Joint PHD Thesis at University Côte d'Azur and Copenhagen university

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Lie group action, approximate invariance and Sub-Riemannian geometry in statistics

At the interface of geometry, statistics, image analysis and medicine, computational anatomy aims at analyzing and modeling the biological variability of the organs shapes and their dynamics at the population level. The goal is to model the mean anatomy, its normal variation, its motion / evolution and to discover morphological differences between normal and pathological groups. Since shapes and deformations live in non-linear spaces, this requires a consistent statistical framework on manifolds and Lie groups, which has motivated the development of Geometric Statistics during the last decade (Pennec 2006, Pennec, Sommer, Fletcher 2019). To consolidate the mathematical bases of geometric statistics, it is now essential to explore geometric structures beyond the classical Riemannian framework.

In computational anatomy, the deformable template theory considers statistics on groups of diffeomorphisms. These groups are endowed with a right-invariant metric in the large deformation diffeomorphic metric mapping (LDDMM) framework of Grenander, Miller, Trouvé and Younes. An alternative framework is based on the canonical symmetric bi-invariant Cartan-Schouten connection (Pennec and Arsigny 2012). This connection corresponds to the canonical symmetric structure of the Lie group and allows defining some affine statistical notions like the mean. In both Riemannian and Lie theoretic cases, we aim at establishing embedded spaces of approximations (flags) that iteratively allow the refined description of the data. The goals of this Phd is to explore how sub-Riemannian and stratified Lie algebra structures can be aligned with statistical subspace decompositions.

When a Lie group acts on a manifold, quotienting to obtain the shape space amounts to assuming that objects are completely equivalent under the action of all transformations. More realistically, we would like objects to be approximately equivalent under transformations weighted by a certain probability distribution that is not uniform. For instance, in the example of computational anatomy where diffeomorphisms act on images, the full invariance leads to consider only the topology of the image iso-levels. However, very large deformations are much less probable than smaller ones, and extremely deformed images may actually represent different anatomies even if they share the same topology. Thus, we need to restrict to a smaller part of the group, or to penalize more the large deformations that are much less probable than smaller ones.

A probability on deformations independent from the data is classically taken to be proportional to $\exp(|v|^2/2\ )$ where |v| is a norm on the Lie algebra of the group. This standard regularization criterion in image registration focuses on the most probable deformations and drastically reduces the search space while avoiding unrealistic local minima. Computing the optimal transformation between each pair of objects (pairwise registration) and taking the set of initial tangent vectors of these deformations gives a sample of the probability distribution in the Lie algebra. Thus, it seems natural to perform Principal Component Analysis (PCA) in order to reduce the distribution support to a small dimensional subspace $\Delta \in I_{id}G$. PCA also gives a sub-Riemannian metric on this subspace. It is noticeable that the spatial discretization of velocity fields which is done in practice to implement diffeomorphic transformations also lead to restrict to a subspace $\Delta \in I_{id}G$ of the full Lie algebra.

The composition of several elements from Δ is performed to obtain the transformation when we consider a discrete number of steps to integrate the trajectory. In such a sub-Riemannian geometric setting, the horizontal trajectories (that stay tangent to the distribution) actually reach a much higher dimensional space (Agrachev et al 2016): because linear subspaces are generically not closed under the Lie bracket, the k-jet of a horizontal curve (its Taylor expansion up to degree k) belongs to the flag of the distribution Δ defined recursively by $\Delta_{\mathrm{k}+1} = \Delta_k + [\Delta, \Delta_k]$ with $\Delta = \ \Delta_1$. The goal is to investigate how this flag interacts with the flag defined by PCA (Pennec 2018) or by the one defined by the discretization, so that statistics and discretization become consistent with this new structure. This structure will also be related to stochastic processes in the frame bundle proposed by (Sommer 2016, Sommer & Svane 2017, Sommer 2019) to model anisotropic covariances on manifolds

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