Optimization-based Anisotropic $hp$-Adaptation for High-Order Methods

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Introduction

High-order methods for compressible flow simulation

- Need for more accurate flow predictions and potential for computational cost improvements
- High-order discretizations on unstructured grids for complex geometries
- Discontinuous finite element: high-order accuracy with compact stencil
- Relies on arbitrary computational space discretization (\(hp\)-mesh)

Anisotropic \(hp\)-adaptation

- Computational cost and robustness limit the practical use of high-order methods
- Mesh adaptation yields significant improvements
- Use local error estimates to seek cost-efficient accuracy improvements
- Anisotropy and \(h\)-refinement/\(p\)-enrichment choices
- Existence of an optimal distribution of the degrees of freedom
Presentation outline

1 Introduction

2 Adaptive High-Order Methods

3 Error Estimate Sensitivities

4 Mesh Optimization

5 Numerical Results

6 Conclusions
High-order finite element methods

Problem formulation

\[ \mathcal{R}(u) = \frac{\partial u}{\partial t} + \nabla \cdot \mathcal{F}(u) = 0 \]

High-order methods seek a discrete approximation satisfying \( \mathcal{R}_h(u_h) = 0 \) in space

\[ \mathcal{V}_{hp} = \left\{ \varphi \in L^2(\Omega) \mid \varphi|_{\Omega_k} \in P_p(\Omega_k), \forall \Omega_k \in T_h \right\} \]

such that \( u_h(x) = \sum_{i=1}^{N_p} u_i \varphi_i(x) \) on every element \( \Omega_k \).

Discontinuous unstructured high-order schemes

- Discontinuous Galerkin

\[ \int_{\Omega_k} \varphi_h \frac{\partial u_h}{\partial t} d\Omega - \int_{\Omega_k} \nabla \varphi_h \cdot \mathcal{F}(u_h) d\Omega + \int_{\partial \Omega_k} \varphi_h \hat{\mathcal{F}}(u_h^+, u_h^-, n) d\Gamma = 0 \]

- Flux Reconstruction

\[ \int_{\Omega_k} \varphi_h \delta_h d\Omega = \int_{\partial \Omega_k} \varphi_h (\hat{\mathcal{F}} - \mathcal{F}(u_h^+) \cdot n) d\Gamma \implies \frac{\partial u_h}{\partial t} + \nabla \cdot \mathcal{F}(u_h) + \delta_h = 0 \]
Error estimates and \( hp \)-adaptation

**Typical \( hp \)-adaptation algorithm**

- Local error estimates \( \eta_k \): feature-based, residual-based and adjoint-based
- Fixed-fraction iterative refinement/coarsening strategy
- \( h \)-refinement/\( p \)-enrichment based on smoothness (e.g. modal decay rate, jumps)
- Isotropic/anisotropic based on directivity (e.g. Hessian, jumps)

**The goal-oriented approach**

Given an output functional of interest

\[
\mathcal{J}(u) = \int_{\Omega} g_\Omega(u) \, d\Omega + \int_{\Gamma} g_\Gamma(u) \, d\Gamma
\]

as well as coarse \( \mathcal{R}_H(u_H) \) and fine \( \mathcal{R}_h(u_h) \) discretizations, we have

\[
\mathcal{J}_h(u_h) - \mathcal{J}_h(u^H_h) \simeq -\psi^T_h \mathcal{R}_h(u^H_h) = \eta_k
\]

with

\[
\left( \frac{\partial \mathcal{R}_h}{\partial u} \bigg|_{u^H_h} \right)^T \psi_h = \left( \frac{\partial \mathcal{J}_h}{\partial u} \bigg|_{u^H_h} \right)^T
\]
Mesh-metric duality

Continuous mesh representation

- Physical element $\iff$ metric matrix $\mathcal{M}_k \in \mathbb{S}_{++}^{d \times d}$

$$\mathcal{M}_k = \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{12} & \mathcal{M}_{22} \end{bmatrix} = R^T(\varphi) \begin{bmatrix} 1/h_1^2 & 0 \\ 0 & 1/h_2^2 \end{bmatrix} R(\varphi)$$

- Global polynomial degree $\mathcal{P}(x)$ and metric $\mathcal{M}(x)$ fields
- Metric-based mesh regeneration / modification
Sensitivities

