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On types of growth for graph-different permutations

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ABSTRACT

We consider an infinite graph G whose vertex set is the set of natural numbers and adjacency depends solely on the difference between vertices. We study the largest cardinality of a set of permutations of $[n]$ any pair of which differ somewhere in a pair of adjacent vertices of G and determine it completely in an interesting special case. We give estimates for other cases and compare the results in case of complementary graphs. We also explore the close relationship between our problem and the concept of Shannon capacity "within a given type."

1. Introduction

The topic of our paper has its origins in the following mathematical puzzle of Körner and Malvenuto [4]. Call two permutations of $[n] := \{1, \dots, n\}$ *colliding* if, represented by linear orderings of $[n]$, they put two consecutive elements of $[n]$ somewhere in the same position. For the maximum cardinality $\rho(n)$ of a set of pairwise colliding permutations of $[n]$ the following conjecture was formulated.

Conjecture 1. *(See [4].) For every $n \in \mathbb{N}$*

$$\rho(n) = \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

This conjecture remains open; for the best bounds the interested reader may consult [5] and [1]. In this paper we initiate a systematic study of similar problems for all the graphs on the countable vertex set $\mathbb{N}$. Not only do we believe that these problems are interesting on their own, but beyond

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this we hope that studying them within a unified framework may shed more light also on the initial problem on colliding permutations.

Let G be an infinite graph whose vertex set is the set $\mathbb{N}$ of the natural numbers. We call two permutations of the elements of $[n]$, the first n natural numbers, G-different if they map some $i \in [n]$ to adjacent vertices of G. (We will often think and write about permutations of $[n]$ as n-length sequences that contain each element of $[n]$ exactly once. In this language, two permutations are G-different if there exists a position where the corresponding two sequences contain the two endpoints of an edge of G.) We denote by $T_G(n)$ the maximum cardinality of a set of pairwise G-different permutations of G. The question about pairwise colliding permutations is that special case of our present problem where the graph is the infinite path L defined by

$$V(L) = \mathbb{N} \qquad E(L) = \big\{\{i, i+1\} \,\big|\, i \in \mathbb{N}\big\}.$$

We will concentrate our attention on the special class of *distance graphs*. Given an arbitrary (finite or infinite) set $D \subseteq \mathbb{N}$ we define the graph $G = G(D)$ by setting

$$V\big(G(D)\big) = \mathbb{N}, \qquad E\big(G(D)\big) = \big\{\{i, i+d\} \,\big|\, i \in \mathbb{N},\ d \in D\big\}.$$

Clearly, $L = G(\{1\})$. We will write

$$T(n, D) = T_{G(D)}(n).$$

In the papers [4] and [5] attention was restricted to those cases where the growth of $T(n, D)$ in n is only exponential. Here we consider various speeds of growth and determine the exact value of $T(n, D)$ for every n in a non-trivial case. We are especially interested in the relationship between the values of $T(n, D)$ and $T(n, \overline{D})$, where $\overline{D} = \mathbb{N} \setminus D$.

2. Superexponential growth

The determination of $T(n, \overline{\{1\}})$ leads to a surprisingly simple formula.

Theorem 1.

$$T\big(n, \overline{\{1\}}\big) = \frac{n!}{2^{\lfloor \frac{n}{2} \rfloor}} \quad \textit{for every } n \in \mathbb{N}.$$

Proof. First we prove the upper bound

$$T\big(n, \overline{\{1\}}\big) \leqslant \frac{n!}{2^{\lfloor \frac{n}{2} \rfloor}}.$$

To this end fix n and define $\sigma_{i,j}$ to be the permutation that exchanges the entries $i \in [n]$ and $j \in [n]$, that is, for any permutation π, $\sigma_{i,j}\pi$ differs from π only in the places where the entries i and j stand, which are exchanged. For any fixed π consider the set of permutations

$$C(\pi) := \big\{\sigma_{1,2}^{\varepsilon_{1,2}} \sigma_{3,4}^{\varepsilon_{3,4}} \dots \sigma_{k,k+1}^{\varepsilon_{k,k+1}} \pi \colon\ \forall i\ \varepsilon_{i,i+1} \in \{0,1\}\big\},$$

where k equals $2\lfloor n/2 \rfloor - 1$, $\sigma_{i,j}^0$ is meant to be the identity permutation, while $\sigma_{i,j}^1 := \sigma_{i,j}$. Let B be a set of permutations of $[n]$ satisfying our condition that for any pair of them there is an $i \in [n]$ they map to numbers at distance at least two and observe that the conditions imply $|C(\pi) \cap B| \leqslant 1$, while $C(\pi) \cap C(\pi') = \emptyset$ if $\pi, \pi' \in B$, $\pi \neq \pi'$. Since $|C(\pi)| = 2^{\lfloor n/2 \rfloor}$ for any π, the foregoing implies

$$|B| \leqslant \frac{n!}{2^{\lfloor \frac{n}{2} \rfloor}},$$

which is the claimed upper bound.

