d$ with equal power to $d+2$ balls of B

$P = \{p_1, ..., p_n\}$: set of centers of the balls of B

Theorem

If B is in general position wrt spheres, the simplicial map

$$f : \mathrm{vert}(\mathrm{Del}(B)) \to P$$

provides a realization of $\mathrm{Del}(B)$

$\mathrm{Del}(B)$ is a triangulation of $P' \subseteq P$ called the Delaunay triangulation of B

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Power diagrams, Delaunay triangulations and polytopes

If B is a set of balls in general position wrt spheres :

$$
\begin{array}{ccc}
\mathcal{V}(B) = h_{b_1}^+ \cap \ldots \cap h_{b_n}^+ & \xrightarrow{\text{duality}} & \mathcal{D}(B) = \mathsf{conv}^-(\{\phi(b_1), \ldots, \phi(b_n)\}) \\
\uparrow & & \downarrow \\
\text{Voronoi Diagram of } B & \xrightarrow{\text{nerve}} & \text{Delaunay Complex of } B
\end{array}
$$

Complexity and algorithm for weighted VD and DT

Number of faces $= \Theta\left(n^{\lfloor \frac{d+1}{2} \rfloor}\right)$ (Upper Bound Th.)

Construction can be done in time $\Theta\left(n \log n + n^{\lfloor \frac{d+1}{2} \rfloor}\right)$ (Convex hull)

Main predicate

$$\texttt{power\_test}(b_0, \ldots, b_{d+1}) = \operatorname{sign}\begin{vmatrix} 1 & \ldots & 1 \\ p_0 & \ldots & p_{d+1} \\ p_0^2 - r_0^2 & \ldots & p_{d+1}^2 - r_{d+1}^2 \end{vmatrix}$$

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Power diagrams are maximization diagrams

Cell of b_i in the power diagram Vor(B)

$$V(b_i) = \{x \in \mathbb{R}^d : \pi(x, b_i) \leq \pi(x, b_j). \forall j \neq i\}$$

$$= \{x \in \mathbb{R}^d : 2p_i x - s_i = \max_{j \in [1,\ldots n]} \{2p_j x - s_j\}\}$$

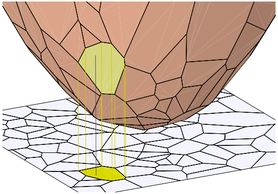
Vor(B) is the maximization diagram of the set of affine functions

$$\{f_i(x) = 2p_i x - s_i, i = 1, \ldots, n\}$$

Affine diagrams (regular subdivisions)

Affine diagrams are defined as the maximization diagrams of a finite set of affine functions

They are equivalently defined as the vertical projections of polyhedra intersection of a finite number of upper half-spaces of $\mathbb{R}^{d+1}$



- Voronoi diagrams and power diagrams are affine diagrams.
- Any affine diagram of $\mathbb{R}^d$ is the power diagram of a set of balls.

- Delaunay and weighted Delaunay triangulations are regular triangulations
- Any regular triangulation is a weighted Delaunay triangulation

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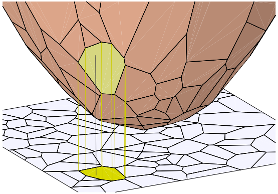

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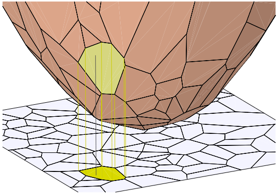

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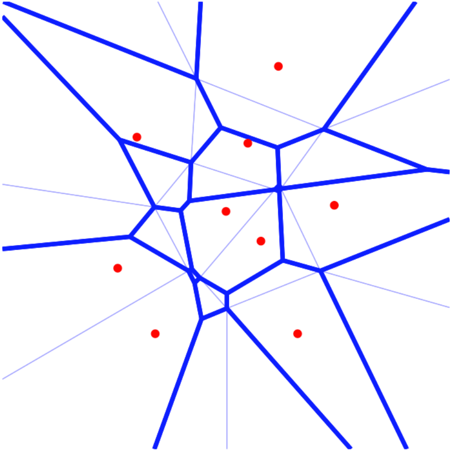
Examples of affine diagrams

