Mesh adaptation for high order finite elements spaces

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Introduction

High order methods

- Discontinuous Galerkin
  Hartmann, Held, Leicht, Prill, *Aerospace Sc. Tech. 2010*
- Spectral differences
  Liu, Vinokur, Wang, *J. Comp. Phys. 2006*
- Residual distributions
  Abgrall, *Computers and Fluids 2006*

Problematic

1. Less efficient if the geometry is not high-ordered approximated
   Bassi, Rebay, *Comp. Phys. 1997*
2. Memory costly

High order mesh adaptation

Find the best mesh $\mathcal{H}_{opt}$ with complexity N which minimizes the $\mathbb{P}_k$ interpolation error of u

Numerical error

$$\|u - \Pi_k u_h\|_{L^p} = \underbrace{\|\Pi_k u - \Pi_k u_h\|_{L^p}}_{\text{Implicit error}} + \underbrace{\|u - \Pi_k u\|_{L^p}}_{\text{Interpolation error}}$$

Outline

1. Metric based mesh adaptation
   - Riemannian metric spaces
   - $\mathbb{P}_1$ adaptation: Hessian based methods
2. From $\mathbb{P}_1$ to $\mathbb{P}_k$
   - Local error estimate: the log-simplex approach
   - Optimal Metric space
3. Numerical applications
   - Smooth analytical functions
   - High order surface mesh generation

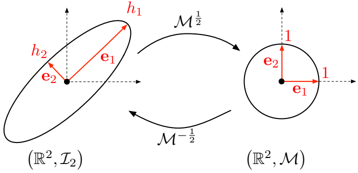
Riemannian metric space

Main idea: change geometric quantities

[George, Hecht and Vallet., Adv. Eng. Software 1991]

Fundamental concept: The notion of **metric** and Riemannian metric space

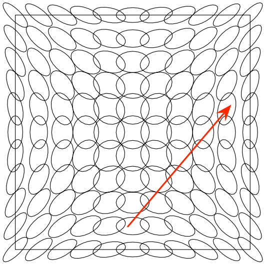
- **Euclidean metric space**
  - $\mathcal{M}$ symmetric definite positive matrix
  - Scalar product $\langle \mathbf{x}\,,\,\mathbf{y}\rangle_{\mathcal{M}} = {}^t\mathbf{x}\,\mathcal{M}\,\mathbf{y}$
  - Length $\ell_{\mathcal{M}}(\mathbf{a},\mathbf{b}) = \sqrt{{}^t\mathbf{ab}\;\mathcal{M}\;\mathbf{ab}}$
  - Unit ball $\mathcal{B}_{\mathcal{M}}$
    - ellipse in $\mathbb{R}^2$
    - ellipsoid in $\mathbb{R}^3$

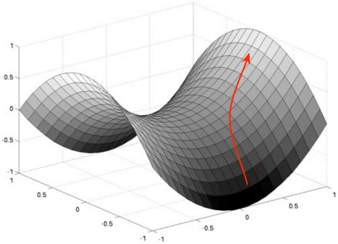


- **Riemannian metric space** $\mathbf{M} = (\mathcal{M}(\mathbf{x}))_{\mathbf{x}\in\Omega}$ continuous in Ω

Riemannian metric space

Computing geometric quantities in Riemannian metric space
$$\mathbf{M} = (\mathcal{M}(\mathbf{x}))_{\mathbf{x}\in\Omega}$$
$$\Updownarrow$$
Computing geometric quantities on $\mathcal{S}$

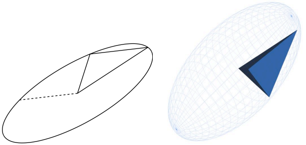




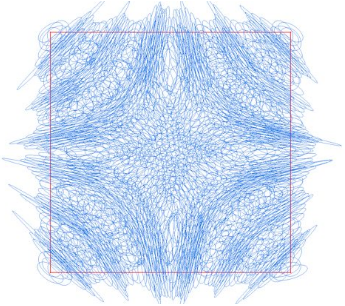
Unit Mesh

Fundamental concept: Generate a **unit mesh** w.r.t $\mathbf{M} = (\mathcal{M}(\mathbf{x}))_{\mathbf{x}\in\Omega}$

