

Class 1: Lagrange mechanics and first steps into numerical integration

Florence Bertails-Descoubes ¹, Thibaut Métivet ², Mélina Skouras ³



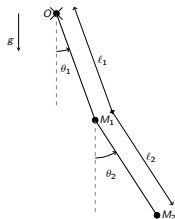
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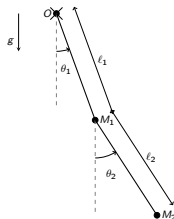
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Example: the double pendulum



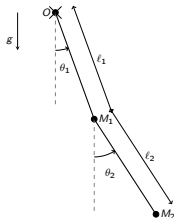
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Questions

- How to formulate the equations of motion of this object?
- How to solve these equations in a “safe” way ? 

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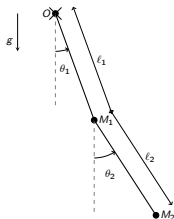


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Goal of this lecture: understand there are many possible choices and learn best practices

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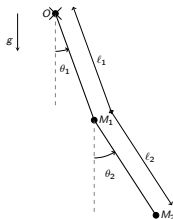
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Outline of the class

- I. Lagrange mechanics: how to **formulate** the equations of motion
- II. Finite differences: how to **solve** the equations of motion

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NB: Once these concepts are known, more complex systems can be considered



Keywords and bibliography

Part I: Lagrange mechanics

- **Analytical Mechanics**

Landau and Lifshitz, Mechanics Vol 1; J. Ferreira, Mécanique analytique

- **Calculus of Variations**

J.-P. Bourguignon, Calcul variationnel ("Variational calculus")

Part II: Finite differences

- **Numerical Analysis**

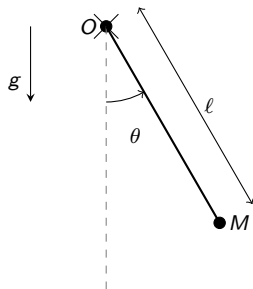
G. Allaire, Analyse numérique et optimisation ("Numerical analysis and optimization")

- **Energy conservation properties**

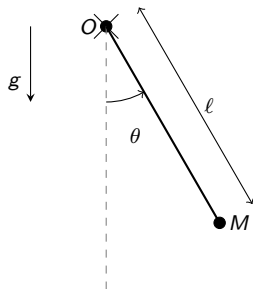
E. Hairer et al., Geometric numerical integration

Part I: Lagrange mechanics

The simple pendulum



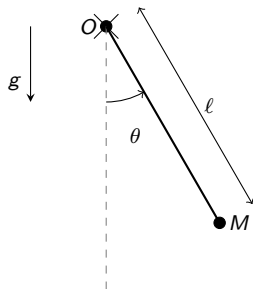
The simple pendulum



Exercise

- Write the equations of motion of the simple pendulum...

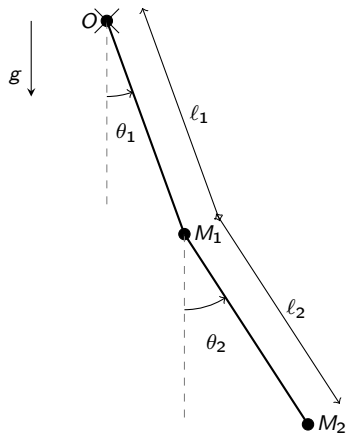
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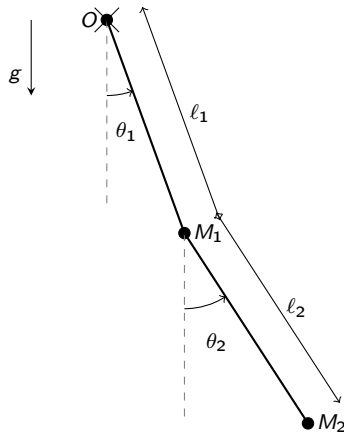
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- Write the equations of motion of the simple pendulum...
- ...using Newton's second law

The double pendulum



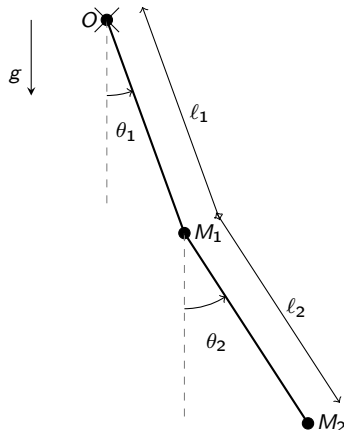
The double pendulum



Exercise

- Write the equations of motion of the double pendulum...

