Simplicial Complexes

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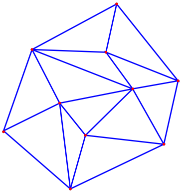
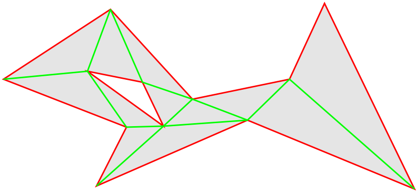
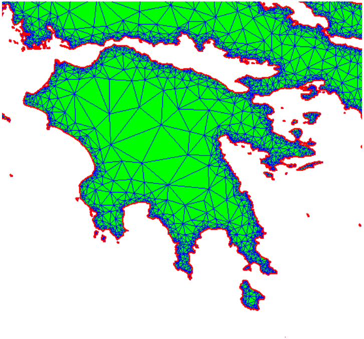
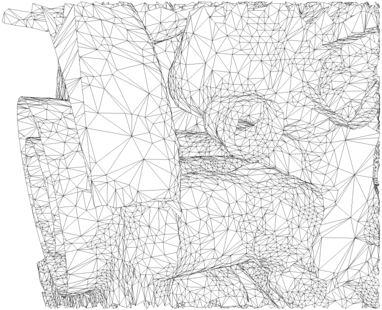
Examples of simplicial complexes









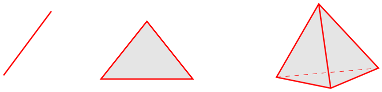
Geometric simplices

A *k-simplex* σ is the convex hull of $k+1$ points of $\mathbb{R}^d$ that are affinely independent

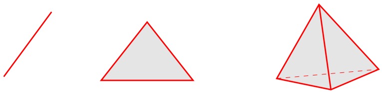
$$\sigma = \mathrm{conv}(p_0, ..., p_k) = \{x \in \mathbb{R}^d,\ x = \sum_{i=0}^{k} \lambda_i\, p_i, \quad \lambda_i \in [0,1],\ \ \sum_{i=0}^{k} \lambda_i = 1\}$$

$k = \dim(\mathrm{aff}(\sigma))$ is called the dimension of σ

1-simplex = line segment
2-simplex = triangle
3-simplex = tetrahedron



Faces of a simplex



$V(\sigma)$ = set of vertices of a k-simplex σ

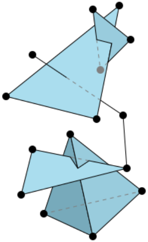
$\forall V' \subseteq V(\sigma)$, $\mathrm{conv}(V')$ is a face of σ

a k-simplex has $\binom{k+1}{i+1}$ faces of dimension i

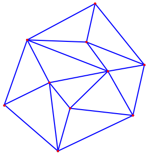
total nb of faces $= \sum_{i=0}^{d} \binom{k+1}{i+1} = 2^{k+1} - 1$

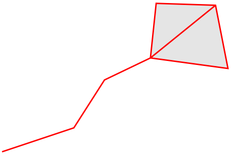
Geometric simplicial complexes

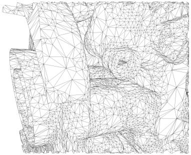
A finite collection of simplices K called the faces of K such that



- $\forall \sigma \in K$, σ is a simplex
- $\sigma \in K, \tau \subset \sigma \Rightarrow \tau \in K$
- $\forall \sigma, \tau \in K$, either $\sigma \cap \tau = \emptyset$ or $\sigma \cap \tau$ is a common face of both







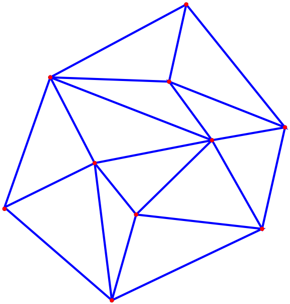
Geometric simplicial complexes

The dimension of a simplicial complex K is the max dimension of its simplices

A subset of K which is a complex is called a subcomplex of K

The underlying space $|K| \subset \mathbb{R}^d$ of K is the union of the simplices of K

Example 1 : Triangulation of a finite point set of $\mathbb{R}^d$



- A simplicial d-complex K is pure if every simplex in K is the face of a d-simplex.
- A triangulation of a finite point set $P \in \mathbb{R}^d$ is a pure geometric simplicial complex K s.t. $\text{vert}(K) = P$ and $|K| = \text{conv(P)}$.