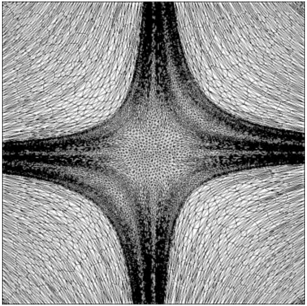
$$\forall \mathbf{e},\ \ell_{\mathcal{M}}(\mathbf{e}) \approx 1 \text{ and } \forall K,\ |K|_{\mathcal{M}} \approx \begin{cases} \sqrt{3}/4 & \text{in 2D} \\ \sqrt{2}/12 & \text{in 3D} \end{cases}$$



Inputs $(\mathcal{H}_0, \mathcal{M}_i)_{i\in\mathcal{H}}$



Output $\mathcal{H}$



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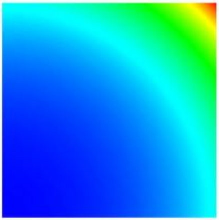
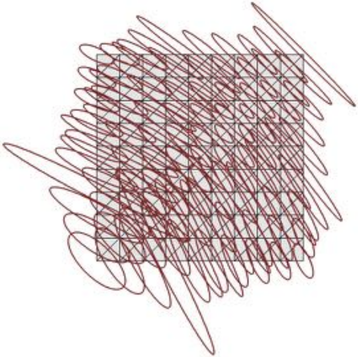
$\mathbb{P}_1$ adaptation: the hessian based methods

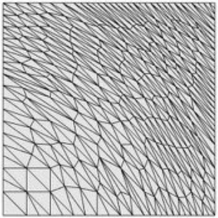
Solution u on Ω



Metric field on a mesh $\mathcal{H}$ of Ω
$\mathcal{M}(x_i) = C(x_i)\,|H_u|\,(x_i)$



Unit Mesh with respect to $\mathbf{M}$



$\mathbb{P}_k$ adaptation problem

What is the metric field $\mathbf{M} = (\mathcal{M}(x))_{x\in\Omega}$ minimizing the P_k interpolation error ?

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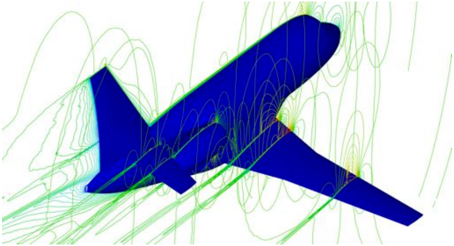
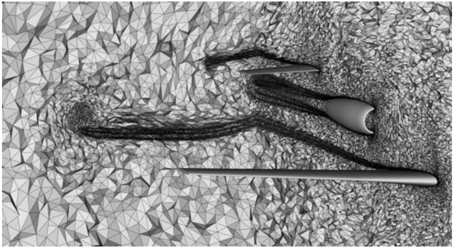
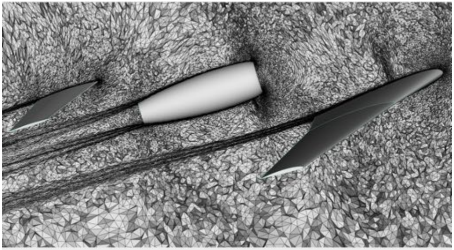
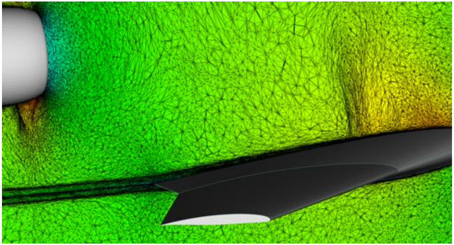
- `metrix`
- `feflo.a`

Adaptive RANS computation



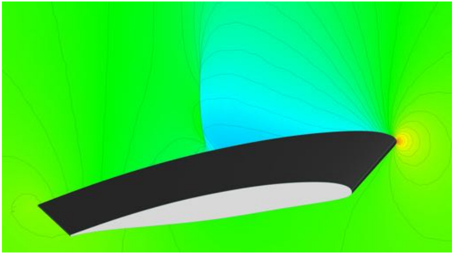
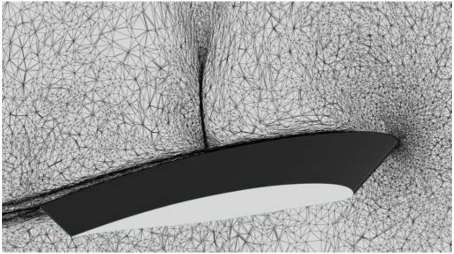






- 10 iterations/12h (20 cores)
- 10 337 483 vertices, 428 464 triangles and 61 629 069 tetrahedra