In order to prove the inequality in the opposite direction, for every n we shall explicitely construct a set of permutations satisfying the requirement. We start by the odd values of n and build our construction in a recursive manner. It will be important for the recursion that for every odd n

the construction be invariant with respect to cyclic shifts. For $n = 1$ the construction consists of the identical permutation. Suppose next to have constructed

$$t_{n-2} := \frac{(n-2)!}{2^{\lfloor \frac{(n-2)}{2} \rfloor}}$$

permutations yielding a set B_{n-2} that satisfies the pairwise relation we need and has the additional property of being closed with respect to cyclic shifts. We will construct a set A_n of $\frac{n-1}{2}t_{n-2}$ permutations of $[n]$ satisfying the same pairwise condition and define B_n to be the set consisting of all the cyclically shifted versions of the elements of A_n. For an arbitrary permutation π of $[n-2]$ and $1 < j \leqslant n$ we define the transformations Ψ^j in the following manner. The permutation $\Psi^j\pi$ is acting on the set $[n]$,

$$\begin{aligned}
&\Psi^j\pi(1) := n, \\
&\Psi^j\pi(i) := \pi(i-1) \quad \text{for every } 1 < i < j, \\
&\Psi^j\pi(j) := n-1, \\
&\Psi^j\pi(i) := \pi(i-2) \quad \text{for every } j < i \leqslant n.
\end{aligned}$$

In other words, the permutation $\Psi^j\pi$ is obtained from π by prefixing n in the position preceding the first number in π and inserting $n-1$ in the jth position of the resulting permutation. For a set A of permutations we denote by $\Psi^j(A)$ the set of the images by Ψ^j of all the permutations of $[n-2]$ belonging to A. As a last element of notation, let us denote by S^j the set of those permutations τ of $[n]$ for which $\tau^{-1}(n-2) < j$. Consider

$$A^j := \Psi^j(B_{n-2}) \cap S^j$$

and set

$$A_n := \bigcup_{j=2}^{n} A^j.$$

(The attentive reader may note that $A^2 = \emptyset$ but we felt it more natural not to exclude this set from the above union.) As every permutation in A_n has n at its first position no two of them can be cyclic shifts of each other, whence $|B_n| = n|A_n|$. Therefore in order to check that we have constructed the right number of permutations it is sufficient to verify that

$$|A_n| = \frac{n-1}{2}t_{n-2}. \tag{1}$$

To this effect, recall that by our hypothesis the set B_{n-2} is invariant with respect to cyclic shifts. This implies that the number of those of its sequences in which a fixed element, in our case $(n-2)$, is confined to any particular subset of the coordinates is proportionate to the cardinality of the coordinate set in question, and thus

$$\left|A^j\right| = \frac{j-2}{n-2}|B_{n-2}|,$$

whence

$$|A_n| = \sum_{j=2}^{n}\left|A^j\right| = \sum_{j=2}^{n}\frac{j-2}{n-2}|B_{n-2}| = \frac{n-1}{2}t_{n-2},$$

which, substituting the value of t_{n-2}, yields

$$|A_n| = \frac{(n-1)!}{2^{\lfloor \frac{n}{2} \rfloor}}.$$

This settles our claim (1) and proves that B_n has the requested number of permutations.

To conclude the proof it remains to show that every pair of sequences from B_n represents a $G(\overline{\{1\}})$-different pair of permutations. We will first prove that such is the case if both sequences are from A_n. If they belong to the same A^j then this is obvious since the two permutations in such a pair must differ somewhere in those coordinates where they feature an element of B_{n-2} and thus the corresponding elements of B_{n-2} must be different sequences. This implies, by our hypothesis, that they differ in some coordinate by strictly more than 1. If the two sequences, π and τ, do not belong to the same A^j, then we must have, say $\pi \in A^j$ and $\tau \in A^k$ with $j < k$. But then in the kth position $\tau(k) = n-1$, while by definition, $\pi(k) < n-2$, settling this case as well.

If π and τ are two permutations that do not belong to A_n but have the value n in the same position, then they are clearly in a similar relation as their respective cyclic shifts in A_n, thus the above argument still applies.

Finally, we must prove that any two of our sequences having the symbol n in different positions also represent a $G(\overline{\{1\}})$-different pair of permutations. Now, unless the symbol n of both of the two sequences meets the symbol $(n-1)$ of the other one, we are done. Otherwise they have their respective subsequences belonging to B_{n-2} positioned in the very same coordinates and it suffices to see that these subsequences are different. For this purpose suppose that the two sequences have their symbol $(n-1)$ in the jth and the kth position, respectively. But then, supposing $j < k$ we can say that they must have their respective symbols $(n-2)$ in different positions since the one having its $(n-1)$ in the kth position has its $(n-2)$ in a position belonging to the open interval (j,k) while the other one has it in the complement of the closed interval $[j,k]$ by construction. This proves our theorem for every odd n.