1. The intersection of a power diagram with an affine subspace (Exercise)
2. A Voronoi diagram defined with a quadratic distance function

$$\|x-a\|_Q = (x-a)^t Q (x-a) \qquad\qquad Q = Q^t$$

3. k order Voronoi diagrams

k-order Voronoi Diagrams



Let P be a set of sites.
Each cell in the k-order Voronoi diagram $\mathrm{Vor}_k(P)$ is the locus of points in $\mathbb{R}^d$ that have the same subset of P as k-nearest neighbors.

k-order Voronoi diagrams are power diagrams

Let $S_1, S_2, \ldots$ denote the subsets of k points of P.
The k-order Voronoi diagram is the minimization diagram of $\delta(x, S_i)$:

$$\begin{aligned} \delta(x, S_i) &= \frac{1}{k} \sum_{p \in S_i} (x-p)^2 \\ &= x^2 - \frac{2}{k} \sum_{p \in S_i} p \cdot x + \frac{1}{k} \sum_{p \in S_i} p^2 \\ &= \pi(b_i, x) \end{aligned}$$

where b_i is the ball

1. centered at $c_i = \frac{1}{k} \sum_{p \in S_i} p$
2. with $s_i = \pi(o, b_i) = c_i^2 - r_i^2 = \frac{1}{k} \sum_{p \in S_i} p^2$
3. and radius $r_i^2 = c_i^2 - \frac{1}{k} \sum_{p \in \mathbb{S}_i} p^2$.

Combinatorial complexity of k-order Voronoi diagrams

Theorem

If P be a set of n points in $\mathbb{R}^d$, the number of vertices and faces in all the Voronoi diagrams $\mathrm{Vor}_j(P)$
of orders $j \leq k$ is:

$$O\left(k^{\lceil \frac{d+1}{2} \rceil} \, n^{\lfloor \frac{d+1}{2} \rfloor}\right)$$

Proof

uses :

- bijection between k-sets and cells in k-order Voronoi diagrams
- the sampling theorem (from randomization theory)

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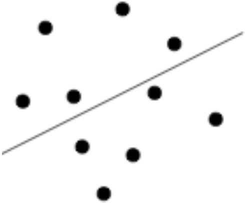
Proof

uses :

- bijection between k-sets and cells in k-order Voronoi diagrams
- the sampling theorem (from randomization theory)

k-sets and k-order Voronoi diagrams

P a set of n points in $\mathbb{R}^d$

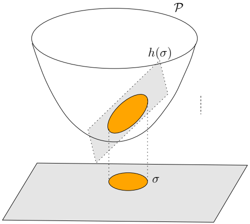


k-sets

A k-set of P is a subset P' of P with size k that can be separated from $P \setminus P'$ by a hyperplane

k-order Voronoi diagrams

k points of P have a cell in $\text{Vor}_k(P)$ iff there exists a ball that contains those points and only those



$\Rightarrow$ each cell of $\text{Vor}_k(P)$ corresponds to a k-set of $\phi(P)$

k-sets and k-order Voronoi diagrams

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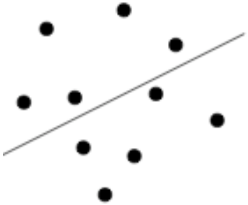

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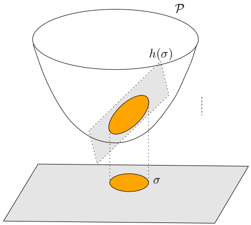

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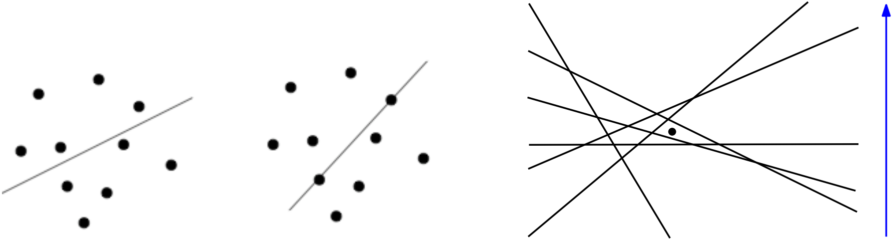
k-sets and k-levels in arrangements of hyperplanes



- For a set of points $P \in \mathbb{R}^d$, we consider the arrangement of the dual hyperplanes $\mathcal{A}(P^*)$
- h defines a k set $P' \quad \Rightarrow \quad h$ separates P' (below h) from $P \setminus P'$ (above h) $\Rightarrow \quad h^*$ is below the k hyperplanes of P'^* and above those of $P^* \setminus P'^*$
- k-sets of P are in 1-1 correspondance with the cells of $\mathcal{A}(P^*)$ of level k, i.e. with k hyperplanes of P^* above it.

