The Physarum Computer

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J. Theoretical Biology

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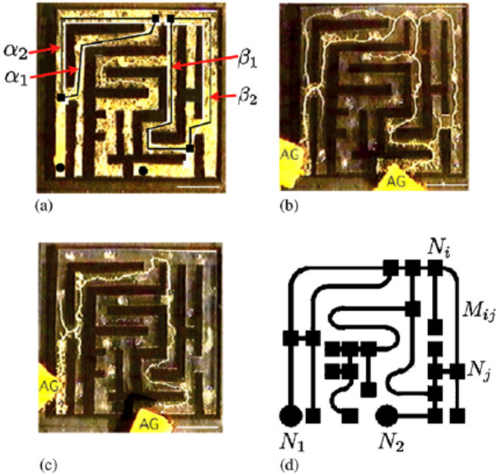


Physarum, a slime mold,

single cell, several nuclei

builds evolving networks

Nakagaki, Yamada, Tóth, Nature 2000

show video

2008 Ig Nobel Prize

For achievements that first make people LAUGH
then make them THINK

COGNITIVE SCIENCE PRIZE: Toshiyuki Nakagaki, Ryo Kobayashi, Atsushi Tero, Ágotá Tóth
for discovering that slime molds can solve puzzles.

REFERENCE: "Intelligence: Maze-Solving by an Amoeboid Organism," Toshiyuki Nakagaki, Hiroyasu Yamada, and Ágota Tóth, Nature, vol. 407, September 2000, p. 470.

Mathematical Model (Tero et al.)

- Physarum is a network of tubes (pipes);
- flow (of liquids and nutrients) through a tube is determined by concentration differences at endpoints of a tube, length of tube, and diameter of tube;
- tubes adapt to the flow through them: if flow through a tube is high (low) relative to diameter of the tube, the tube grows (shrinks) in diameter.
- mathematics is the same as for flows in an electrical network with time-dependent resistors.
- Tero et al., J. of Theoretical Biology, 553 – 564, 2007

Mathematical Model (Tero et al.)

- $G = (V, E)$ undirected graph
- each edge e has a positive length L_e (fixed) and a positive diameter $D_e(t)$ (dynamic)
- send one unit of current (flow) from s_0 to s_1 in an electrical network where resistance of e equals

$$R_e(t) = L_e / D_e(t).$$

- $Q_e(t)$ is resulting flow across e at time t
- Dynamics:

$$\dot{D}_e(t) = \frac{dD_e(t)}{dt} = |Q_e(t)| - D_e(t).$$

The Questions

Does system convergence for all (!!!) initial conditions?

How fast does it converge?

Details of the convergence process?

Beyond shortest paths?

Inspiration for distributed algorithms?

Convergence against Shortest Path

Theorem (Convergence (SODA 12, J. Theoretical Biology))

Dynamics converge against shortest path, i.e.,

- *potential difference between source and sink converges to length of shortest source-sink path,*
- $D_e \to 1$ *for edges on shortest source-sink path,*
- $D_e \to 0$ *for edges not on shortest source sink path*

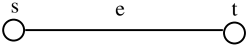
this assumes that shortest path is unique; otherwise ...

Miyaji/Onishi previously proved convergence for planar graphs with source and sink on the same face

Our Approach

- analytical investigation of simple systems, in particular, parallel links
- experimental investigation (computer simulation) of larger systems
  - to form intuition about the dynamics
  - to kill conjectures
  - to support conjectures
- proof attempts for conjectures surviving experiments

A Single Link (Miyaji/Ohnishi)



e has length *L* and diameter *D*

$$Q = 1$$
$$\dot{D} = 1 - D$$
$$D = 1 + (D(0) - 1)e^{-t} \to 1$$

Diameter of *e* converges to 1, resistance of *e* converges to *L*.