In order to prove our claim also for even values of n, it is enough to consider the set A_{n+1} (now $n+1$ is odd) and delete the first entry, which is $(n+1)$, from each of the permutations in this set. This way we get the right number of permutations of $[n]$ and their pairwise relations satisfy the requirement by the previous part of the proof. □

Remark 1. The construction presented in the proof above (in fact, a family of similar constructions that includes it) can also be described in the following short way. Let n be odd and take all those permutations of $[n]$ in which the cyclic order within each of the triples $\{1,2,3\}, \{3,4,5\}, \dots, \{n-2, n-1, n\}$ is the same as in some given fixed permutation; say, the identity. Verifying the relevant properties of the construction is also simple using this description. The case of even n follows the same way as in the above proof.

Remark 2. Consider the graph whose vertex set is the set of permutations of $[n]$ and such that two permutations form an edge if and only if they satisfy the requirement we dealt with in Theorem 1. Denote this graph by $H_{\overline{\{1\}}}(n)$. Observe that its clique number $\omega(H_{\overline{\{1\}}}(n)) = T(n, \overline{\{1\}})$ by definition and notice that by the proof above its chromatic number $\chi(H_{\overline{\{1\}}}(n))$ has the same value. (The sets $C(\pi)$ defined in the proof can serve as color classes of an optimal coloring.) This observation will be used in the proof of the subsequent corollary.

With some additional argument the above theorem gives the exact value of $T(n, \overline{\{q\}})$ also for $q \neq 1$. We will need the following well-known lemma, the roots of which go back to Shannon [9]. We give a short proof for the sake of completeness.

Lemma 1. *Let $G_1, \dots, G_k$ be graphs and let $G_1 \cdot \ldots \cdot G_k$ denote their co-normal product, i.e., the graph with vertex set $V(G_1) \times \cdots \times V(G_k)$ in which two vertices $\boldsymbol{x}, \boldsymbol{y}$ are adjacent if there is an i such that the respective ith entries x_i, y_i of these sequences satisfy $\{x_i, y_i\} \in E(G_i)$. If $\chi(G_i) = \omega(G_i)$ holds for every i, then $\omega(G_1 \cdot \ldots \cdot G_k) = \prod_{i=1}^{k} \omega(G_i)$.*

Proof. It is easy to verify that $\omega(G_1 \cdot \ldots \cdot G_k) \geqslant \prod_{i=1}^{k} \omega(G_i)$ always holds. To prove the reverse inequality observe that $\chi(G_1 \cdot \ldots \cdot G_k) \leqslant \prod_{i=1}^{k} \chi(G_i)$. By $\omega(G_1 \cdot \ldots \cdot G_k) \leqslant \chi(G_1 \cdot \ldots \cdot G_k)$ the conditions $\chi(G_i) = \omega(G_i)$ imply the statement. □

Corollary 1. *Let q be an arbitrary fixed natural number and let n have the form $aq+m$, where $m \in \{0,\dots,q-1\}$. Then*

$$T\big(n,\overline{\{q\}}\big)=\frac{n!}{(2^{\lfloor \frac{a}{2}\rfloor})^{q-m}(2^{\lfloor \frac{a+1}{2}\rfloor})^{m}}.$$

Proof. Let S_n be the set of all permutations of $[n]$ represented as sequences and consider a largest possible set B_n of sequences from S_n which satisfies the requirements for $D=\overline{\{q\}}$. Let $h:\mathbb{N}\to\{0,\dots,q-1\}$ be the residue map modulo q, or, in fact, any map for which $h(k)=h(\ell)$ if and only if q divides $|k-\ell|$. For sequences $\boldsymbol{x}=x_1\dots x_n$ extend h as $h(\boldsymbol{x}):=h(x_1)\dots h(x_n)$. Partition S_n according to the image of h, i.e., put $\boldsymbol{x}$ and $\boldsymbol{y}$ into the same partition class iff $h(\boldsymbol{x})=h(\boldsymbol{y})$. The number of partition classes so obtained is

$$t:=\frac{n!}{(a!)^{q-m}((a+1)!)^{m}}=\binom{n}{a,\dots,a,a+1,\dots,a+1}.$$

We call the classes $W_1,\dots,W_t$. If two sequences $\boldsymbol{x},\boldsymbol{y}$ belong to different W_j's then there must be a position i for which $|x_i-y_i|$ is not divisible by q, in particular, it is not equal to q. Thus $T(n,\overline{\{q\}})$ is just the sum of the maximum possible cardinalities of sets of sequences one can find within each W_j such that each pair of these sequences satisfies the condition.