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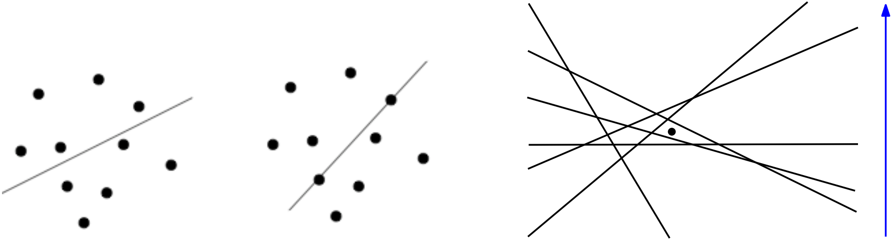

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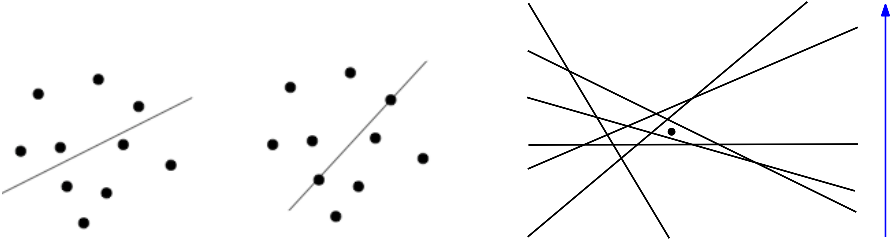

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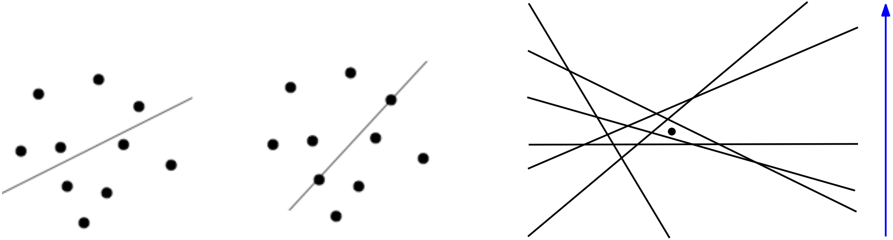
Bounding the number of k-sets



$c_k(P)$: Number of k-sets of P = Number of cells of level k in $\mathcal{A}(P^*)$
$c_{\leq k}(P) = \sum_{l \leq k} c_l(P)$
$c'_{\leq k}(P)$: Number of vertices of $\mathcal{A}(P^*)$ with level at most k
$c_{\leq k}(n) = \max_{|P|=n} c_{\leq k}(P)$ $c'_{\leq k}(n) = \max_{|P|=n} c'_{\leq k}(P)$

Hyp. in general position : each vertex $\in d$ hyperplanes incident to 2^d cells

Vertices of level k are incident to cells with level $\in [k, k+d]$

Cells of level k have incident vertices with level $\in [k-d, k]$

$$c_{\leq k}(n) = O\left(c'_{\leq k}(n)\right)$$

Regions, conflicts and the sampling theorem

O a set of n objects.

$\mathcal{F}(O)$ set of configurations defined by O

- each configuration is defined by a subset of b objects
- each configuration is in conflict with a subset of O

$\mathcal{F}_j(O)$ set of configurations in conflict with j objects
$|\mathcal{F}_{\leq k}(O)|$ number of configurations defined by O
in conflict with at most k objects of O
$f_0(r) = \text{Exp}(|\mathcal{F}_0(R|)$ expected number of configurations
defined and without conflict on a random r-sample of O.