Thus, potential difference between source and sink converges to *L* (= length of shortest source-sink path)

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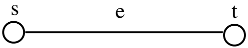

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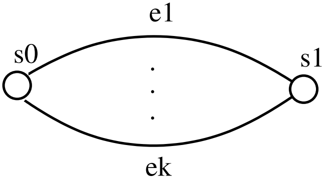
Parallel Links (Miyaji/Ohnishi 07)



parallel links with lengths $L_1 < L_2 < \ldots < L_k$

$D_1 \rightarrow 1$, $D_2, \ldots, D_k \rightarrow 0$

$p_{s_0} - p_{s_1} \rightarrow L_1$

but $D_2, \ldots, D_{k-1}$ do not necessarily converge monotonically

What did Evolution Optimize?

Evolution optimized dynamics so as to achieve a global objective. Which? (Lyapunov Function)

First idea: the energy of the flow $\sum_e Q_e \Delta_e$ decreases over time

not true, even for parallel links

Theorem

For the case of parallel links:

$$\sum_i Q_i L_i, \quad \frac{\sum_i D_i L_i}{\sum_i D_i}, \quad \text{and } (p_s - p_t) \sum_i D_i L_i$$

decrease over time

computer experiment: the obvious generalizations (replace i by e) to general graphs do not work

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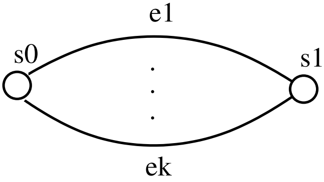
A not so Obvious Generalization



$$\frac{\sum_i D_i L_i}{\sum_i D_i} \quad \Rightarrow \quad \frac{\sum_e D_e L_e}{\text{minimum total diameter of a } s_0\text{-}s_1 \text{ cut}}$$

What did Evolution Optimize?

Computer experiment:

$$V := \frac{\sum_e D_e L_e}{\text{minimum total diameter of a } s_0\text{-}s_1 \text{ cut}} \quad \text{decreases}$$

Theorem (Lyapunov Function)

$$V + \left(\sum_{e \in \delta(\{s_0\})} D_e - 1\right)^2 \quad \textit{decreases.}$$

Derivative of V (essentially) satisfies

$$\dot{V} \le -c \cdot \sum_e (D_e - |Q_e|)^2.$$

Proof uses min-cut-max-flow and ...

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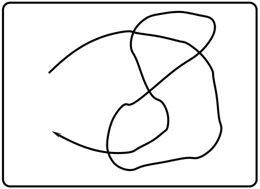
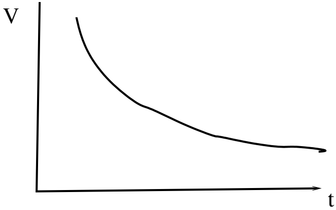
Statespace = R^E





- V decreases and stays positive $\Rightarrow \dot{V} \rightarrow 0$
- $\dot{V} \leq -c \cdot \sum_e (D_e - |Q_e|)^2$
- $|D_e - |Q_e||$ goes to zero for all e
- $Q_e = D_e \Delta_e / L_e$ and hence $\Delta_e \approx L_e$ for $Q_e(t)$ non-vanishing and t large
- $\Delta_{s_0 s_1}$ converges to length of some source-sink path
- $\Delta_{s_0 s_1}$ converges to length of shortest path
- …

Convergence against Shortest Path

Corollary (Convergence)

Dynamics converge against shortest path, i.e.,

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Discretization and Speed of Convergence

$$D_e(t+1) = D_e(t) + h(|Q_e(t)| - D_e(t))$$

Theorem

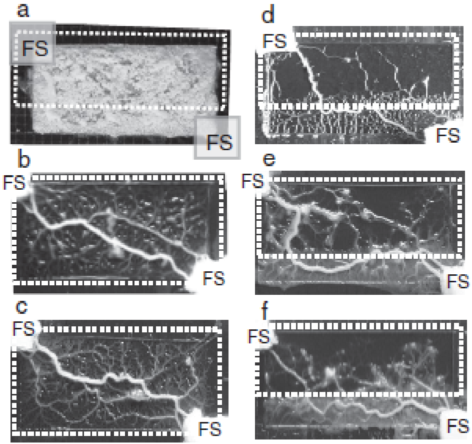
Let *opt* be the length of shortest source-sink path.