Fix any class W_j. For each $\boldsymbol{x}\in W_j$ and each position i the value $h(x_i)$ is the same by definition. Let h_i^j denote this common value. For $k\in\{0,\dots,q-1\}$ set $E_k=\{i \mid h_i^j=k\}$. Consider the subsequence of each $\boldsymbol{x}\in W_j$ given by the entries at the positions belonging to E_k. Note that the size of $|E_k|$ is either a or $a+1$. Let H_k be the following graph. Its vertex set consists of $|E_k|$-length sequences of different numbers from $[n]\cap\{\ell \mid h(\ell)=k\}$. Two such sequences $\boldsymbol{x}$ and $\boldsymbol{y}$ are adjacent in H_k iff at some coordinate i we have $|x_i-y_i|\neq q$. It is straightforward that H_k is isomorphic to the graph $H_{\overline{\{1\}}}(|E_k|)$ defined in Remark 2. Whence its clique number is $T(|E_k|,\overline{\{1\}})$, while, by Remark 2, its chromatic number has this same value. Let $\hat{H}_j$ be the graph with vertex set W_j where two vertices are adjacent if they satisfy the requirement that at some position their difference is neither 0 nor q. One easily verifies that $\hat{H}_j$ is isomorphic to the co-normal product (for the definition see Lemma 1) of the graphs $H_0,\dots,H_{q-1}$, which is, by the foregoing, isomorphic to $\prod_{k=0}^{q-1}H_{\overline{\{1\}}}(|E_k|)$. We are interested in the clique number of this graph. By Lemma 1 and Remark 2 this value is equal to $\prod_{k=0}^{q-1}\omega(H_{\overline{\{1\}}}(|E_k|))$. Noticing that $q-m$ of the sets E_k have size a and m of them have size $a+1$, this is further equal to $(a!/2^{\lfloor \frac{a}{2}\rfloor})^{q-m}((a+1)!/2^{\lfloor \frac{a+1}{2}\rfloor})^{m}$ by Theorem 1.

The latter value is the same for all sets W_j and the number of these sets is $\binom{n}{a,\dots,a,a+1,\dots,a+1}$ (with a and $a+1$ appearing $q-m$ and m times, respectively). Thus we have obtained

$$T\big(n,\overline{\{q\}}\big)=\binom{n}{a,\dots,a,a+1,\dots,a+1}\left(\frac{a!}{2^{\lfloor \frac{a}{2}\rfloor}}\right)^{q-m}\left(\frac{(a+1)!}{2^{\lfloor \frac{a+1}{2}\rfloor}}\right)^{m}$$
$$=\frac{n!}{(2^{\lfloor \frac{a}{2}\rfloor})^{q-m}(2^{\lfloor \frac{a+1}{2}\rfloor})^{m}}. \qquad \square$$

3. Graph pairs

It seems interesting to study the relationship of the values of $T(n,D)$ for pairs of disjoint sets (graphs) and their union, especially in case of pairs of complementary sets.

Let us define

$$\phi(D,\overline{D}):=\limsup_{n\to\infty}\frac{1}{n}\log\frac{T(n,D)T(n,\overline{D})}{n!}$$

and call it the *split strength* of the partition $\{D,\overline{D}\}$ of the natural numbers. (All logarithms in the paper are of base 2.) Consider the case $D:=\{1\}$. We know from [5] that

$$10^{\frac{n-4}{4}}\leqslant T\big(n,\{1\}\big)\leqslant 2^{n}.$$

(We do not need the sharper form of the upper bound here. For an exponential improvement in the above lower bound the reader is invited to consult Brightwell and Fairthorne [1].) Using this in combination with Theorem 1 yields

Proposition 1.

$$0.33 < \phi\bigl(\{1\}, \overline{\{1\}}\bigr) \leqslant \frac{1}{2}.$$

We continue with other examples. Denoting by $2\mathbb{N}$ the set of the even numbers, we would like to determine $\phi(2\mathbb{N}, \overline{2\mathbb{N}})$. To this end, notice first that

$$T(n, \overline{2\mathbb{N}}) = \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

In fact, this easily follows, as in [4], by observing that two permutations differ in every position by an even number if and only if the even numbers occupy the same set of positions in both.

Somewhat surprisingly, $T(n, 2\mathbb{N})$ seems hard to determine and we only have some easy bounds.

Proposition 2.

$$\frac{n!(\lceil \frac{n}{2} \rceil + 1)}{2\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leqslant T(n, 2\mathbb{N}) \leqslant \frac{n!}{2^{\lfloor \frac{n}{2} \rfloor}}.$$

Proof. The upper bound is a trivial consequence of (the upper bound part of) Theorem 1. Although the lower bound follows from the lower bound on $\kappa(K_n)$ in [5], yet for the reader's convenience we give the details without explicit reference to said paper. (Those needing more details may however consult [5].) We consider the set $[n]$ as the disjoint union of its respective subsets of odd and even numbers. Correspondingly, we divide the coordinate set in two (with a little twist). In the first $\lceil \frac{n}{2} \rceil + 1$ coordinates we write the even permutations of the set A consisting of all the odd numbers from $[n]$ with the addition of the extra symbol $\star$. (More precisely, first we represent these $\lceil \frac{n}{2} \rceil + 1$ many symbols bijectively by the first natural numbers up to their cardinality, then extend this bijection to the permutations of both sets and consider only those permutations of the elements of A that correspond to the even permutations of the first $|A|$ natural numbers). We represent an arbitrary permutation of A in form of a sequence $\boldsymbol{x}$ and similarly let $\boldsymbol{y}$ be an arbitrary permutation of the set B of the even elements of $[n]$. We will say that $\boldsymbol{y}$ is *hooked up to* $\boldsymbol{x}$ if we replace the $\star$ in $\boldsymbol{x}$ by the first coordinate of $\boldsymbol{y}$ and concatenate the rest of $\boldsymbol{y}$ as a suffix to the resulting sequence. Let us denote by $\boldsymbol{x} \leftharpoondown \boldsymbol{y}$ the permutation of $[n]$ so obtained. Define $A \leftharpoondown B$ to be the set of all these permutations as $\boldsymbol{x}$ and $\boldsymbol{y}$ take all of their possible values. Clearly,