Adaptive RANS computation

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Local error

Local interpolation error

$$|u(x) - \Pi_k u(x)|_2 \leq C \left| d^{(k+1)} u(x_0)(x - x_0) \right| + o\left(|x - x_0|_2^{k+1} \right), \text{ for all } x, x_0 \in \Omega$$

$\mathbb{P}_1$ adaptation

Alauzet, Frey, *Comp. Meth. App. Mech. Eng. 2005*

Loseille, Dervieux, Frey, Alauzet *Ame. Inst. Aero. Astr. 2007*

- Optimal metric $\mathcal{M}_{opt}(x) = C(p, N, x)\,|H_u(x)|$, with $C(p, N, x) > 0$

$\mathbb{P}_k$ adaptation, $k > 1$

Cao, *Soc. Indus. App. Math. 2007*

Mirebeau, *Num. Math. 2011*

Hecht, Kuate, *J. Comput. Appl. Math. 2014*

- Find a metric $\mathbf{Q} = (\mathcal{Q}(x))_{x \in \Omega}$ such that

$$\left({}^t y \mathcal{Q}(x) y\right)^{\frac{k+1}{2}} \geq d^{(k+1)} u(x)(y), \text{ for all } x \in \Omega, y \in \mathbb{R}^n$$

- Optimal metric $\mathcal{M}_{opt}(x) = C(p, N, x)\mathcal{Q}(x)$, with $C(p, N, x) > 0$

Log-simplex method

- Optimisation problem for $\mathcal{L} = \log(\mathcal{Q})$
- Approximated linear problem, solved by a simplex method

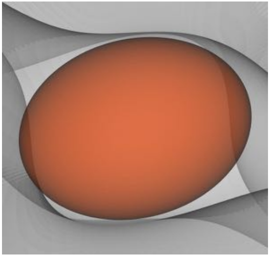
Geometrical interpretation

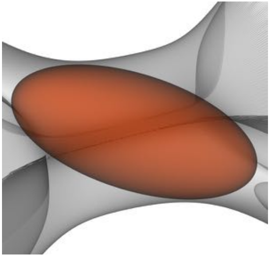
The problem can finally be reduced to the following one

Local problem

For p an homogeneous polynomial of degree $k+1$, find a metric $\mathcal{Q}$ such that

- $x \in \mathcal{B}_{\mathcal{Q}} \Rightarrow |p(x)| < 1$
- $\mathcal{B}_{\mathcal{Q}}$ has the largest volume $\Leftrightarrow \det(\mathcal{Q})$ is minimal



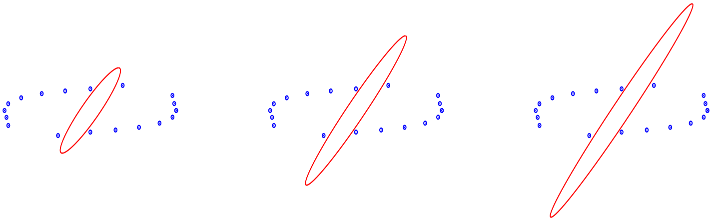


Local discrete problem

For a set $\{x_1, ..., x_m\}$ such that $p(x_i) = 1$, find a metric $\mathcal{Q}$ such that

- ${}^t x_i \mathcal{Q} x_i \geq 1$, for all $x_i \in \{x_1, ..., x_m\}$
- $\det(\mathcal{Q})$ is minimal

This problem is always ill-posed



Approximated problem

Main ideas

1. Problem for $\mathcal{L} = \log(\mathcal{Q})$
2. Notice that
$$ {}^t x \, \mathcal{L} \, x \geq - \|x\|^2 \log\left(\|x\|^2\right) \quad \Longrightarrow \quad {}^t x \, \mathcal{Q} \, x \geq 1 $$
3. Notice that
$$ \det(\mathcal{Q}) = \exp\left(\mathrm{trace}(\mathcal{L})\right) $$

Approximated log discrete problem

For a set $\{x_1, \ldots, x_n\}$ such that $p(x_i) = 1$, find a symmetric matrix $\mathcal{L}$ such that

- ${}^t x_i \mathcal{L} x_i \geq - \|x_i\|^2 \log\left(\|x_i\|^2\right)$, for all $i \in \{1, \ldots, n\}$
- $\mathrm{trace}(\mathcal{L})$ is minimal

