# Class 1: Lagrange mechanics and first steps into numerical integration

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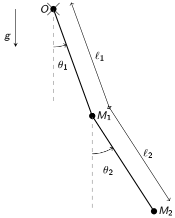
# Example: the double pendulum



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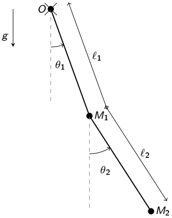


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- How to solve these equations in a “safe” way ?

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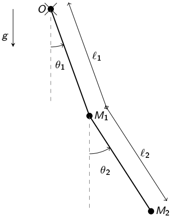


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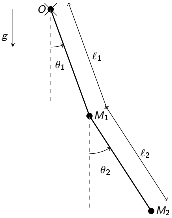


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- II. Finite differences: how to solve the equations of motion

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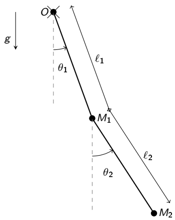


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NB: Once these concepts are known, more complex systems can be considered

# Keywords and bibliography

## Part I: Lagrange mechanics

- Analytical Mechanics

  Landau and Lifshitz, Mechanics Vol 1; J. Ferreira, Mécanique analytique

- Calculus of Variations

  J.-P. Bourguignon, Calcul variationnel ("Variational calculus")

## Part II: Finite differences

- Numerical Analysis

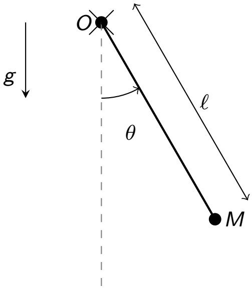
  G. Allaire, Analyse numérique et optimisation ("Numerical analysis and optimization")

- Energy conservation properties

  E. Hairer et al., Geometric numerical integration

# Part I: Lagrange mechanics

# The simple pendulum



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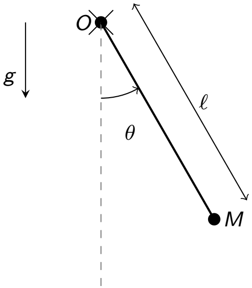


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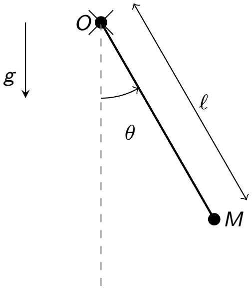


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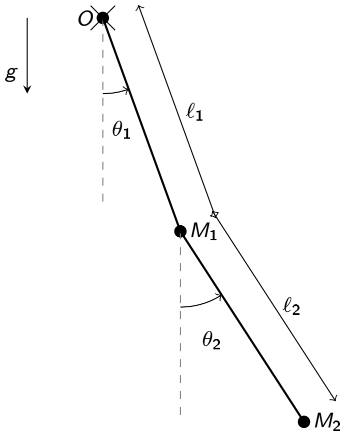

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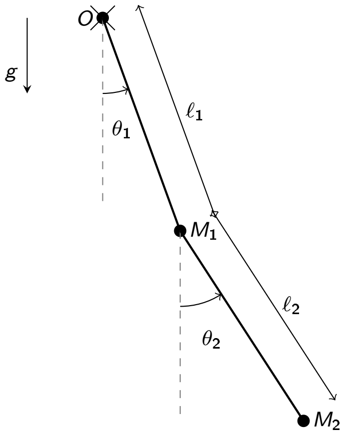


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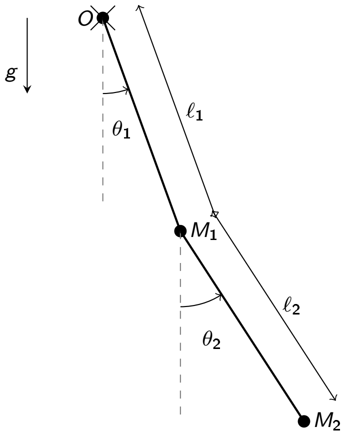


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# Generalized coordinates

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Generalized coordinates, denoted $q_i$, are $n$ independent variables (functions of time) which allow to characterize the configuration of a system possessing $n$ degrees of freedom. The generalized velocities of the system are defined by $\frac{\mathrm{d}}{\mathrm{d}t} q_i = \dot{q}_i$.

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*Example:* For the simple pendulum, the position of the mass $M$ can be computed as $\mathrm{OM} = \ell\,\mathrm{e}_r$ and its velocity $\mathrm{v}_M$ as $\mathrm{v}_M = \ell\dot{\theta}\,\mathrm{e}_\theta$ (in the basis $(\mathrm{e}_r, \mathrm{e}_\theta)$ defined by $\theta$).

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At instant $t$, having both the $q_i(t)$ and the $\dot{q}_i(t)$ is necessary, and also sufficient, to determine the accelerations $\ddot{q}_i(t)$ of the system at $t$, and thus predict the trajectory $q_i(t_+)$ forward in time, $t_+ > t$.