$$|A \leftharpoondown B| = \frac{1}{2}\left(\left\lceil \frac{n}{2} \right\rceil + 1\right)! \left\lfloor \frac{n}{2} \right\rfloor !$$

which in turn equals the claimed lower bound in the statement of the proposition. It is very easy to see on the other hand that all the pairs of permutations from $A \leftharpoondown B$ differ by an even number in some coordinate. $\square$

Corollary 2.

$$0 \leqslant \phi(2\mathbb{N}, \overline{2\mathbb{N}}) \leqslant \frac{1}{2}.$$

Next we quickly review the following immediate consequence of our hitherto results on split strength.

Proposition 3. *Let q be an arbitrary but fixed natural number. Then $\phi(\{q\}, \overline{\{q\}})$ is independent of the actual value of q.*

Proof. We prove more, namely that the asymptotics of $T(n,\{q\})$ is independent of the value of q we fix and the same is true for $T(n,\overline{\{q\}})$. For the latter it follows immediately from the formula given in Corollary 1.

Now we turn to $T(n,\{q\})$. Consider the distance graph $G(\{q\})$ of the set $\{q\}$ and look at the graph it induces on $[n]$. Since the latter is isomorphic to a subgraph of P_n, the path on n vertices that the analogous distance graph $G(\{1\})$ induces on the same set, we immediately see that

$$T\big(n,\{q\}\big) \leqslant T\big(n,\{1\}\big). \tag{2}$$

In the reverse direction, we just have to observe that, for every $m \in \{0,1,\dots,q-1\}$, the graph $G(\{q\})$ induces an infinite path on the residue class $q\mathbb{N}+m$ of the numbers congruent to m modulo q. This implies

$$T\big(n,\{q\}\big) \geqslant \prod_{m=0}^{q-1} T\left(\left\lfloor \frac{n-m}{q} \right\rfloor, \{1\}\right) \tag{3}$$

by concatenating the respective constructions of permutations for each fixed m. Whence it is immediate that $T(n,\{q\})$ and $T(n,\{1\})$ have the same exponential growth rate. □

We know very little about split strength and thus there are many questions to ask. Is it always true that $\phi(D,\overline{D})$ is finite and non-negative as it seems by these examples? In order to see the greater picture, we have to look at different kind of growth rates as well.

4. Intermediate growth

So far we have only seen growth rates at an exponential factor away from either 1 or $n!$. We intend to show here, however, that in between growth rates are also possible. In particular, we will see that $T(n,D)$ and $T(n,\overline{D})$ can have essentially the same growth rate, while their product is still about $n!$.

Let $ex(n)$ denote the largest exponent s for which 2^s is a divisor of n. We define

$$E := \big\{n \bigm| n \in \mathbb{N},\ ex(n) \equiv 0 \pmod 2\big\}. \tag{4}$$

Theorem 2. *If n is a power of 4, then we have*

(a) $$(\sqrt{n})!^{\sqrt{n}} \leqslant T(n,E) \leqslant \frac{n!}{(\sqrt{n})!^{\sqrt{n}}},$$

(b) $$(\sqrt{n})!^{\sqrt{n}} \leqslant T(n,\bar{E}) \leqslant \frac{n!}{(\sqrt{n})!^{\sqrt{n}}}.$$

Proof. We prove the lower bound part of (a) first. It will be convenient to consider the elements of $[n]$ as binary sequences of length $t := \lceil \log n \rceil$, with each natural number from $[n]$ represented by its *binary expansion*. (Integer parts could be deleted by our assumption on n, moreover, we also know that t is an even number.) In fact, instead of permuting the n integers in $\{1,\dots,n\}$, now we will permute the n numbers in $\{0,\dots,n-1\}$. With a shift by 1, the two are obviously equivalent for our purposes. For simplicity, we will index the coordinates of the binary expansions from right to left. Hence in particular m is odd if in its binary expansion $\boldsymbol{x} = x_t x_{t-1} \dots x_1$ the rightmost coordinate x_1 is 1 and even else. Let further $\boldsymbol{x}^{odd}$ and $\boldsymbol{x}^{even}$ denote the subsequence of the odd and the even indexed coordinates of $\boldsymbol{x}$, respectively. Finally, let $\nu(\boldsymbol{x})$ be the smallest (i.e., rightmost) index i for which $x_i = 1$. By a slight abuse of notation we will consider the various subsets of $\{0,\dots,n-1\}$ as subsets of $\{0,1\}^t$. Quite clearly, for every $\boldsymbol{x} \in \{0,1\}^t$ we have

$$\nu(\boldsymbol{x}) = ex(\boldsymbol{x}) + 1$$

and, in particular, $\boldsymbol{x} \in E$ if and only if $\nu(\boldsymbol{x}) \equiv 1$ modulo 2. In order to prove the lower bound, let us consider the partition induced on $\{0,1\}^t$ (i.e., on $\{0,\dots,n-1\}$) by the mapping $f : \{0,1\}^t \to \{0,1\}^{\frac{t}{2}}$ where

$$f(\boldsymbol{x}) := \boldsymbol{x}^{even} \quad \text{for every } \boldsymbol{x} \in \{0,1\}^t.$$