Resolution of the Log discrete problem

Advantages of the *log*−approximation

- The problem is linear in $\mathcal{L}$
  $\Rightarrow$ solved by a linear simplex algorithm
- It has solutions if the points x_i are well chosen

Proposition

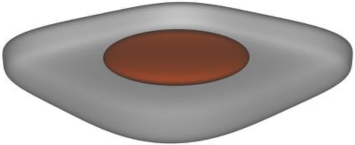
Let $(x_1, \dots, x_n)$ be an orthogonal basis of $\mathbb{R}^n$, $n = 2, 3$. If $\mathcal{L}$ is a symmetric matrix such that

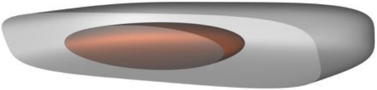
$$ {}^t x_i \mathcal{L} x_i \geq - \|x_i\|^2 \log\left(\|x_i\|^2\right), \text{ for all } i \in \{1, \dots, n\}, $$

then there exists $C(x_1, \dots, x_n) \in \mathbb{R}$ such that

$$ trace(\mathcal{L}) \geq C(x_1, \dots, x_n). $$

How to recover the initial constraints ?





Iterative process

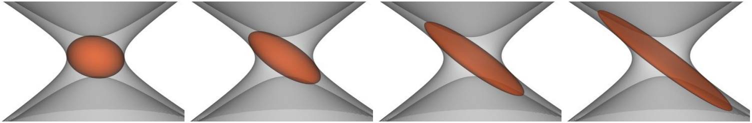
- choose a set $\{x_1, ...x_m\} \in \mathbb{R}^n$ such that $|p(x_i)| = 1$
- compute the optimal metric $\mathcal{Q}$ solution of the $log-$simplex problem
- perform the change of variable $y = \mathcal{Q}^{\frac{1}{2}}x$
- replace p by $p \circ \mathcal{Q}^{-\frac{1}{2}}$

$\Longrightarrow$ If $\mathcal{Q}$ converges, the log contraints and the initial ones are **equivalent**

Infinite branches

How to deal with infinite branches ?

1 iteration | 10 iterations | 20 iterations | 30 iterations



Polynomial reduction

[Borouchaki, George]

There exist symmetric matrices $\{H_{ijk}\}_{i+j+k=d-2}$ such that

$$p(x,y,z) = \sum_{i+j+k=d-2} x^i y^j z^k \left({}^tXH_{ijk}X\right), \quad X = (x,y,z)$$

We replace p by $q(x,y,z) = \sum_{i+j+k=d-2} \left|x^i y^j z^k\right| \left({}^tX\,|H_{ijk}|\,X\right)$.

Log-simplex $\mathbb{P}_k$ algorithm

Input

- Mesh $\mathcal{H}$ of Ω
- Solution u at the nodes of $\mathcal{H}$

Output

- Metric field $\mathbf{Q}$ at the vertices of $\mathcal{H}$

For each vertex $x_0 \in \mathcal{H}$

1. Compute $p = d^{k+1}u(x_0)$
   - On each tetrahedron, build u_k the interpolation of u of order k
   - Compute $d^k u_k$ on each tetrahedron
   - Differentiate $d^k u_k$ on the whole mesh by L^2-projection
2. Factorize p in order to get rid of the infinite branches
3. Perform the iterative log-simplex method with constraint points on the level set of $p \circ \mathcal{Q}^{-\frac{1}{2}}$ at each step

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Optimal continuous metric field

Continuous problem

- Given $\mathbf{Q} = (\mathcal{Q}(\mathbf{x}))_{\mathbf{x}\in\boldsymbol{\Omega}}$ such that

$$({}^t x \mathcal{Q}(x_0) x)^{\frac{k+1}{2}} \geq d^{k+1} u(x_0)(x), \qquad \text{for all } x_0 \in \Omega,\ x \in \mathbb{R}^n,$$

- Find $\mathbf{M}_{\mathbf{L}^p} = (\mathcal{M}_{\mathbf{L}^p}(\mathbf{x}))_{\mathbf{x}\in\Omega}$ with complexity N which minimizes

$$E_{\mathbf{L}^p}(\mathbf{M}) \quad = \left(\int_\Omega \left| \text{trace}\left(\mathcal{M}^{-\frac{1}{2}}(x) \mathcal{Q}(x) \mathcal{M}^{-\frac{1}{2}}(x) \right) \right|^{\frac{p(k+1)}{2}} dx \right)^{\frac{1}{p}}.$$

$\Longrightarrow$ Solved by a calculus of variations

Remark

$$E_{\mathbf{L}^p}(\mathbf{M}) \simeq \| u - \Pi_k u \|_{\mathcal{H}, \mathbf{L}^p(\Omega)},$$

for $\mathcal{H}$ a unit mesh with respect to $\mathbf{M}$.