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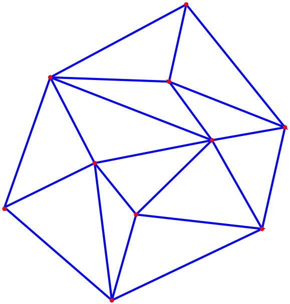

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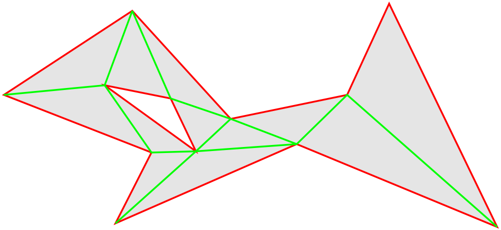
Example 2 : triangulation of a polygonal domain of $\mathbb{R}^2$



A triangulation of a polygonal domain $\Omega \subset \mathbb{R}^2$ is a pure geometric simplicial complex K s.t. $\quad \mathrm{vert}(K) = \mathrm{vert}(\Omega) \quad$ and $\quad |K| = \Omega$.

Exercises

- Show that such a triangulation exists for any Ω
- Propose an algorithm of complexity $O(n \log n)$ to compute it where $n = \sharp \mathrm{vert}(\Omega)$
- Show that some polyhedral domains of $\mathbb{R}^3$ do not admit a triangulation

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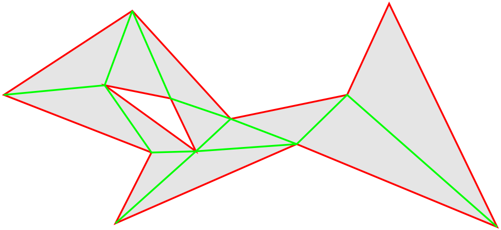

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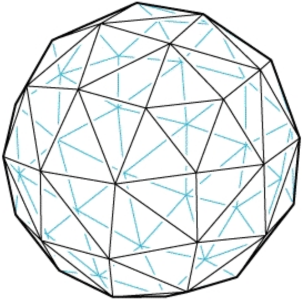
Example 3 : the boundary complex of the convex hull of a finite set of points in general position



Polytope

$$\text{conv}(P) = \{x \in \mathbb{R}^d,\ x = \sum_{i=0}^{k} \lambda_i\, p_i,\ \lambda_i \in [0,1],\ \sum_{i=0}^{k} \lambda_i = 1\}$$

Supporting hyperplane H :
$H \cap P \neq \emptyset$, P on one side of H

Faces : $\text{conv}(P) \cap H$, H supp. hyp.

- P is in general position iff no subset of $k+2$ points lie in a k-flat
- If P is in general position, all faces of $\text{conv}(P)$ are simplices

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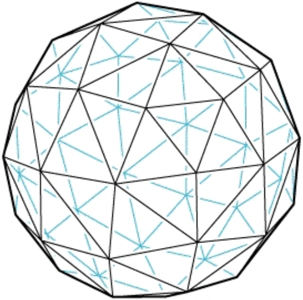

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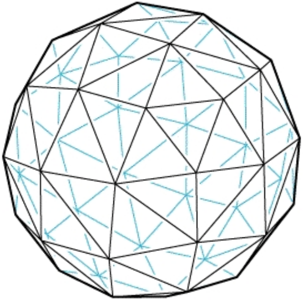

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Abstract simplicial complexes

Given a finite set of elements P, an abstract simplicial complex K with vertex set P is a set of subsets of P s.t.

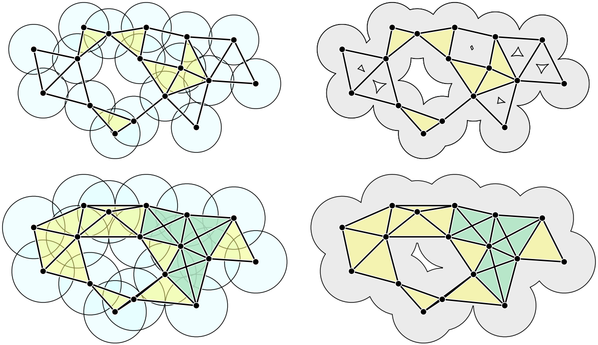
1. $\forall p \in P, \quad p \in K$
2. if $\sigma \in K$ and $\tau \subseteq \sigma$, then $\tau \in K$