A unifying approach

- Optimal flexibility is obtained via node density modifications

\[ \rho(\mathcal{M}, \mathcal{P}) := \frac{1}{|\Omega_{\text{ref}}|} N(\mathcal{P}) \sqrt{\det \mathcal{M}} \quad \Rightarrow \quad N_{dof}(\mathcal{M}, \mathcal{P}) := \int_{\Omega} \rho(\mathcal{M}, \mathcal{P}) d\Omega \]

Error estimate sensitivities

- Information on where and how to refine \( \frac{d\eta}{dp}, \frac{d\eta}{d\mathcal{M}} \)

- Based on \( \psi(x) = \sum_{i=1}^{N_p} \psi_i L_i(x) \) and \( \mathcal{R}(x) = \sum_{i=1}^{N_p} \mathcal{R}_i L_i(x) \)

\[ \frac{d\eta}{dp} \simeq - \left( \psi^T \frac{\partial \mathcal{R}}{\partial p} + \frac{\partial \psi}{\partial p} \right) \]

\[ \frac{d\eta}{d\mathcal{M}} = \sum_{k=1}^{d} \sum_{n=1}^{N_v} \frac{d\eta}{dx_n^k} \frac{dx_n^k}{d\mathcal{M}} \simeq - \sum_{k=1}^{d} \sum_{n=1}^{N_v} \left( \frac{\partial \psi}{\partial x_n^k} \mathcal{R} + \psi^T \frac{\partial \mathcal{R}}{\partial x_n^k} \right) \frac{dx_n^k}{d\mathcal{M}} \]
Sensitivities (continued)

Evaluation of $\frac{\partial \psi}{\partial p}$ and $\frac{\partial R}{\partial p}$

- Projection to the monomial basis yields an expression with explicit dependence on $p$

$$\psi(x, y) = \sum_{i=1}^{N_p} \psi_i L_i(x, y) = \sum_{j=1}^{N_p} \tilde{\psi}_j x^m(p,j) y^m(p,j)$$

Evaluation of $\frac{\partial \psi}{\partial x_n}$, $\frac{\partial R}{\partial x_n}$ and $\frac{\partial x_n}{\partial M}$

- From the Lagrange basis expansion of $\psi$ and $R$, we get

$$\frac{\partial \psi}{\partial x_n}(x, y) = \sum_{i=1}^{N_p} \frac{\partial \psi(x_i, y_i)}{\partial x_n} L_i(x, y) + \psi(x_i, y_i) \frac{\partial L_i}{\partial x_n}(x, y)$$

- Based on the mapping

$$x_n = N\tilde{x}_n + \text{cst} = M^{-1/2}\tilde{x}_n + \text{cst} = R^T\Lambda^{-1/2}R\tilde{x}_n + \text{cst},$$

$$\frac{\partial x_n}{\partial M} = \frac{\partial N(h_1, h_2, \varphi)}{\partial M(h_1, h_2, \varphi)} \tilde{x}_n$$
**hp-mesh optimization**

**Continuous problem formulation**

\[
\text{minimize} \quad \|\eta(M, P)\|_2^2 = \int_{\Omega} \eta^2 \, d\Omega \\
\text{subject to} \quad N_{dof}(M, P) \leq \kappa N_{dof}(M_0, P_0) \\
\quad \quad \quad P(x) \geq 0 \\
\quad \quad \quad \lambda_i(M(x)) \geq \epsilon, \quad i = 1, \ldots, d
\]

**Discretization over** \(N_c\) **control points**

\[
\text{minimize} \quad \|\eta\|_2^2 = \sum_{k=1}^{N_e} |\Omega_k| \eta_k^2 \\
\text{subject to} \quad \sum_{k=1}^{N_c} c_k N_k(P_k) \sqrt{\det M_k} \leq \kappa \sum_{k=1}^{N_c} c_k N_k(P_{0,k}) \sqrt{\det M_{0,k}} \\
\quad \quad \quad P_k \geq 0, \quad \forall k = 1, \ldots, N_c \\
\quad \quad \quad \lambda_i(M_k) \geq \epsilon, \quad \forall k = 1, \ldots, N_c \text{ and } i = 1, \ldots, d
\]
Set $(\mathcal{M}, \mathcal{P}) = (\mathcal{M}_0, \mathcal{P}_0)$ and $\mathcal{N}_{dof}$ constraint

Create $hp$-mesh from $(\mathcal{M}, \mathcal{P})$ fields

Compute flow and adjoint solutions

Compute error estimate $\eta$ and sensitivities $\left( \frac{d\eta}{dp}, \frac{d\eta}{dM} \right)$