(The classes of the partition are the full inverse images corresponding to the various values of f.) It follows by construction that

$$f(\boldsymbol{x}) = f(\boldsymbol{y}) \quad \text{implies} \quad |\boldsymbol{x} - \boldsymbol{y}| \in E \tag{5}$$

where by the difference of the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ we mean the difference in ordinary arithmetics of the natural numbers they represent. Indeed, executing the subtraction in the binary number system we are using here one sees that both $\boldsymbol{x} - \boldsymbol{y}$ and $\boldsymbol{y} - \boldsymbol{x}$ have their rightmost 1 in the position where, scanning the binary expansions of $\boldsymbol{x}$ and $\boldsymbol{y}$ from right to left, we find the first position in which they differ. Now, since $\boldsymbol{x}^{even} = \boldsymbol{y}^{even}$ by assumption, the position in question must have an odd index. In other words, $\nu(|\boldsymbol{x} - \boldsymbol{y}|) \equiv 1$ modulo 2. For every $\boldsymbol{z} \in \{0,1\}^{\frac{t}{2}}$ we denote by $S(\boldsymbol{z})$ the set of all the permutations of the elements of the full inverse image $f^{-1}(\boldsymbol{z})$ of $\boldsymbol{z}$. Thus, by our previous argument, all these permutations are pairwise $G(E)$-different. Consider the Cartesian product

$$C := \prod_{\boldsymbol{z} \in \{0,1\}^{\frac{t}{2}}} S(\boldsymbol{z}). \tag{6}$$

Note that the elements of C are permutations of the numbers in $\{0, \dots, n-1\}$. The above consideration implies that C is a set of pairwise $G(E)$-different permutations. Further, observing that for every $\boldsymbol{z} \in \{0,1\}^{\frac{t}{2}}$

$$\left|f^{-1}(\boldsymbol{z})\right| = 2^{\frac{t}{2}}$$

we have

$$|C| = \left(2^{\frac{t}{2}}\right)!^{2^{\frac{t}{2}}} \tag{7}$$

proving the lower bound in (a). (One might get a somewhat larger set by using the hookup operation instead of straightforward direct product but we do not intend to increase the complexity of the presentation for this relatively small gain here.)

Next we prove the upper bound part of (b). Notice that the set C we have constructed above has a stronger property than needed so far. In fact, in every coordinate, the absolute difference of our permutations is either 0 or else it belongs to E. This observation will be the basis for our upper bound.

Consider the auxiliary graph $H_{\overline{E}}$ the vertices of which are the permutations of $[n]$ and two are adjacent if they satisfy the requirement that at some position they have two numbers such that their difference is in $\overline{E}$. Clearly, $T(n, \overline{E}) = \omega(H_{\overline{E}})$, the clique number of this graph by definition. The above observation about C implies that its independence number $\alpha(H_{\overline{E}})$ is at least $|C|$. Note that $H_{\overline{E}}$ is vertex transitive (any permuting of the coordinates in the vertices gives an automorphism), thus by a well-known fact (cf., e.g., in [8]) its fractional chromatic number $\chi_f(H_{\overline{E}})$ is equal to the ratio of the number of vertices and the independence number. Using also that the clique number cannot acceed the fractional chromatic number (cf. [8]) we obtained

$$T(n, \overline{E}) = \omega(H_{\overline{E}}) \leqslant \chi_f(H_{\overline{E}}) = \frac{|V(H_{\overline{E}})|}{\alpha(H_{\overline{E}})} \leqslant \frac{n!}{|C|} = \frac{n!}{(2^{\frac{t}{2}})!^{2^{\frac{t}{2}}}}$$

proving the upper bound in part (b).

Exchanging the role of even and odd above we obtain the lower bound in (b) and the upper bound in (a) in a similar way. □

Theorem 2 shows that the investigated values of $T(n, E)$ and $T(n, \overline{E})$ have essentially the same growth rate at around $\sqrt{n!}$. The following statement is a straightforward generalization of the above.