The double pendulum



Exercise

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Generalized coordinates

Definition

Generalized coordinates, denoted q_i , are n independent variables (functions of time) which allow to characterize the configuration of a system possessing n degrees of freedom. The **generalized velocities** of the system are defined by $\frac{d}{dt}q_i = \dot{q}_i$.

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Example: For the simple pendulum, the position of the mass M can be computed as $OM = \ell \mathbf{e}_r$ and its velocity v_M as $v_M = \ell \dot{\theta} \mathbf{e}_\theta$ (in the basis $(\mathbf{e}_r, \mathbf{e}_\theta)$ defined by θ).

Equations of motion

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At instant t , having both the $q_i(t)$ and the $\dot{q}_i(t)$ is necessary, and also sufficient, to determine the accelerations $\ddot{q}_i(t)$ of the system at t , and thus predict the trajectory $q_i(t_+)$ forward in time, $t_+ > t$.

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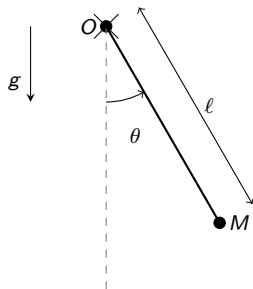
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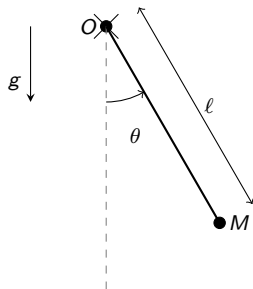
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- Principle of **least action**: more general settings, beyond Newtonian dynamics!

→ Both principles lead to the so-called **Euler-Lagrange** equations
which take a specific form in Newtonian dynamics

Back to the simple pendulum



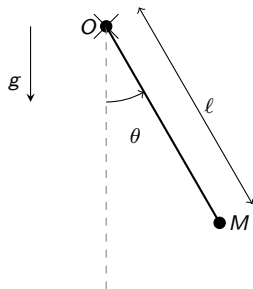
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Exercise

- Write the equations of motion of the simple pendulum...

Back to the simple pendulum



Exercise

- Write the equations of motion of the simple pendulum...
- ...using the principle of virtual work

General method: Least action principle

Let $q = \{q_1, \dots, q_i, \dots, q_n\}$ and $\dot{q} = \frac{d}{dt}q$.

We consider a system subject to **holonomic** constraints (the q_i are independent) and **conservative** forces. Let T be the **kinetic energy** of the system, and U its **potential energy**.

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Principle of least action (Hamilton principle)

The actual trajectory $q(t)$ followed by the system between two instants a and $b > a$ should be such that the **action** of the system,

$$S(q, \dot{q}) = \int_a^b \underbrace{\mathcal{L}(q(t), \dot{q}(t), t)}_{T-U} dt,$$

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$$\mathcal{L} : \begin{array}{l|l} \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} & \longrightarrow \mathbb{R} \\ (x, u, t) & \longmapsto \mathcal{L}(x, u, t), \end{array}$$

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Remark: S is a *functional*, as it takes as arguments the two functions q and $\dot{q}$.