The elements of K are called the (abstract) simplices or faces of K

The dimension of a simplex σ is $\dim(\sigma) = \sharp\text{vert}(\sigma) - 1$

Nerve of a finite cover $\mathcal{U} = \{U_1, ..., U_n\}$ of X

An example of an abstract simplicial complex



The nerve of $\mathcal{U}$ is the simplicial complex $K(U)$ defined by

$$\sigma = [U_{i_0}, ..., U_{i_k}] \in K(U) \quad \Leftrightarrow \quad \cap_{i=1}^{k} U_{i_j} \neq \emptyset$$

Realization of an abstract simplicial complex

- A realization of an abstract simplicial complex K is a geometric simplicial complex K_g whose corresponding abstract simplicial complex is isomorphic to K, i.e.

$$\exists \text{ bijective } f : \text{vert}(K) \to \text{vert}(K_g) \quad \text{s.t.} \quad \sigma \in K \quad \Rightarrow \quad f(\sigma) \in K_g$$

- Any abstract simplicial complex K can be realized in $\mathbb{R}^n$

  Hint : $v_i \to p_i = (0, ..., 0, 1, 0, ...0) \in \mathbb{R}^n$ $\quad (n = \sharp\text{vert}(K))$
  $\sigma = \text{conv}(p_1, ..., p_n)$ (canonical simplex)
  $K_g \subseteq \sigma$

- Realizations are not unique but are all topologically equivalent (homeomorphic)

Topological equivalence

Two subsets X and Y of $\mathbb{R}^d$ are said to be topologically equivalent or homeomorphic if there exists a continuous, bijective map $f : X \to Y$ with continuous inverse f^{-1}

Topological disks

Not a topological disk

No need for the condition f^{-1} to be continuous
if X is compact and Y is Hausdorff (e.g. a metric space)

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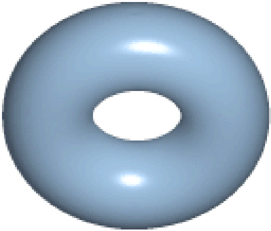
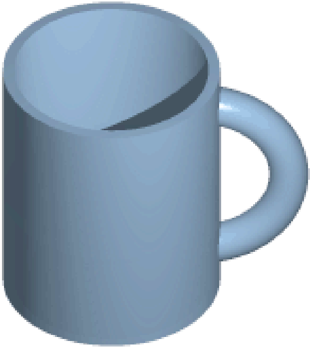
Are these objects homeomorphic ?





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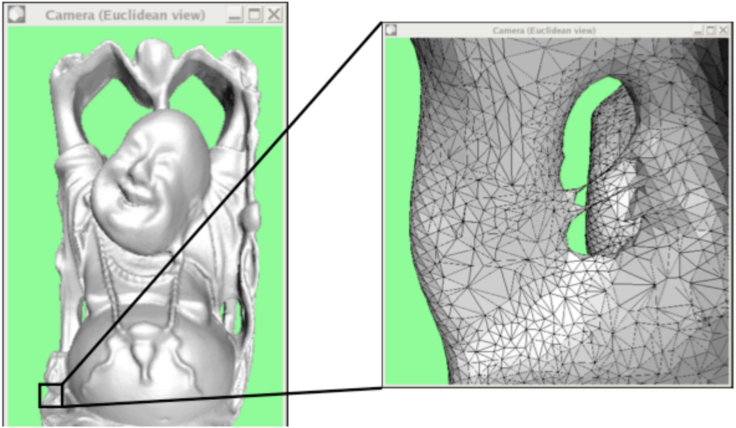

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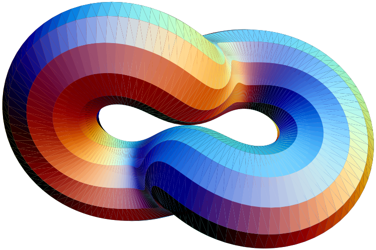

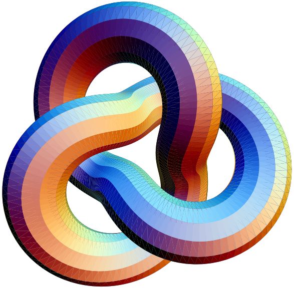

Triangulated balls and spheres

A triangulated d-ball ($(d-1)$-sphere) is a simplicial complex whose realization is homeomorphic to the unit d-ball ($(d-1)$-sphere) of $\mathbb{R}^d$