The sampling theorem [Clarkson & Shor 1992]

For $2 \leq k \leq \frac{n}{b+1}$, $|\mathcal{F}_{\leq k}(O)| \leq 4\,(b+1)^b\, k^b\, f_0(\lfloor \frac{n}{k} \rfloor)$

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Proof of the sampling theorem

$$f_0(r) = \sum_j |\mathcal{F}_j(O)| \frac{\binom{n-b-j}{r-b}}{\binom{n}{r}} \geq |\mathcal{F}_{\leq k}(O)| \frac{\binom{n-b-k}{r-b}}{\binom{n}{r}}$$

then,we prove that
for $r = \frac{n}{k}$

$$\frac{\binom{n-b-k}{r-b}}{\binom{n}{r}} \geq \frac{1}{4(b+1)^b k^b}$$

$$\frac{\binom{n-b-k}{r-b}}{\binom{n}{r}} = \underbrace{\frac{r!}{(r-b)!} \frac{(n-b)!}{n!}}_{\geq \frac{1}{(b+1)^b k^b}} \underbrace{\frac{(n-r)!}{(n-r-k)!} \frac{(n-b-k)!}{(n-b)!}}_{\geq \frac{1}{4}}$$

Proof of the sampling theorem
end

$$\frac{(n-r)!}{(n-r-k)!}\frac{(n-b-k)!}{(n-b)!} = \prod_{j=1}^{k}\frac{n-r-k+j}{n-b-k+j} \geq \left(\frac{n-r-k+1}{n-b-k+1}\right)^k$$

$$\geq \left(\frac{n-n/k-k+1}{n-k}\right)^k$$

$$\geq (1-1/k)^k \geq 1/4 \text{ pour } (2 \leq k),$$

$$\frac{r!}{(r-b)!}\frac{(n-b)!}{n!} = \prod_{l=0}^{b-1}\frac{r-l}{n-l} \geq \prod_{l=1}^{b}\frac{r+1-b}{n}$$

$$\geq \prod_{l=1}^{b}\frac{n/k-b}{n}$$

$$\geq 1/k^b(1-\frac{bk}{n})^b \geq \frac{1}{k^b(b+1)^b} \text{ pour } (k \leq \frac{n}{b+1}).$$

Bounding the number of k-sets

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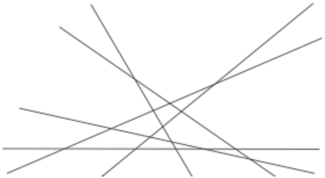
Objects O: n hyperplanes of $\mathbb{R}^d$

Configurations : vertices in $\mathcal{A}(O)$, $b = d$

Conflict between v and h : $v \in h^+$



Sampling th: $c'_{\leq k}(P) \leq 4(d+1)^d k^d f_0\left(\left\lfloor \frac{n}{k} \right\rfloor\right)$

Upper bound th: $f_0(\left\lfloor \frac{n}{k} \right\rfloor) = O\left(\frac{n^{\left\lfloor \frac{d}{2} \right\rfloor}}{k^{\left\lfloor \frac{d}{2} \right\rfloor}}\right)$

$\Rightarrow c'_{\leq k}(n) = O\left(k^{\left\lceil \frac{d}{2} \right\rceil} n^{\left\lfloor \frac{d}{2} \right\rfloor}\right)$

Combinatorial complexities

- Number of vertices of level $\leq k$ in an arrangement of n hyperplanes in $\mathbb{R}^d$

  Number of cells of level $\leq k$ in an arrangement of n hyperplanes in $\mathbb{R}^d$

  Total number of $j \leq k$ sets for a set of n points in $\mathbb{R}^d$

$$\left(k^{\lceil \frac{d}{2} \rceil} n^{\lfloor \frac{d}{2} \rfloor}\right)$$

- Total number of faces in the Voronoi diagrams of order $j \leq k$ for a set of n points in $\mathbb{R}^d$

$$\left(k^{\lceil \frac{d+1}{2} \rceil} n^{\lfloor \frac{d+1}{2} \rfloor}\right)$$

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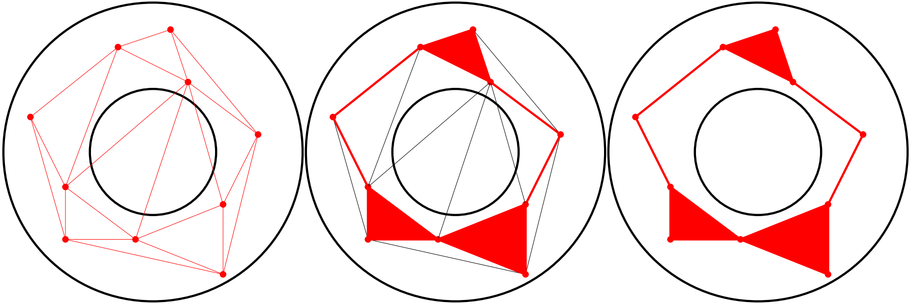
Restriction of Delaunay triangulation



Let $\Omega \subseteq \mathbb{R}^d$ and $P \in \mathbb{R}^d$ a finite set of points.