Optimal continuous metric field

Optimal metric for P_k adaptation

$$\mathcal{M}_{\mathbf{L}^p}^{k} \quad = N^{\frac{2}{n}} \underbrace{\left(\int_\Omega (\det \mathcal{Q})^{\frac{p(k+1)}{2p(k+1)+2n}} \right)^{-\frac{2}{n}}}_{1} \quad \underbrace{(\det \mathcal{Q})^{\frac{-1}{p(k+1)+n}}}_{2} \quad \underbrace{\mathcal{Q}}_{3}$$

1 Global normalization: to reach the constraint complexity N

2 Local normalization: sensitivity to small solution variations, depends on $\mathbf{L}^p$ norm

3 Optimal directions and optimal lengths

Properties

- $E_{\mathbf{L}^p}(\mathbf{M}_{\mathbf{L}^p}^{k}) = n^{\frac{k+1}{2}} \; N^{-\frac{k+1}{n}} \left(\int_\Omega (\det \mathcal{Q})^{\frac{p(k+1)}{2p(k+1)+2n}} \right)^{\frac{p(k+1)+n}{np}}$
- Asymptotic convergence $\quad E_{\mathbf{L}^p}(\mathbf{M}_{\mathbf{L}^p}^{k}) \leq \dfrac{C}{N^{\frac{k+1}{n}}}, \quad$ when $N \gg 1$

Remark

If $k = 1$, then $\mathbf{Q} = |H_u|$, with H_u the hessian matrix of u

P_k adaptation algorithm

Input

- Mesh $\mathcal{H}_{in}$ of Ω
- Solution u at the nodes of $\mathcal{H}_{in}$
- Complexity N

Output

- Adapted mesh $\mathcal{H}_{out}$

Do

1. For each $x \in \mathcal{H}_{in}$, compute $d^{k+1}u(x)$
2. For each $x \in \mathcal{H}_{in}$, compute $\mathcal{Q}(x)$ which approximates $d^{k+1}u(x)$
3. Normalize $\mathbf{Q} = (\mathcal{Q}(\mathbf{x}))_{\mathbf{x}\in\mathcal{H}_{\mathbf{in}}}$ and get $\mathbf{M} = (\mathcal{M}(\mathbf{x}))_{\mathbf{x}\in\mathcal{H}_{\mathbf{in}}}$ with complexity N
4. Remesh $\mathcal{H}_{in}$ and get $\mathcal{H}_{out}$, which is unit with respect to $\mathbf{M}$

`pkmetrix`	1. 2. 3.
`feflo.a`	4.

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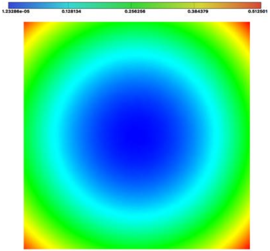
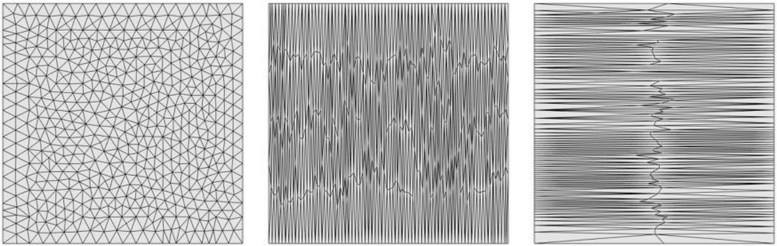
2D example

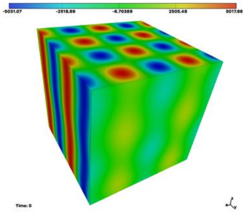
- $\Omega = \left[-\frac{1}{2}, \frac{1}{2}\right]^2$
- $u(x,y) = x^2 + y^2 + \dfrac{x^3}{10} + \dfrac{y^4}{10^5}$