Examples

- a triangulated simple polygon
- the boundary complex of a simplicial d-polytope is a triangulated $(d-1)$-sphere
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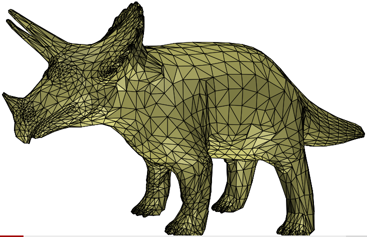
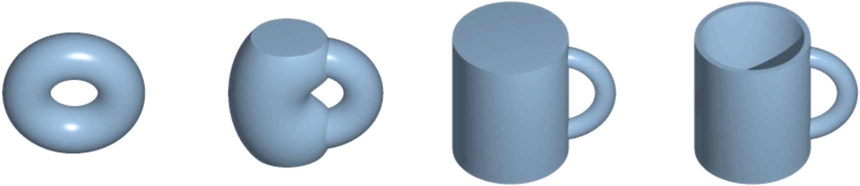


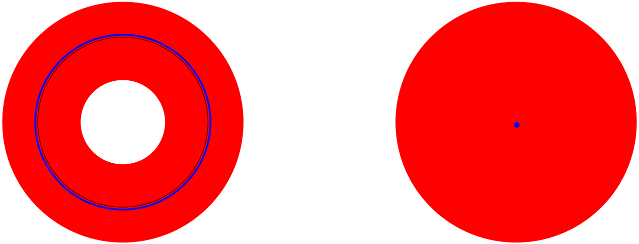
A weaker notion of topological equivalence

Let X and Y be two subsets of $\mathbb{R}^d$. Two maps $f_0, f_1 : X \to Y$ are said to be homotopic if there exists a continuous map $H : [0, 1] \times X \to Y$ s.t.

$$\forall x \in X, \quad H(0, x) = f_0(x) \quad \wedge \quad H(1, x) = f_1(x)$$



Homotopy equivalence

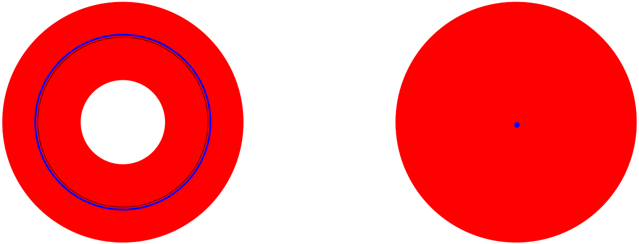


X and Y are said to be homotopy equivalent if there exist two continuous maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that $f \circ g$ $(g \circ f)$ is homotopic to the identity map in Y (X)

Deformation retract : $r : X \rightarrow Y \subseteq X$ is a d.r. if it is homotopic to Id
X and Y then have the same homotopy type

X is said to be contractible if it has the same homotopy type as a point

Homotopy equivalence



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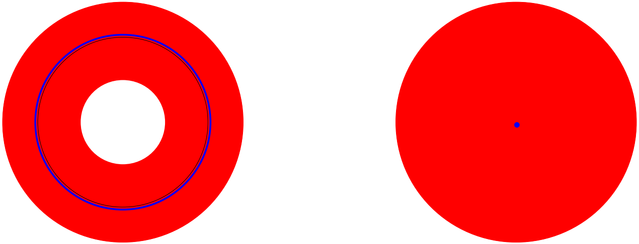

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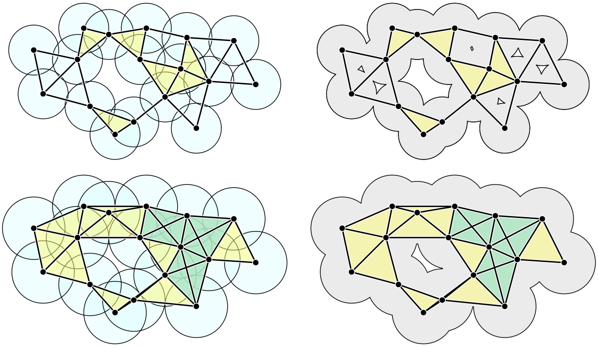
Nerve of a finite cover $\mathcal{U} = \{U_1, ..., U_n\}$ of X



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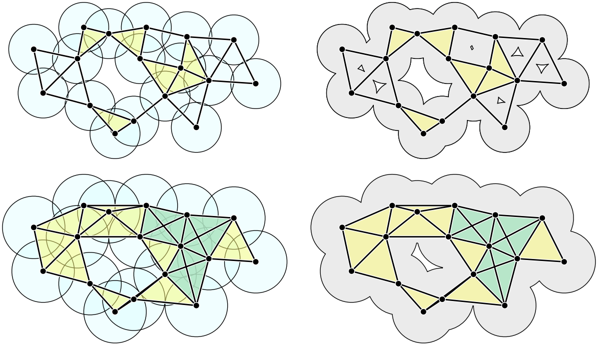
Nerve of a cover