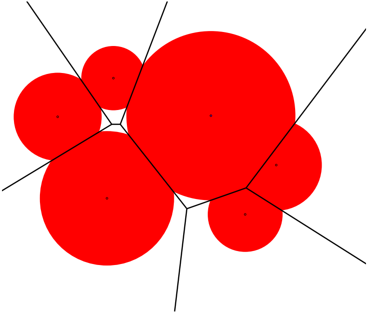
Vor$(E) \cap \Omega$ is a cover of Ω. Its nerve is called the Delaunay triangulation of E restricted to Ω, noted $\mathrm{Del}_{|\Omega}(P)$

If Vor$(E) \cap \Omega$ is a good cover of Ω, $\mathrm{Del}_{|\Omega}(P)$ is homotopy equivalent to Ω (Nerve theorem)

Union of balls

- What is the combinatorial complexity of the boundary of the union U of n balls of $\mathbb{R}^d$?
- Compare with the complexity of the arrangement of the bounding hyperspheres
- How can we compute U ?
- What is the image of U in the space of spheres ?

Restriction of $\mathrm{Del}(B)$ to $U = \bigcup_{b \in B} b$



- $U = \bigcup_{b\in B} b \cap V(b)$ and $\partial U \cap \partial b = V(b) \cap \partial b$.
- The nerve of $\mathcal{C}$ is the restriction of $\mathrm{Del}(B)$ to U, i.e. the subcomplex $\mathrm{Del}_{|U}(B)$ of $\mathrm{Del}(B)$ whose faces have a circumcenter in U
- $\forall b, \ \ b \cap V(b)$ is convex and thus contractible
- $\mathcal{C} = \{b \cap V(b), b \in B\}$ is a good cover of U
- The nerve of $\mathcal{C}$ is a deformation retract of U

homotopy equivalent (Nerve theorem)

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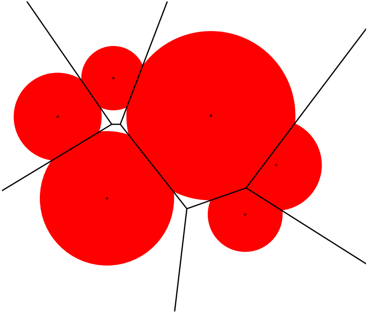

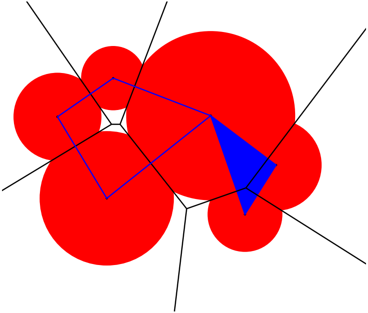

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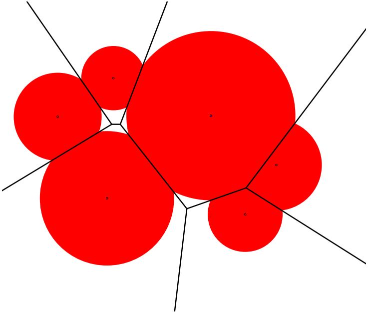

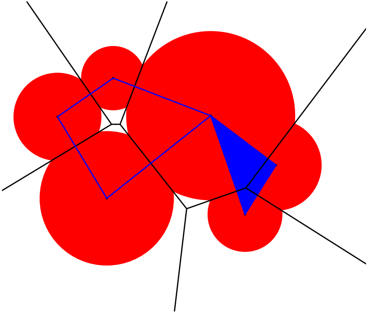

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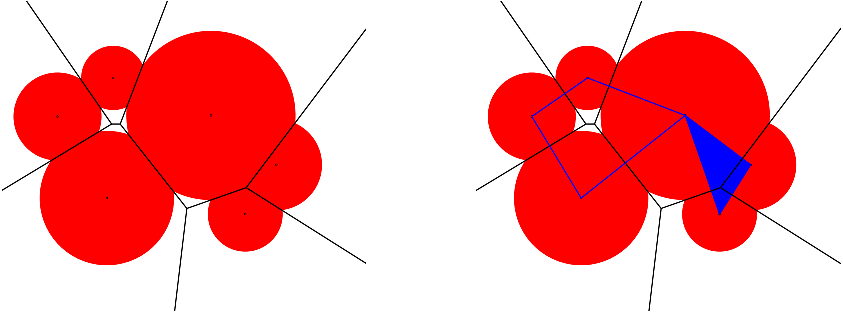