Euler-Lagrange equations

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Theorem

A necessary condition for the action S to be minimal is that q satisfies for all $t \in [a, b]$ the so-called Euler-Lagrange equations,

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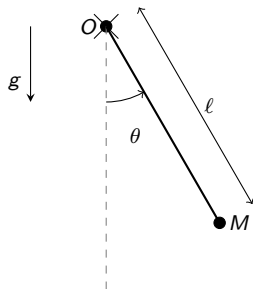
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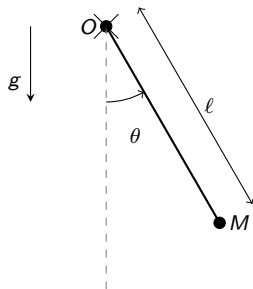
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Remark: This is a **variational** principle (minimum condition on a functional). It can be applied beyond dynamics, for instance to find object shapes with minimal weight, or to compute shapes at static equilibrium.

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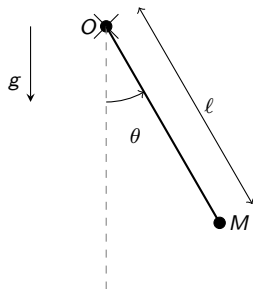
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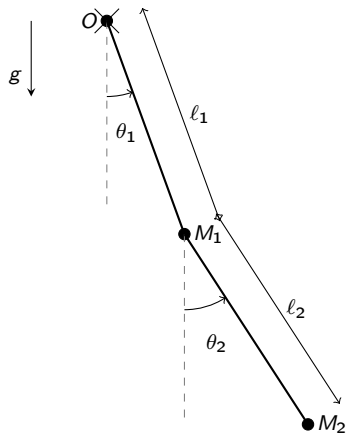
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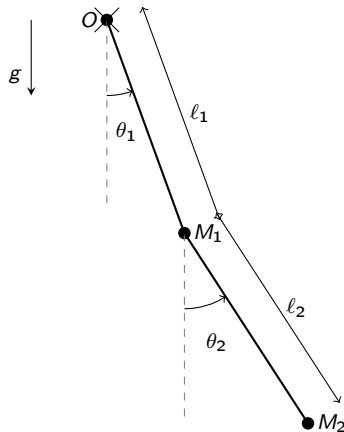
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- ...using the Euler-Lagrange formalism

The double pendulum



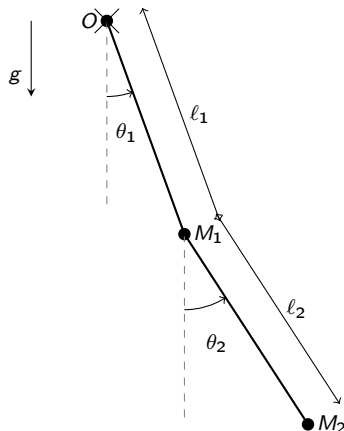
The double pendulum



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- Write the equations of motion of the double pendulum...

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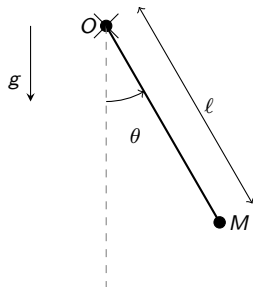


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- ...using the Euler-Lagrange formalism

Part II: Finite differences

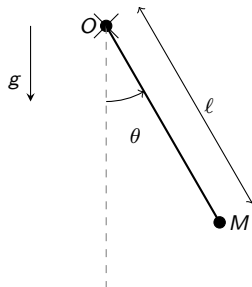
Back to the simple pendulum



Computing the dynamics

$$\ddot{\theta} + \frac{g}{\ell} \sin \theta = 0 \quad \text{with } \theta(0) = \theta_0 \text{ and } \dot{\theta}(0) = \lambda_0$$

Back to the simple pendulum

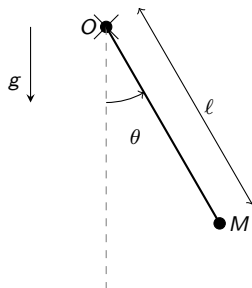


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- Nonlinear equation, no explicit solution

Back to the simple pendulum



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- Nonlinear equation, no explicit solution
- $\rightarrow$ Recourse to **numerical integration**

Cauchy problem

We consider the following first-order differential equation with initial value,

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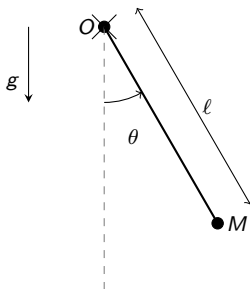
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where f is a continuous function from $\mathbb{R}^p \times \mathbb{R}$ to $\mathbb{R}^p$, $p \geq 1$. Such a problem belongs to the class of **Cauchy** problems. The **unknown** of the problem is the function x from $\mathbb{R}$ to $\mathbb{R}^p$.