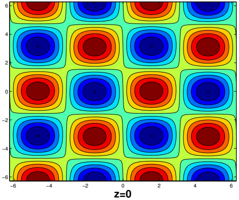


$\mathbb{P}_1, \mathbb{P}_2$ and $\mathbb{P}_3$ adapted meshes

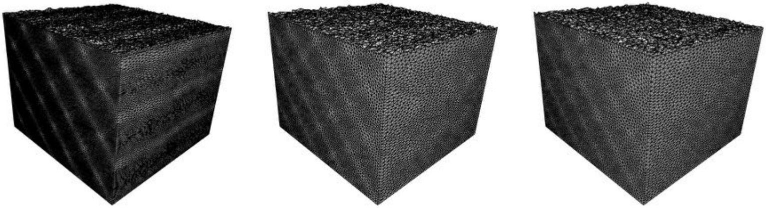


- $\Omega = [-2\pi, 2\pi]^3$
- $u(x, y, z) = 5000\cos(x)\sin(y) + 500\cos(y)\sin(z) + 100\cos(z)\sin(x)$
  $-2\cos(2x)\cos(2y) + \cos(2y)\cos(2z) + 2\cos(2x)\cos(2z) - 10$

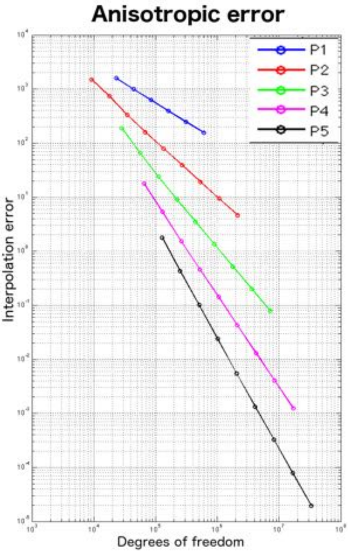


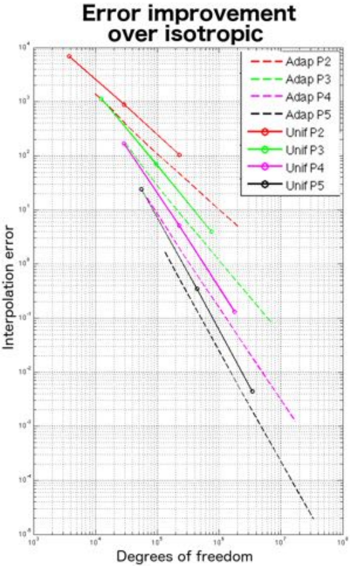


$\mathbb{P}_1, \mathbb{P}_3$ and $\mathbb{P}_5$ adapted meshes



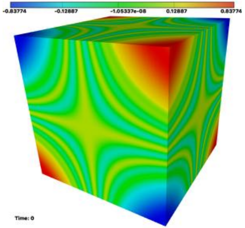
$\mathbb{P}_k$ interpolation error: gyroide

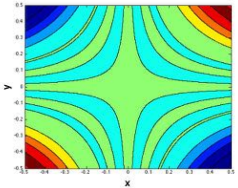




Various frequencies

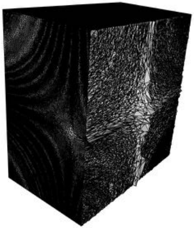
- $\Omega = \left[-\frac{1}{2}, \frac{1}{2}\right]^3$
- $u(x, y, z) = 8 \left\|xyz\right\|_2^2 \sin\left(5\pi \left\|xyz\right\|_2^2\right)^4 + \frac{1}{10}\left(1 - sin(5\pi \left\|xyz\right\|_2^8\right)^8 cos\left(100\pi \left\|xyz\right\|_2^2\right)$

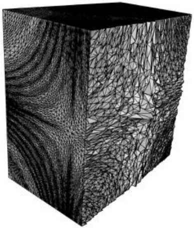


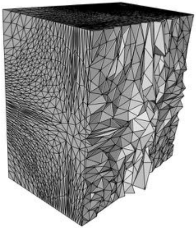


$x = 0.5$

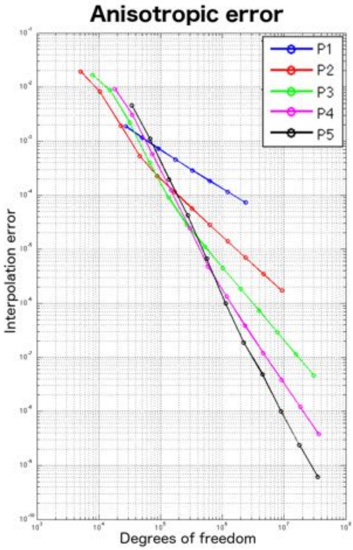
$\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_4$ adapted meshes