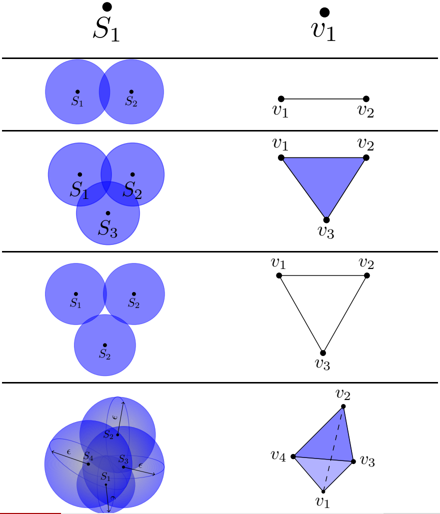


Nerve Theorem (Leray)

If any intersection of the U_i is either empty or contractible, then X and $K(U)$ have the same homotopy type

Example 1: Cech complex of a point set $P \subset \mathbb{R}^d$

$$\sigma \subseteq P \in C(P, \alpha) \quad \Leftrightarrow \quad \cap_{p \in \sigma} B(p, \alpha) \neq \emptyset$$



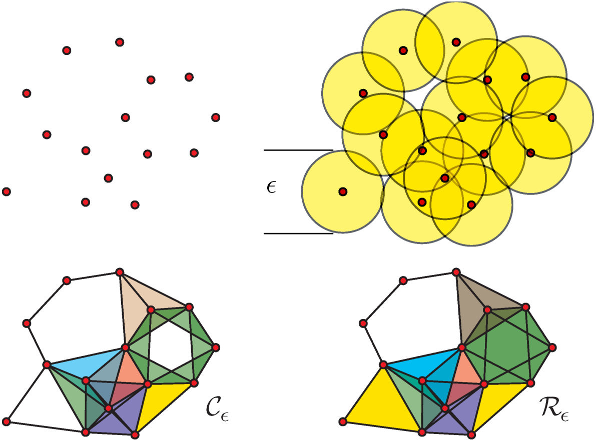
Exercises

- Show that $\sigma \in C(P, \alpha) \quad \Leftrightarrow \quad R(\text{minball}(P)) \leq \alpha$
- Propose an algorithm to compute minball(P)

  ($O(\#P)$ time complexity for fixed dimension d)
- Involves the computation of radii of spheres circumscribing d-simplices

Example 2 : Rips complex of P

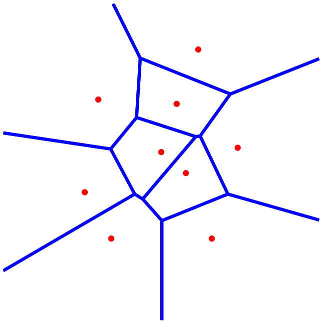
$$\sigma \subseteq P \in R(P, \alpha) \quad \Leftrightarrow \quad \forall p, q \in \sigma \ \|p - q\| \leq \alpha \quad \Leftrightarrow \quad B(p, \frac{\alpha}{2}) \cap B(q, \frac{\alpha}{2}) \neq \emptyset$$



Exercises

- Show that $R(P, \alpha) \subseteq C(P, \alpha) \subseteq R(P, 2\alpha)$
- Computing $R(P, \alpha)$ reduces to computing the graph G (vertices+edges) of $R(P, \alpha)$ and computing the cliques of G

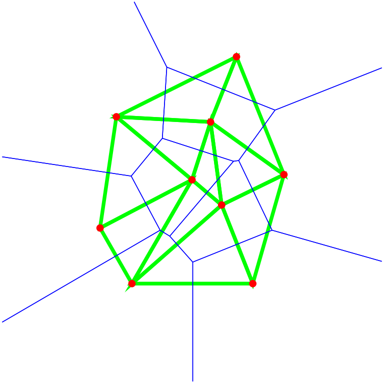
Nerves of Euclidean Voronoi diagrams



Voronoi cell $\quad V(p_i) = \{x : \|x - p_i\| \leq \|x - p_j\|, \ \forall j\}$

Voronoi diagram $(\mathcal{P})$ $\quad = \{$ collection of all cells $V(p_i)$, $p_i \in \mathcal{P}$ $\}$

Nerves of Euclidean Voronoi diagrams



The nerve of $\mathrm{Vor}(P)$ is called the Delaunay complex $\mathrm{Del}(P)$

$\mathrm{Del}(P)$ cannot always be realized in $\mathbb{R}^d$

Triangulation of a finite point set of $\mathbb{R}^d$

- A simplicial k-complex K is pure if every simplex in K is the face of a k-simplex.