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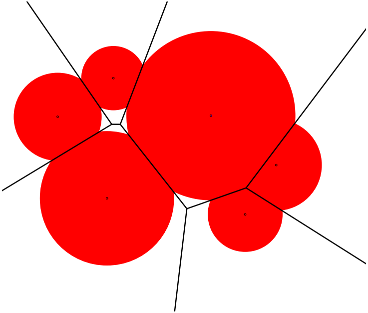

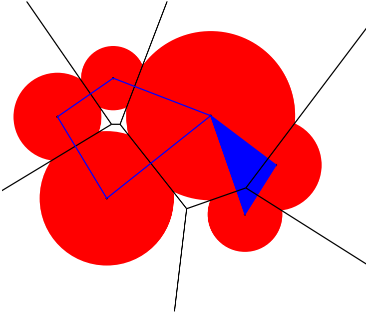

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homotopy equivalent (Nerve theorem)

Cech complex versus $\mathrm{Del}_{|U}(B)$

- Both complexes are homotopy equivalent to U
- The size of Cech(B) is $\Theta(n^d)$
- The size of $\mathrm{Del}_{|U}(B)$ is $\Theta(n^{\lceil \frac{d}{2} \rceil})$

Filtration of a simplicial complex

1. A filtration of K is a sequence of subcomplexes of K

$$\emptyset = K^0 \subset K^1 \subset \cdots \subset K^m = K$$

such that: $K^{i+1} = K^i \cup \sigma^{i+1}$, where σ^{i+1} is a simplex of K

2. Alternatively a filtration of K can be seen as an ordering $\sigma_1, \ldots \sigma_m$ of the simplices of K such that the set K^i of the first i simplices is a subcomplex of K

The ordering should be consistent with the dimension of the simplices

Filtration plays a central role in topological persistence

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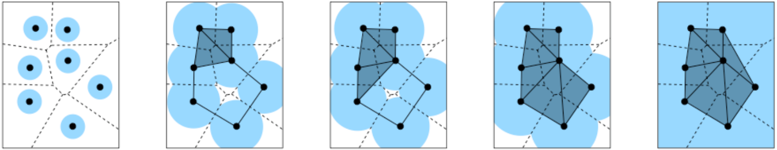
Filtration plays a central role in topological persistence

α-filtration of Delaunay complexes

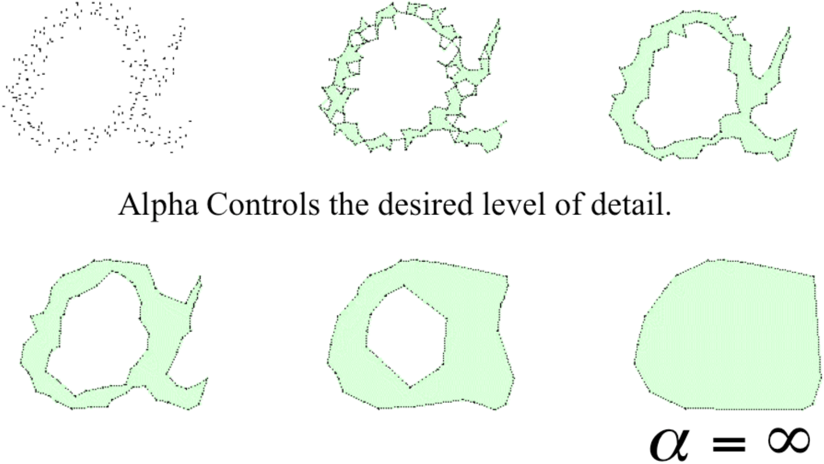
P a finite set of points of $\mathbb{R}^d$ $\qquad U(\alpha) = \bigcup_{p\in P} B(p,\alpha)$