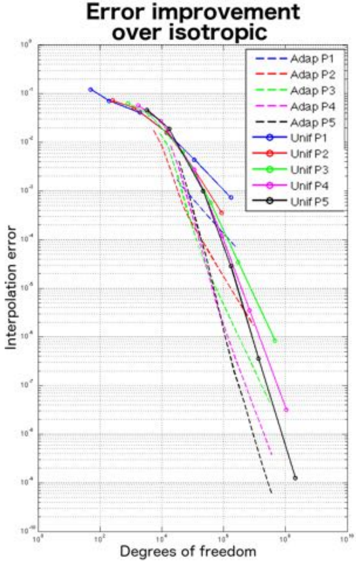






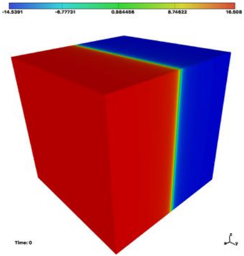
$\mathbb{P}_k$ interpolation error: various frequencies

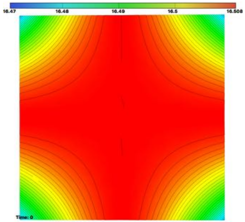




Smooth shock

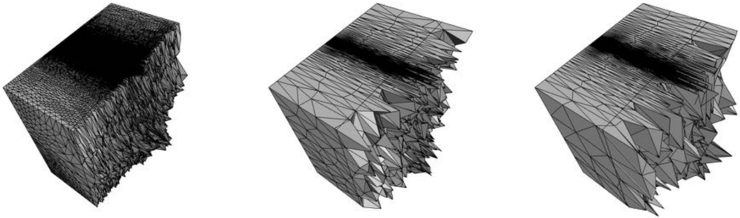
- $\Omega = \left[-\frac{1}{2}, \frac{1}{2}\right]^3$
- $u(x, y, z) = 10\,\mathrm{arctan}(100x) + \cos(yz)$



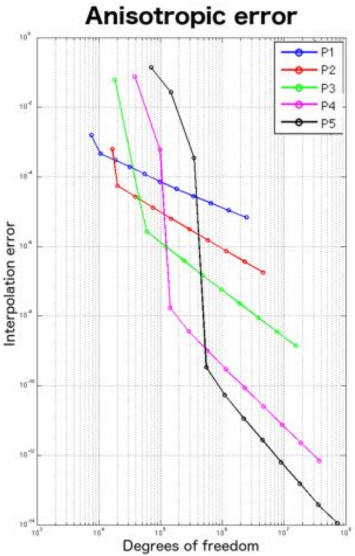


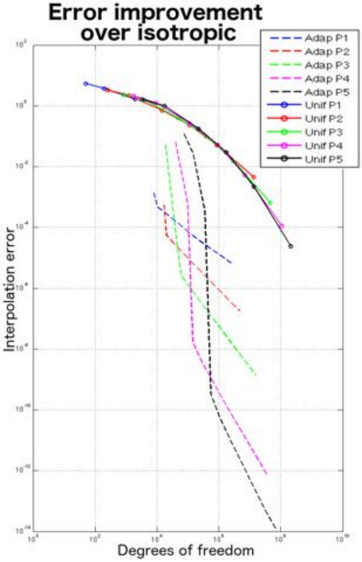
$x = 0.5$

$\mathbb{P}_1$, $\mathbb{P}_3$ and $\mathbb{P}_4$ adapted meshes



$\mathbb{P}_k$ interpolation error: smooth shock





CPU times

The gyroide function

	P1	P2	P3	P4	P5
degrees of freedom	606 834	535 177	444 435	523 941	510 630
interpolation error	155.271	19.274	3.513	0.458	0.101
total CPU time (s)	264	82	31	26	24
derivatives (s)	1	14	8	12	12
metric field (s)	6	48	14	9	9
remeshing (s)	247	15	7	4	2

The function with various frequencies

	P1	P2	P3	P4	P5
degrees of freedom	2 374 794	2 376 164	1 989 277	2 329 110	2 220 443
interpolation error	7.2×10^{-5}	6.9×10^{-6}	1.8×10^{-6}	3.9×10^{-7}	1.8×10^{-7}
total CPU time (s)	365	604	153	120	115
derivatives (s)	2	102	37	40	45
metric field (s)	26	409	100	70	64
remeshing (s)	330	63	11	6	4