Cauchy problem

Exercise

Show that the simple pendulum equation of motion enters the formalism above.



Equation of motion (reminder)

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Existence and uniqueness of a solution

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A function $g : \mathbb{R}^p \rightarrow \mathbb{R}^p$ is **Lipschitz continuous** if there exists a constant $K \geq 0$ such that

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Exercise

- Show that a Lipschitz continuous function is necessarily continuous.
- Is the opposite true?

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If f is locally Lipschitz continuous with respect to its first (state) variable $x(t)$, then there **exists** a **unique** solution $\bar{x}$ to the Cauchy problem, and $\bar{x}$ is C^1 continuous.

Numerical scheme

Recall that the Cauchy problem $\begin{cases} \dot{x} = f(x(t), t) & t \in [a, b] \\ x(a) = x_0 \end{cases}$

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→ The greater m , the faster the method converges to the exact solution.

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A **single-step numerical scheme** is defined as the recurrence

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Remark: Convergence is generally hard to prove directly. Instead, one usually prefers to prove **consistency** + **stability w.r.t. errors** instead.

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Recall that $x_{k+1} = x_k + h\Phi(x_k, h, t_k)$, i.e. $\frac{x_{k+1} - x_k}{h} - \Phi(x_k, t_k, h) = 0$.

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Consistency of a numerical scheme

Exercise

Show that Explicit Euler is a **consistent** scheme of **first** order.

Reminder: Explicit Euler: $x_{k+1} = x_k + hf(x_k, t_k)$

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$$\forall h, \forall \varepsilon_k < \bar{\varepsilon}, \quad \sup_k \|x_k - \tilde{x}_k\| < M \sup_k \|\varepsilon_k\| \leq M \bar{\varepsilon}$$

Stability with respect to errors

We consider that at each time step, the computation of x_k may be altered by a **perturbation** ε_k on the increment, i.e., instead of computing the exact sequence x_k as previously, we compute $\tilde{x}_k$ such that

$$\tilde{x}_0 = x_0 + \varepsilon_0 \quad \tilde{x}_{k+1} = \tilde{x}_k + h(\Phi(\tilde{x}_k, t_k, h) + \varepsilon_{k+1}).$$

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- **Easy criterion:** stability w.r.t. errors is guaranteed when Φ satisfies some regularity properties: typically, when Φ is **Lipschitz continuous** w.r.t. x .

Convergence

Theorem

Convergence = consistency + stability

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Exercise

Show that Explicit Euler is a **convergent** scheme.

Other type of stability: Stability of the numerical solution (A-stability)

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Exercise

Derive the Explicit Euler scheme for the simple scalar Cauchy problem

$$x(a) = x_0 \quad \dot{x}(t) = -\lambda x(t) \quad \forall t \in [a, b] \quad \text{with } \lambda > 0.$$

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→ To keep in mind: stability w.r.t. errors is useful for proving **convergence**. Stability as defined above is useful when considering integration of systems on **moderate or large time intervals**, or when using a **large timestep** (which is often useful in practice!).

Explicit vs. Implicit Euler

Exercise 1

Consider the linearized pendulum problem (valid for small angle θ),

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0 \quad \text{with } \theta(0) = \theta_0 \text{ and } \dot{\theta}(0) = \lambda_0,$$

and express the condition on the time step h for Explicit Euler to be stable.

Explicit vs. Implicit Euler

Exercise 2

Same question for Implicit Euler.

Explicit vs. Implicit Euler

In the general case:

Stability of Explicit Euler

Explicit Euler is **conditionally** stable.

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In a nutshell

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- References: G. Allaire, Analyse numérique et optimisation (“Numerical analysis and optimization”), E. Hairer et al., Geometric numerical integration.