Interpolate sensitivities at control points

Update $(\mathcal{M}, \mathcal{P})$
1D linear advection

- Manufactured solution with sharp features designed to require both $h$- and $p$- adaptivity
- Scheme: Flux Reconstruction
- Functional of interest: $J(u) = \int_0^1 u^2 dx$
- Main challenge: error propagation

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Exact solution
Initial coarse P1 solution
Sensitivities verification

![Graph](image1)

![Graph](image2)

<table>
<thead>
<tr>
<th>Polynomial degree</th>
<th>Difference for $\frac{d\eta}{dp}$ $\Delta h = 1.0e-02$</th>
<th>Difference for $\frac{d\eta}{dM}$ $\Delta h = 2.5e-03$</th>
<th>Difference for $\frac{d\eta}{dM}$ $\Delta h = 6.25e-04$</th>
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Numerical Results

Optimization-based $hp$-adaptation

- Functional error convergence for $hp$-adaptation

![Graph showing convergence plots for various methods: Adjoint-based hp (0.3/1), Adjoint-based hp (0.3/0.01), Adjoint-based hp (0.3/0.001), Adjoint-based hp (0.3/0.0001), Adjoint-based hp (0.5/1), Adjoint-based hp (modal), Sensitivity-based hp (0.3), Optimization-based hp.](image-url)
Numerical Results

2D linear advection-diffusion

- Manufactured solution designed to require both $h$- and $p$- adaptivity
- $U_\infty = [1, 1]$ and $\mu = 10^{-2}$
- Functional of interest: $J(u) = \int_{\Gamma_e} u(x, y) \, d\Gamma$
- Scheme: Discontinuous Galerkin

![Solution $u(x, y)$](image1)

![Adjoint-based error indicator $\eta$](image2)
Optimization-based $p$-adaptation

- Functional error convergence for $p$-adaptation

![Graph showing functional error convergence for $p$-adaptation]
Numerical Results

Optimization convergence for $p$-adaptation

- Objective function and norm of the first optimality condition throughout the optimization

![Graph showing objective function and norm of the first optimality condition](image)
Comparison of \( p \)-adapted meshes

- Adapted \( p \)-distributions using the adjoint-based and optimization-based approaches
- Similar number of degrees of freedom but vastly different error levels

Adjoint-based

Optimization-based
Comparison of $p$-adapted meshes

**Adjoints-based**

**Optimization-based**
Comparison of $p$-adapted meshes

Adjoint-based

Optimization-based
Control grid and constant dofs optimization

![Graph showing control grid and functional error comparison. The x-axis represents the number of degrees of freedom, ranging from $10^{-7}$ to $10^5$. The y-axis represents the functional error, ranging from $10^{-6}$ to $10^1$. The plot compares uniform metric refinement and optimization-based $h$-adaptation.]
Optimization-based \(h\)-adaptation

- Functional error convergence for \(h\)-adaptation

![Diagram showing functional error convergence for different adaptation methods and settings.](image_url)
Objective function and norm of the first optimality condition throughout the optimization.
Comparison of $h$-adapted meshes

- Adapted meshes using the adjoint-based and optimization-based methods
- A less clustered grid results in a significantly lower error for a similar number of elements

Adjoint-based

Optimization-based
Comparison of $h$-adapted meshes

Adjoint-based

Optimization-based
Optimization-based $hp$-adaptation

- Functional error convergence for $hp$-adaptation
Numerical Results

Optimization convergence for $hp$-adaptation

- Objective function and norm of the first optimality condition throughout the optimization
Comparison of $hp$-adapted meshes

- Adapted $hp$-meshes using the adjoint-based and optimization-based approaches
- Significantly reduced functional error at an equivalent number of degrees of freedom

**Adjoint-based**

**Optimization-based**
Comparison of $hp$-adapted meshes

Adjacent-based

Optimization-based
Comparison of \(hp\)-adapted meshes

\begin{align*}
\text{Adjoint-based} & & \text{Optimization-based}
\end{align*}
Conclusions

A novel optimization-based framework for anisotropic $hp$-adaptation

- A global approach unifying $r$-, $h$- and $p$-adaptation
- Implicitly handles $h/p$, anisotropy and refinement/coarsening choices
- Features a reduced dependence on user-defined parameters
- Showed promising results on 1D and 2D linear benchmark cases
- Generalizes to non-linear governing equations, other discretizations and 3D problems

Further research

- Numerical assessment on non-linear 2D benchmark cases with curved geometries
- Experiments on 2D/3D inviscid and viscous compressible flows
- Comparisons with INRIA’s grid adaptation framework