The smooth shock function

	P1	P2	P3	P4	P5
degrees of freedom	2 463 091	2 299 983	1 926 453	2 299 810	2 219 674
interpolation error	6.8×10^{-6}	3.7×10^{-7}	2.3×10^{-8}	8.5×10^{-11}	1.2×10^{-11}
total CPU time (s)	547	517	382	138	144
derivatives (s)	3	96	99	61	70
metric field (s)	19	327	228	61	66
remeshing (s)	501	62	31	11	7

Outline

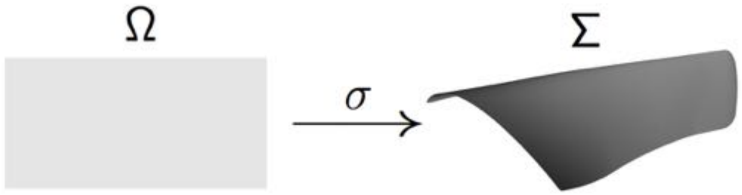
1. Metric based mesh adaptation
    - Riemannian metric spaces
    - $\mathbb{P}_1$ adaptation: Hessian based methods
2. From $\mathbb{P}_1$ to $\mathbb{P}_k$
    - Local error estimate: the log-simplex approach
    - Optimal Metric space
3. Numerical applications
    - Smooth analytical functions
    - High order surface mesh generation

High order surface mesh generation

Notations

- $\Sigma \subset \mathbb{R}^3$ surface of $\mathbb{R}^3$
- $\Omega \subset \mathbb{R}^2$ parameters space
- Surface parametrization $\sigma : \Omega \rightarrow \Sigma$

Surface parameterization



High order surface mesh generation

Local coordinates system $(\sigma(u_0, v_0), \vec{u}, \vec{v}, \vec{n})$

- $\vec{u} = \dfrac{\partial_u \sigma}{\|\partial_u \sigma\|_2}(u_0, v_0)$
- $\vec{v} = \dfrac{\partial_v \sigma - \langle \partial_v \sigma, \vec{u} \rangle}{\|\partial_v \sigma - \langle \partial_v \sigma, \vec{u} \rangle\|_2}(u_0, v_0)$
- $\vec{n} = \dfrac{\partial_u \sigma \wedge \partial_v \sigma}{\|\partial_u \sigma \wedge \partial_v \sigma\|_2}(u_0, v_0)$

Local mapping in $(\sigma(u_0, v_0), \vec{u}, \vec{v}, \vec{n})$

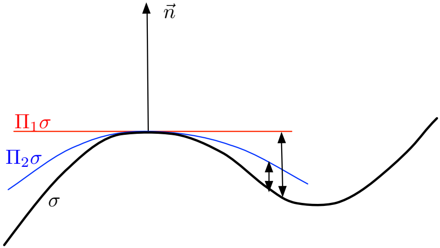
- $\mathcal{V} \subset \mathbb{R}^2$ neighborhood of $(0,0)$
- $\mathcal{U} \subset \mathbb{R}^2$ neighborhood of $(0,0,0)$
- $\xi : \mathcal{V} \subset \mathbb{R}^2 \to \mathbb{R}$ such that $\xi(0,0) = 0$
- $\Sigma \cap \mathcal{U} = \{(u, v, \xi(u,v)), (u,v) \in \mathcal{V}\}$

High order surface mesh generation

Local error model

$$|\xi(u,v) - \Pi_k \xi(u,v)| \leq \left| d^{k+1}\xi(0,0)(u,v) \right| + o\left(\|(u,v)\|_2^{k+1} \right)$$

Geometrical point of view



Log-simplex method $\Rightarrow$ Find $\mathcal{Q}$ such that

$$\left({}^t x\, \mathcal{Q}\, x\right)^{\frac{k+1}{2}} \geq d^{(k+1)}\xi(0,0)(x), \text{ for all } x = (u,v) \in \mathbb{R}^2$$

An example of surface curved mesh

$\mathbb{P}_1$ mesh : 2417 vertices, 4611 triangles

$\mathbb{P}_2$ mesh : 2921 vertices and interpolation nodes, 1408 curved triangles

Conclusions and Perspectives

Achievements

1. Theoretical extension of the Hessian methods for P_k adaptation
2. Numerical implementation for smooth analytical functions
3. Application to the generation of surface curved meshes

Perspectives

1. Treat the discontinuities of the interpolated function
2. Application to high-order resolution of PDE
3. Surface adaptation with several patches

Thank you !