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→ Both principles lead to the so-called Euler-Lagrange equations

which take a specific form in Newtonian dynamics

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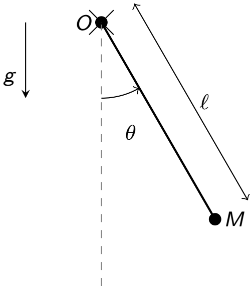

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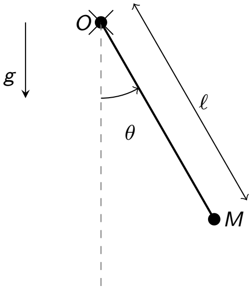


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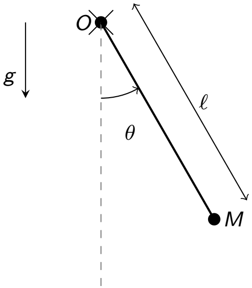


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## General method: Least action principle

Let $q = \{q_0, \ldots, q_i, \ldots, q_{n-1}\}$ and $\dot{q} = \frac{\mathrm{d}}{\mathrm{d}t} q$.
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## Principle of least action (Hamilton principle)

The actual trajectory $q(t)$ followed by the system between two instants $a$ and $b > a$ should be such that the action of the system,

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*Remark:* $\mathcal{S}$ is a *functional*, as it takes as arguments the two functions $q$ and $\dot{q}$.

# Euler-Lagrange equations

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## Theorem

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*Remark:* This is a variational principle (minimum condition on a functional). It can be applied beyond dynamics, for instance to find object shapes with minimal weight, or to compute shapes at static equilibrium.

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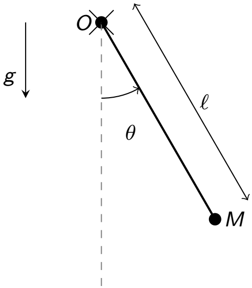

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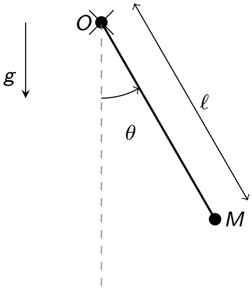


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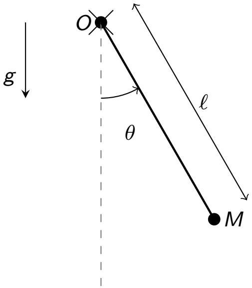


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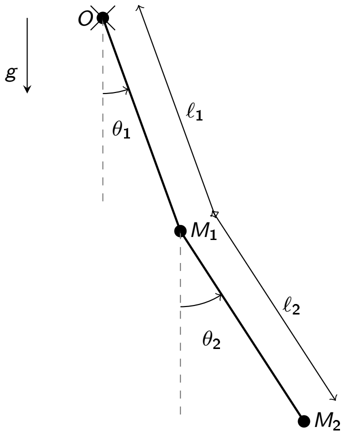

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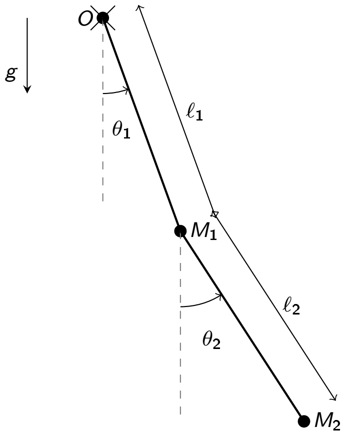
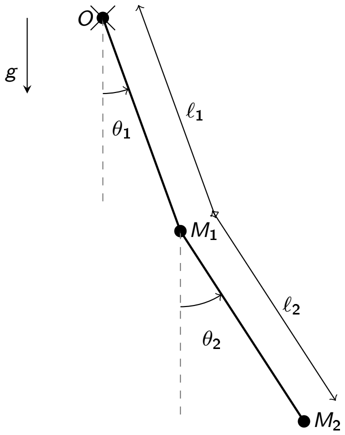



## Exercise

- Write the equations of motion of the double pendulum...