- A triangulation of a finite point set $P \in \mathbb{R}^d$ is a pure geometric simplicial complex K s.t. $\text{vert}(K) = P$ and $|K| = \text{conv}(P)$.

Problem : show that the Delaunay triangulation of a finite point set of $\mathbb{R}^d$ is a triangulation under some mild genericity assumption

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Stars and links

- Let K be a simplicial complex with vertex set P. The star of $p \in P$ is the set of simplices of K that have p as a vertex
- The link of p is the set of simplices $\tau \subset \sigma$ such that $\sigma \in \mathrm{star}(p, K)$ but $\tau \notin \mathrm{star}(p, K)$

If K is a triangulation of a point set

- the link of any vertex of $K \setminus \partial K$ is a triangulated $(k-1)$-sphere
- the link of any vertex of ∂K is a triangulated $(k-1)$-ball

Data structures to represent simplicial complexes

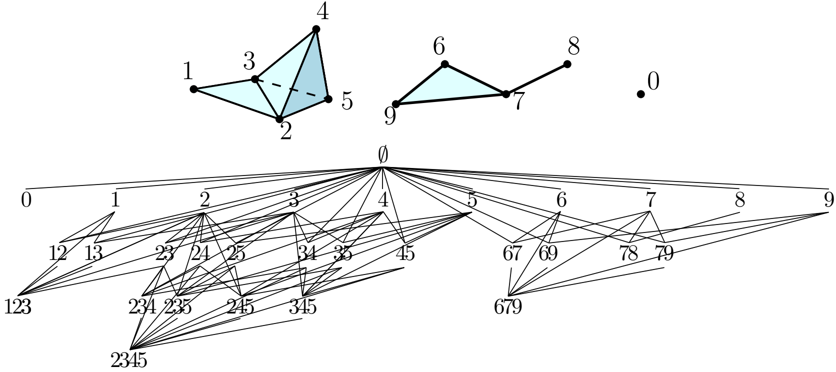
Atomic operations

- **Look-up/Insertion/Deletion** of a simplex
- The **facets** and **subfaces** of a simplex
- The **cofaces** of a simplex
- **Edge contractions**
- **Elementary collapses**

Explicit representation of all simplices ? of all incidence relations ?

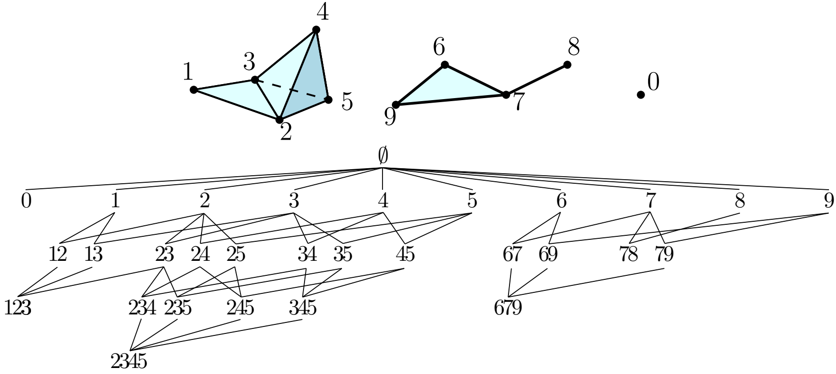
The incidence graph

$$
\begin{array}{llcl}
G(V,E) & \sigma \in V & \Leftrightarrow & \sigma \in K \\
 & (\sigma,\tau) \in E & \Leftrightarrow & \sigma \subset \tau
\end{array}
$$