Theorem 3. *For every rational number $\alpha \in (0,1)$, there is a set $E_\alpha \subseteq \mathbb{N}$ such that for infinitely many values of n we have*

$$\left(n^{\alpha}\right)!^{n^{1-\alpha}} \leqslant T(n, E_\alpha) \leqslant \frac{n!}{(n^{1-\alpha})!^{n^{\alpha}}}$$

and

$$\left(n^{1-\alpha}\right)!^{n^{\alpha}} \leqslant T(n, \overline{E_\alpha}) \leqslant \frac{n!}{(n^{\alpha})!^{n^{1-\alpha}}}.$$

Remark 3. Notice that taking logarithm and using the estimate $\log(k!) \approx k \log k$ the above inequalities give that $\log T(n, E_\alpha)$ is about $\alpha \log(n!)$, while $\log T(n, \overline{E}_\alpha)$ is about $(1-\alpha)\log(n!)$.

Proof. Let

$$E_\alpha := \{m \mid m \in \mathbb{N},\ ex(m) \equiv 0, 1, \ldots, p-1 \pmod q\},$$

where $\alpha = p/q$ and suppose n is a power of 2^q. The reasoning is essentially the same as in Theorem 2, which is the case $\alpha = 1/2$, $q = 2$. Instead of $[n]$ we again permute the elements of $\{0, \ldots, n-1\}$ and represent each of these numbers by their binary expansion. We collect into one group those numbers of $\{0, \ldots, n-1\}$ whose binary expansion has the very same subsequence in those positions which are indexed by numbers congruent to $p+1, \ldots, q$ modulo q. There are $n^{\frac{q-p}{q}} = n^{1-\alpha}$ different such groups each containing n^α different numbers. Permuting the numbers within a group we get $(n^\alpha)!$ permutations of those numbers and these are bound to differ at some position by the difference of two different numbers in the group. Such a value belongs to E_α by construction. Putting all permutations of all our groups together we obtain $(n^\alpha)!^{n^{1-\alpha}}$ different permutations altogether that not only satisfy the requirements given by the set E_α but no two of which satisfy the requirements prescribed by $\overline{E}_\alpha$. This gives the upper bound for $T(n, \overline{E}_\alpha)$ in a similar way as the upper bound on $T(n, \overline{E})$ is proven in Theorem 2. The rest is also similar to what we have seen there. □

5. Exponential growth and Shannon capacity

In this section we return to the more familiar territory of distance graphs with finite chromatic number. The relevance of this parameter is shown by the following simple observation.

Proposition 4. *Let G be an infinite graph with finite chromatic number $\chi(G)$. Then*

$$T_G(n) \leqslant \left(\chi(G)\right)^n.$$

Proof. Let $c : V(G) \to [\chi(G)]$ be an optimal coloring of the vertices of G and let $c_n : V(G)^n \to [\chi(G)]^n$ be its usual extension to sequences. Notice that none of the full inverse images c_n^{-1} of the elements of $[\chi(G)]^n$ can contain two pairwise G-different permutations of $[n]$. □

In particular, distance graphs of "rare" sets of distances have finite chromatic number. More precisely, by a result of Ruzsa, Tuza and Voigt [7], if the set $D := \{d_1, d_2, \ldots, d_n, \ldots\}$ has the density of a geometric progression in the sense that $\liminf_{n\to\infty} \frac{d_{n+1}}{d_n} > 1$, then the distance graph $G(D)$ has finite chromatic number. Clearly, this density condition is sufficient but not necessary for the chromatic number to be finite (cf., e.g., the set of odd numbers as differences that result in a bipartite graph).

However, for some graphs G with finite chromatic number one can give a better upper bound on $T_G(n)$. This bound is easily obtained once we realize the tight connection of our present problem with the classical concept of Shannon capacity of a graph [9].

Given a sequence $\boldsymbol{x} \in V^n$ we shall denote by $P_{\boldsymbol{x}}$ the probability distribution on the elements of V defined by

$$P_{\boldsymbol{x}}(a) = \frac{1}{n}\left|\{i \mid x_i = a,\ i = 1, 2, \ldots, n\}\right|$$

for every $a \in V$; here $\boldsymbol{x} = x_1 \ldots x_n$. The probability distribution $P_{\boldsymbol{x}}$ is called the *type* of $\boldsymbol{x}$. Let $V^n(P, \varepsilon)$ denote the set of those $\boldsymbol{x} \in V^n$ for which

$$|P_{\boldsymbol{x}} - P| = \max_{a \in V}\left|P_{\boldsymbol{x}}(a) - P(a)\right| \leqslant \varepsilon.$$

Let G be a finite graph with vertex set $V = V(G)$ and edge set $E(G)$. As always, we will say that the sequences $\boldsymbol{x} = x_1 x_2 \ldots x_n \in V^n$ and $\boldsymbol{y} = y_1 y_2 \ldots y_n \in V^n$ are G-different if there is at least one index $i \in [n]$ with $\{x_i, y_i\} \in E(G)$. Let $\omega(G, n)$ and $\omega(G, P, \varepsilon, n)$ be the largest cardinality of any set $C \subseteq V^n$ and $C' \subset V^n(P, \varepsilon)$, respectively, of pairwise G-different sequences. The Shannon capacity of G (or of $\overline{G}$ in the more usual notational convention, cf. [9]) can be defined as

$$C(G) = \limsup_{n\to\infty} \frac{1}{n} \log \omega(G, n),$$

while the *capacity* $C(G, P)$ *of* G *within the type* P is given (cf. [2]) as

$$C(G, P) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \omega(G, P, \varepsilon, n).$$

It is immediate from the definitions that $C(G, P) \leqslant C(G)$ always holds and using the methods of [3] it is easy to prove that in fact $C(G) = \max_P C(G, P)$.