# The double pendulum

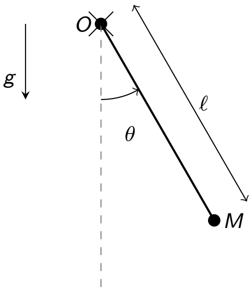



## Exercise

- Write the equations of motion of the double pendulum...
- ...using the Euler-Lagrange formalism

# Part II: Finite differences

# Back to the simple pendulum




## Computing the dynamics

$$\ddot{\theta} + \frac{g}{\ell} \sin \theta = 0 \qquad \text{with } \theta(0) = \theta_0 \text{ and } \dot{\theta}(0) = \lambda_0$$

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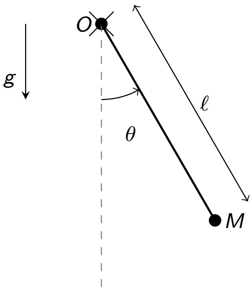


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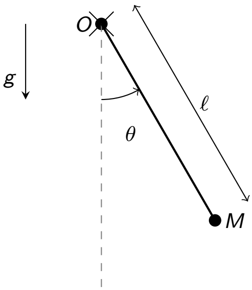


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- Nonlinear equation, no explicit solution
- $\rightarrow$ Recourse to numerical integration

## Cauchy problem

We consider the following first-order differential equation with initial value,

$$\begin{cases} \dot{x} = f(x(t), t) & t \in [a, b] \\ x(a) = x_0 \end{cases}$$

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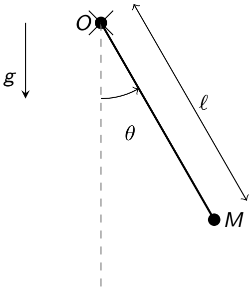
where $f$ is a continuous function from $\mathbb{R}^p \times \mathbb{R}$ to $\mathbb{R}^p$, $p \geq 1$. Such a problem belongs to the class of Cauchy problems. The unknown of the problem is the function $x$ from $\mathbb{R}$ to $\mathbb{R}^p$.

# Cauchy problem

## Exercise

Show that the simple pendulum equation of motion enters the formalism above.




## Equation of motion (reminder)

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# Existence and uniqueness of a solution

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A function $g : \mathbb{R}^p \to \mathbb{R}^p$ is Lipschitz continuous if there exists a constant $K \geq 0$ such that

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### Exercise

- Show that a Lipschitz continuous function is necessarily continuous.
- Is the opposite true?

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If $f$ is locally Lipschitz continuous with respect to its first (state) variable $x(t)$, then there exists a unique solution $\bar{x}$ to the Cauchy problem, and $\bar{x}$ is $C^1$ continuous.

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Recall that the Cauchy problem $\begin{cases} \dot{x} = f(x(t), t) & t \in [a, b] \\ x(a) = x_0 \end{cases}$

admits the unique solution $\bar{x}(t)$ for $t \in [a, b]$ (under some regularity assumptions on $f$). In general, this solution has no explicit form, so we search for a numerical approximation of $\bar{x}(t)$.

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*Remark:* Convergence is generally hard to prove directly. Instead, one usually prefers to prove consistency + stability w.r.t. errors instead.

## Consistency of a numerical scheme

Recall that $x_{k+1} = x_k + h\Phi(x_k, h, t_k)$, i.e. $\frac{x_{k+1} - x_k}{h} - \Phi(x_k, t_k, h) = 0$.

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# Consistency of a numerical scheme

## Exercise

Show that `Explicit Euler` is a consistent scheme of first order.

Reminder: `Explicit Euler`: $x_{k+1} = x_k + hf(x_k, t_k)$

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$$\tilde{x}_0 = x_0 + \varepsilon_0 \qquad \tilde{x}_{k+1} = \tilde{x}_k + h\left(\Phi(\tilde{x}_k, t_k, h) + \varepsilon_{k+1}\right).$$

### Definition

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*Remark:*

- Means that a perturbation on the initial condition and on the increment $\Phi$ only yields a bounded perturbation on the numerical scheme, and so the scheme does not amplify errors.
- Easy criterion: stability w.r.t. errors is guaranteed when $\Phi$ satisfies some regularity properties: typically, when $\Phi$ is Lipschitz continuous w.r.t. $x$.

# Convergence

**Theorem**

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*Convergence = consistency + stability*

**Exercise**

Show that `Explicit Euler` is a convergent scheme.

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## Exercise

Derive the `Explicit Euler` scheme for the simple scalar Cauchy problem

$$x(a) = x_0 \quad \dot{x}(t) = -\lambda\, x(t) \quad \forall t \in [a, b] \quad \text{with } \lambda > 0.$$

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$\rightarrow$ To keep in mind: stability w.r.t. errors is useful for proving convergence. Stability as defined above is useful when considering integration of systems on moderate or large time intervals, or when using a large timestep (which is often useful in practice!).

# Explicit vs. Implicit Euler

## Exercise 1

Consider the linearized pendulum problem (valid for small angle $\theta$),

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0 \qquad \text{with } \theta(0) = \theta_0 \text{ and } \dot{\theta}(0) = \lambda_0,$$

and express the condition on the time step $h$ for `Explicit Euler` to be stable.

# Explicit vs. Implicit Euler

**Exercise 2**

Same question for `Implicit Euler`.

# Explicit vs. Implicit Euler

In the general case:

**Stability of `Explicit Euler`**

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- References: G. Allaire, Analyse numérique et optimisation ("Numerical analysis and optimization"), E. Hairer et al., Geometric numerical integration.