α-complex $= \mathrm{Del}_{|U(\alpha)}(P)$

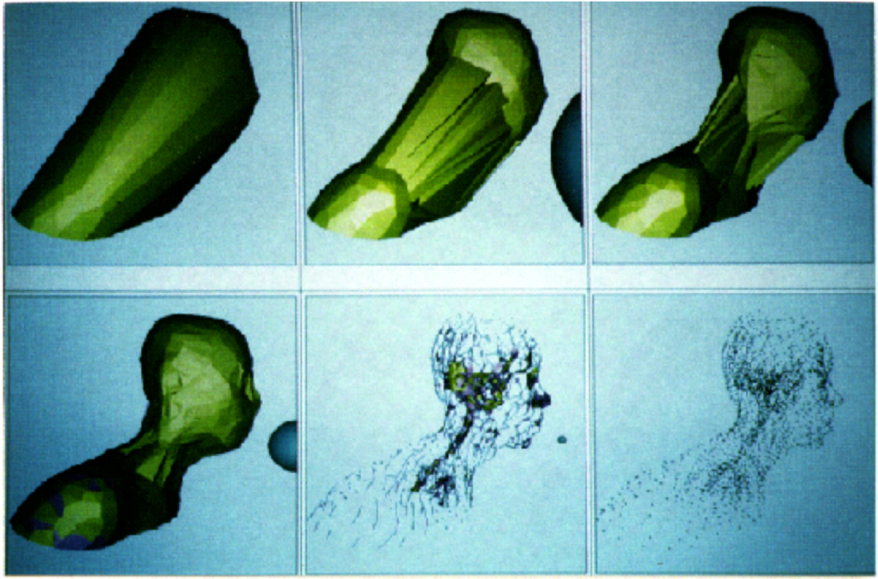


The filtration $\{\mathrm{Del}_{|U(\alpha)}(P),\ \alpha \in \mathbb{R}^+\}$ is called the α-filtration of $\mathrm{Del}(P)$

Shape reconstruction using α-complexes (2d)

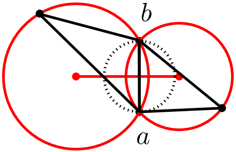


Shape reconstruction using α-complexes (3d)

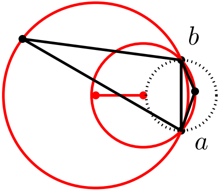


Constructing the α-filtration of $\mathrm{Del}(P)$

$\sigma \in \mathrm{Del}(P)$ is said to be Gabriel iff $\sigma \cap \sigma^* \neq \emptyset$



A Gabriel edge



A non Gabriel edge

Algorithm

for each d-simplex $\sigma \in \mathrm{Del}(P)$: $\alpha_{min}(\sigma) = r(\sigma)$

for $k = d-1, ..., 0,$

for each k-face $\sigma \in \mathrm{Del}(P)$

$\alpha_{med}(\sigma) = \min_{\sigma \in \mathrm{coface}(\sigma)} \alpha_{min}(\sigma)$

if σ is Gabriel then $\alpha_{min}(\sigma) = r(\sigma)$

else $\alpha_{min}(\sigma) = \alpha_{med}(\sigma)$

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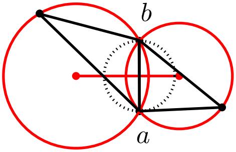

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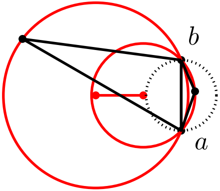

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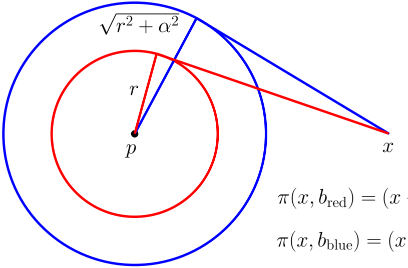
α-filtration of weighted Delaunay complexes

$$B = \{b_i = (p_i, r_i)\}_{i=1,\ldots,n} \qquad W(\alpha) = \bigcup_{i=1}^{n} B\left(p_i, \sqrt{r_i^2 + \alpha^2}\right)$$



$$\pi(x, b_{\mathrm{red}}) = (x-p)^2 - r^2$$

$$\pi(x, b_{\mathrm{blue}}) = (x-p)^2 - r^2 - \alpha^2$$

α-complex $= \mathrm{Del}_{W(\alpha)}(B)$ Filtration : $\{\mathrm{Del}_{W(\alpha)}(B),\ \alpha \in \mathbb{R}^+\}$