Let $\epsilon > 0$ be arbitrary. Set $h = \epsilon/(2mL)$, where L is largest edge length. Assume $L_{P^*} \geq 1$.

After $\widetilde{O}(nmL^2/\epsilon^3)$ iterations, solution is $1+\epsilon$-optimal, i.e., $V = \sum_e L_e D_e$ is at most $(1+\epsilon)opt$.

Arithmetic with $O(\log(nL/\epsilon))$ bits suffices.

Nonuniform Physarum



$$\dot{D}_e(t) = |Q_e(t)| - a_e D_e(t)$$

a_e reactivity of e

No convergence proof

Nonuniform Directed Physarum

$$\dot{D}_e(t) = Q_e(t) - a_e D_e(t)$$

Theorem

Ito/Johansson/Nakagaki/Tero (2011) prove convergence for uniform case ($a_e = 1$ for all e). We generalize their proof

converges to shortest path according to length function $a_e L_e$

discretization converges in $\widetilde{O}(nmL^2/\epsilon^3)$ iterations to $1 + O(\epsilon)$ optimal solution (our proof requires uniformity)

The Transportation Problem

- undirected graph $G = (V, E)$
- $b : V \to \mathbb{R}$ such that $\sum_v b_v = 0$
- v supplies flow b_v if $b_v > 0$
- v extracts flow $|b_v|$ if $b_v < 0$
- find a cheapest flow where cost of sending f units across an edge of length L is $L \cdot f$

Dynamics of Physarum solves transportation problem.

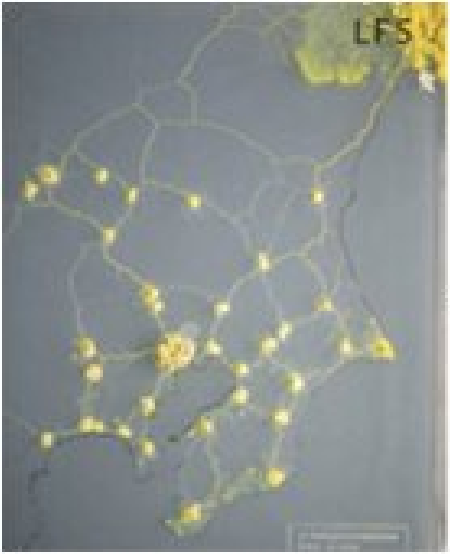
D_e's converge against a mincost solution of transportation problem.

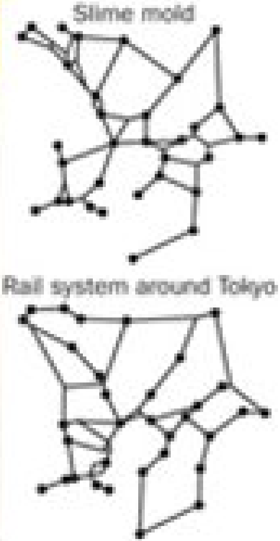
proof requires a non-degeneracy assumption

Open Problems I

- nonuniform Physarum, convergence, discretization, complexity
- nonuniform directed Physarum, discretization, complexity
- dependency on L or $\log L$?
- Physarum apparently can do more, e.g., network design.
- inspiration for the design of distributed algorithms and/or approximation algs for NP-complete problems

Network Design: Science 2010





My Current Projects

Understand the principles of network formation. What does the network optimize?

Nonuniform Versions of Physarum

Can I use Physarum as an inspiration for approximation algorithms?

Thank you for Listening