The Hasse diagram

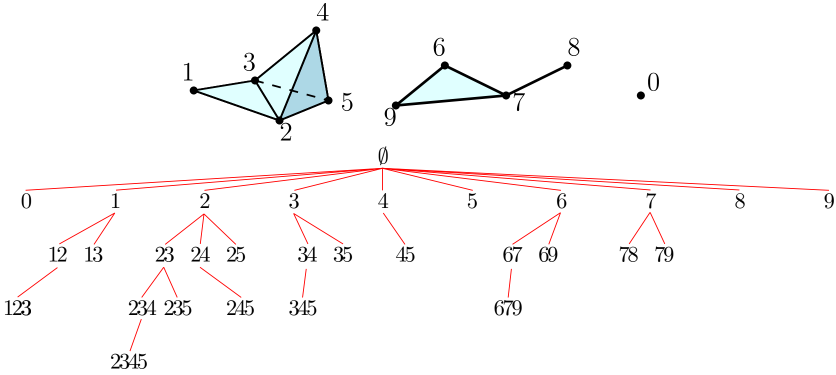
$$G(V,E) \qquad \begin{array}{lcl} \sigma \in V & \Leftrightarrow & \sigma \in K \\ (\sigma,\tau) \in E & \Leftrightarrow & \sigma \subset \tau \;\; \wedge \;\; \dim(\sigma) = \dim(\tau) - 1 \end{array}$$



The simplex tree

[B., Maria 2014]

1. Select a specific spanning tree of the Hasse diagram s.t. the chosen incidences respect the lexicographic order
2. Keep only the biggest vertex in each simplex. The vertices of a simplex are encountered in the path from the root to its node



The simplex tree

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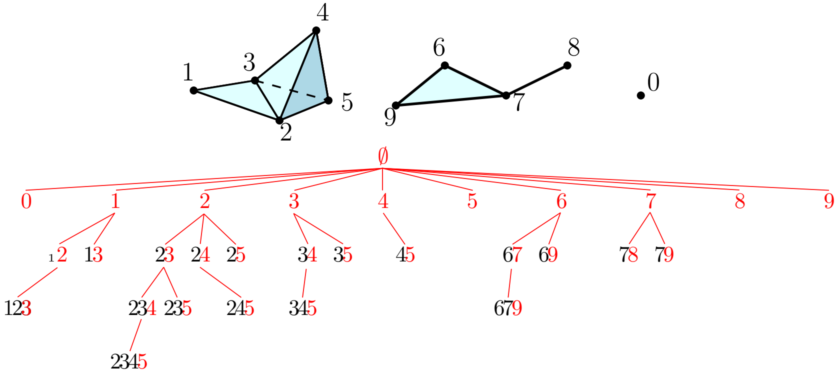

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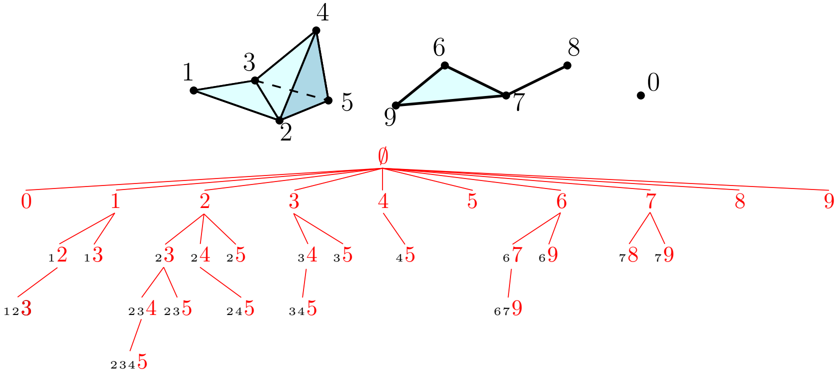

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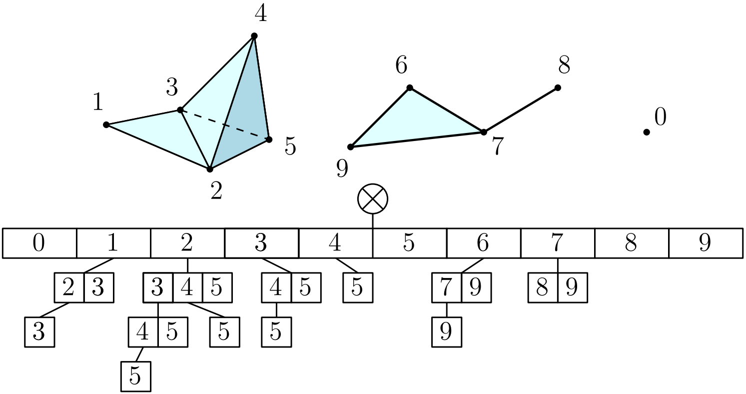
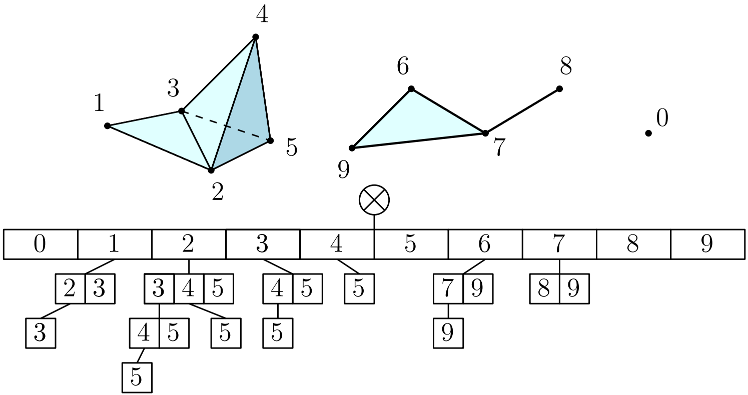