In what follows we will restrict attention to graphs we call *residue graph*s. We say that an infinite graph G with vertex set $\mathbb{N}$ is a residue graph if there exists a natural number r and a finite graph $M = M(G)$ with vertex set $\{0, 1, \ldots, r-1\}$ such that

$$\{a, b\} \in E(G) \quad \text{if and only if} \quad \bigl\{(a)_{\text{mod } r}, (b)_{\text{mod } r}\bigr\} \in E(M).$$

Let Q be the uniform distribution on $\{0, 1, \ldots, r-1\}$. We have

Theorem 4.

$$\lim_{n\to\infty} \frac{1}{n} \log T_G(n) = C\bigl(M(G), Q\bigr).$$

Proof. To prove

$$\limsup_{n\to\infty} \frac{1}{n} \log T_G(n) \geqslant C\bigl(M(G), Q\bigr)$$

consider, for every n those sequences $\boldsymbol{x} \in \{0, 1, \ldots, r-1\}^n$ whose type Q_n satisfies

$$Q_n(a) = \frac{1}{n} \bigl|\bigl\{m \bigm| m \leqslant n,\ (m)_{\text{mod } r} = a\bigr\}\bigr|$$

for every $a \in \{0, 1, \ldots, r-1\}$. Let M^n be the graph whose vertices are the n-length sequences of vertices of M and whose vertices are adjacent if the corresponding sequences are M-different. For every n let C_n denote a complete subgraph of maximum cardinality M^n induces on the set of sequences of type Q_n. Notice that $\lim_{n\to\infty} \frac{1}{n} \log |C_n| = C(M(G, Q))$, because any sequence in $[V(M)]^n(Q, \varepsilon)$ can be extended to one of type $Q_{n'}$, where $n \leqslant n' \leqslant n + 2|V(M)|\varepsilon n$, i.e., by adding at most $2|V(M)|\varepsilon n$ new coordinates. This shows $|C_{n'}| \geqslant \omega(G, Q_n, \varepsilon, n)$ and since $|V(M)|$ is constant and ε can be chosen arbitrarily small, it proves that $\frac{1}{n} \log |C_n|$ tends to $C(M(G, Q))$ as n goes to infinity. Now for any fixed n and to any sequence $\boldsymbol{x} \in V(C_n)$ we associate a permutation of $[n]$ by replacing the occurrences of a in the sequence by the different numbers congruent to a modulo r, in a strictly increasing order. The result is a set of permutations which is G-different and has cardinality $|C_n|$. With the observation above this proves the desired inequality.

For the reverse inequality let n be a multiple of r and consider any construction achieving $T_G(n)$, i.e., a set of permutations of the elements of $[n]$ that are pairwise G-different, while the cardinality of the set is $T_G(n)$. Substitute the occurrence of the number i in each of these permutations by the unique $j \in \{0, \ldots, r-1\}$ which is congruent to it modulo r. Doing this for all $i \in [n]$ we get $T_G(n)$ different sequences of vertices of M each having type Q that actually form a clique in M^n. □

The following corollary is immediate.

Corollary 3.

$$\lim_{n\to\infty} \frac{1}{n} \log T_G(n) \leqslant C\bigl(M(G)\bigr).$$

Let us consider the following

Example. Let G have vertex set $\mathbb{N}$ and set

$$\{a,b\} \in E(G) \quad \text{if } |a-b| \equiv 1 \text{ or } 4 \pmod 5.$$

As an easy consequence of Lovász' celebrated formula [6] for the Shannon capacity of the pentagon graph we obtain, using the last theorem (and also that the Shannon capacity of C_5 is obtained by sequences the type of which is the uniform distribution), that

$$\lim_{n\to\infty} \frac{1}{n} \log T_G(n) = \frac{1}{2} \log 5.$$

It is an easy observation that for any graph M and any rational probability distribution P on its vertex set one can construct (by simply substituting each vertex by an independent set of appropriate size) a graph $\hat{M}$ for which $C(M,P) = C(\hat{M},Q)$, where Q is again the uniform distribution. It is then easy to construct an infinite graph G, which is a residue graph with respect to $\hat{M}$ and thus the asymptotics of $T_G(n)$ is in an analogous relationship with the capacity $C(M,P)$ as the one expressed in Theorem 4. Taking into account the remark that $C(M)$ can be expressed as the maximum of the values $C(M,P)$ over P, we can conclude that the class of problems asking for the asymptotics of $T_G(n)$ for various infinite graphs G contains the Shannon capacity problem of all such graphs for which Shannon capacity is attained as the capacity within a type for some rational distribution.

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