The simplex tree is a trie

1. index the vertices of K
2. associate to each simplex $\sigma \in K$, the sorted list of its vertices
3. store the simplices in a trie.



Performance of the simplex tree

- Explicit representation of all simplices
- #nodes $=$ $\#\mathcal{K}$
- **Memory complexity**: $O(1)$ per simplex.
- depth $=$ dim$(\mathcal{K}) + 1$
- #children(σ) $\leq$ #cofaces(σ) $\leq$ deg(last(σ))

Data	$\|\mathcal{P}\|$	D	d	r	k	T_{g}	$\|E\|$	T_{Rips}	$\|\mathcal{K}\|$	T_{tot}	$T_{\mathrm{tot}}/\|\mathcal{K}\|$
Bud	49,990	3	2	0.11	3	1.5	1,275,930	104.5	354,695,000	104.6	$3.0 \cdot 10^{-7}$
Bro	15,000	25	?	0.019	25	0.6	3083	36.5	116,743,000	37.1	$3.2 \cdot 10^{-7}$
Cy8	6,040	24	2	0.4	24	0.11	76,657	4.5	13,379,500	4.61	$3.4 \cdot 10^{-7}$
Kl	90,000	5	2	0.075	5	0.46	1,120,000	68.1	233,557,000	68.5	$2.9 \cdot 10^{-7}$
S4	50,000	5	4	0.28	5	2.2	1,422,490	95.1	275,126,000	97.3	$3.6 \cdot 10^{-7}$

Exercises

- Show how to implement the atomic operations on a ST

  [B., Maria 2014]

- Show how to represent a Rips complex

Computing the min. enclosing ball mb(P) of $P \subset \mathbb{R}^d$

Properties

- mb(P) is unique
- mb(P) is determined by at most $d+1$ points
- If $B = \mathrm{mb}(P \setminus \{p\})$ and $p \notin B$, then $p \in \partial$ mb(P)
- same results for mb (P, Q), the min ball B such that

  $P \subset \mathrm{int} B$ and $Q \in \partial B$ (if it exists)

  If $B = \mathrm{mb}(P \setminus \{p\}, Q))$ and $p \notin B$, then

  - $p \in \partial$ mb(P, Q) (if it exists)
  - $\Leftrightarrow$ mb(P, Q) = mb$(P \setminus \{p\}, Q \cup \{p\})$

Computing the min. enclosing ball mb(P) of $P \subset \mathbb{R}^d$

Algorithm

input P

$Q := \emptyset$ // points on ∂ mb(P)

mb(P) := miniball(P, Q)

stop

Algorithm miniball(P, Q) // $Q =$ points that have to be on ∂minball

1. if $P = \emptyset$ then compute directly $B :=$ mb(Q)
2. else
   1. choose a random $p \in P$
   2. $B :=$ miniball($P \setminus \{p\}, Q$)
   3. if $p \notin B$ then $B :=$miniball($P \setminus \{p\}, Q \cup \{p\}$) // $p \in \partial B$
3. return B

Complexity analysis

Let $T(n,j)$ = expected number of tests $p \notin B$, $p \in P$
with $\#P = n$ and $j = d+1-\#Q$

$T(0,j) = 0$ and $T(n,0) = 0$

since p is any point among P and $\#(P \cap \partial B) = j$,

proba ($p \notin B = \text{miniball}(P \setminus \{p\}, Q)) \leq \frac{j}{n}$

$$T(n,j) \leq T(n-1,j) + O(1) + \frac{j}{n}\, T(n-1,j-1)$$

$$\Rightarrow \quad T(n,j) \leq (j+1)!\, n$$

Complexity of mb(P) $= O(d)\, T(n, d+1) = O(n)$ for fixed d