Class 1: Lagrange mechanics
and first steps into numerical integration

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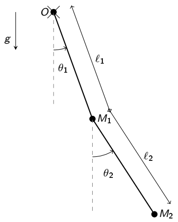
Example: the double pendulum



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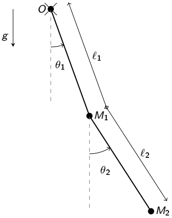

Questions

- How to formulate the equations of motion of this object?
- How to solve these equations in a “safe” way ?

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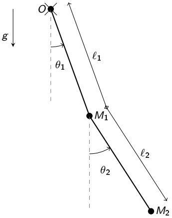

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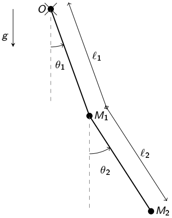

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Outline of the class

- I. Lagrange mechanics: how to formulate the equations of motion
- II. Finite differences: how to solve the equations of motion

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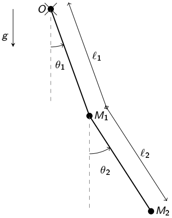

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NB: Once these concepts are known, more complex systems can be considered

Keywords and bibliography

Part I: Lagrange mechanics

- Analytical Mechanics

  Landau and Lifshitz, Mechanics Vol 1; J. Fereira, Mécanique analytique

- Calculus of Variations

  J.-P. Bourguignon, Calcul variationnel ("Variational calculus")

Part II: Finite differences

- Numerical Analysis

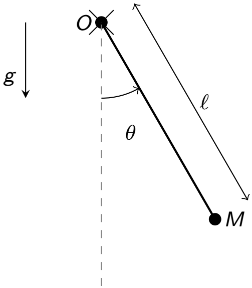
  G. Allaire, Analyse numérique et optimisation ("Numerical analysis and optimization")

- Energy conservation properties

  E. Hairer et al., Geometric numerical integration

Part I: Lagrange mechanics

The simple pendulum



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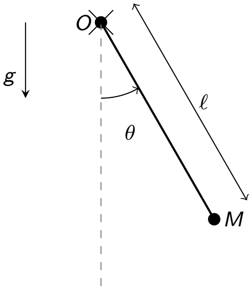

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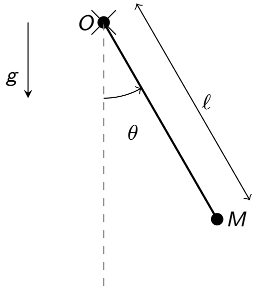

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- ...using Newton's second law

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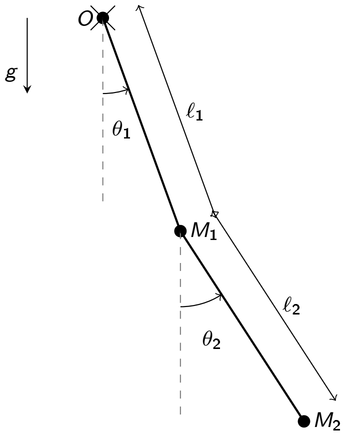

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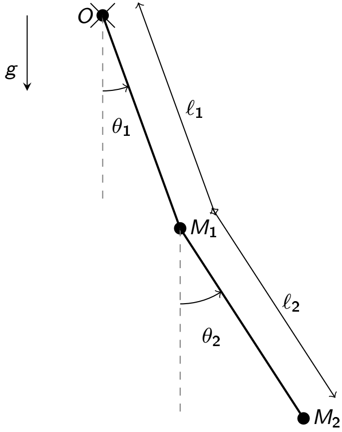

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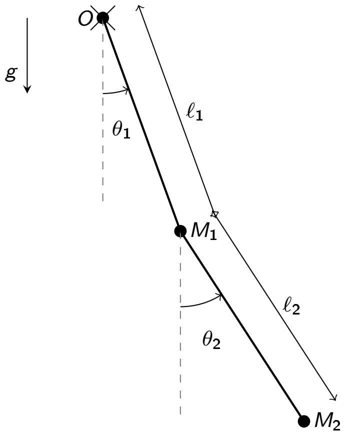

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Generalized coordinates, denoted q_i, are n independent variables (functions of time) which allow to characterize the configuration of a system possessing n degrees of freedom. The generalized velocities of the system are defined by $\frac{\mathrm{d}}{\mathrm{d}t} q_i = \dot{q}_i$.

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Example: For the simple pendulum, the position of the mass M can be computed as $\mathrm{OM} = \ell\, \mathrm{e}_r$ and its velocity v_M as $\mathrm{v}_M = \ell\dot{\theta}\, \mathrm{e}_\theta$ (in the basis $(\mathrm{e}_r, \mathrm{e}_\theta)$ defined by θ).

Equations of motion

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which take a specific form in Newtonian dynamics

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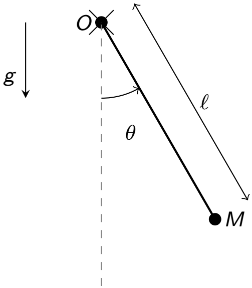

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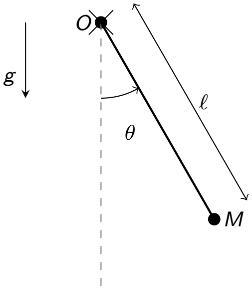

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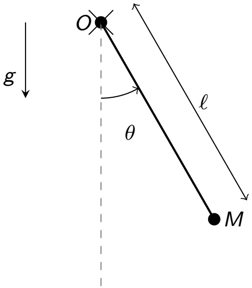

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General method: Least action principle

Let $q = \{q_0, \ldots, q_i, \ldots, q_{n-1}\}$ and $\dot{q} = \frac{\mathrm{d}}{\mathrm{d}t} q$.
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Principle of least action (Hamilton principle)

The actual trajectory $q(t)$ followed by the system between two instants a and $b > a$ should be such that the action of the system,

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Remark: $\mathcal{S}$ is a *functional*, as it takes as arguments the two functions q and $\dot{q}$.

Euler-Lagrange equations

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A necessary condition for the action $\mathcal{S}$ to be minimal is that q satisfies for all $t \in [a, b]$ the so-called Euler-Lagrange equations,

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Remark: This is a variational principle (minimum condition on a functional). It can be applied beyond dynamics, for instance to find object shapes with minimal weight, or to compute shapes at static equilibrium.

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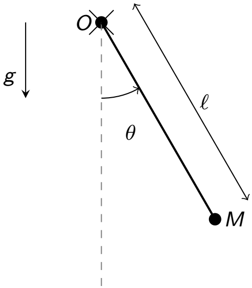

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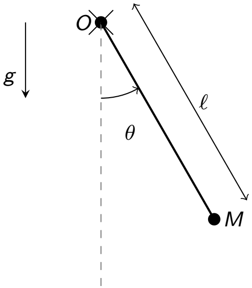

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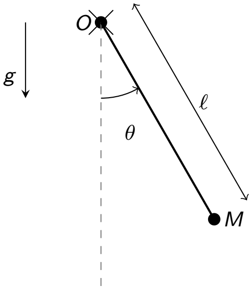

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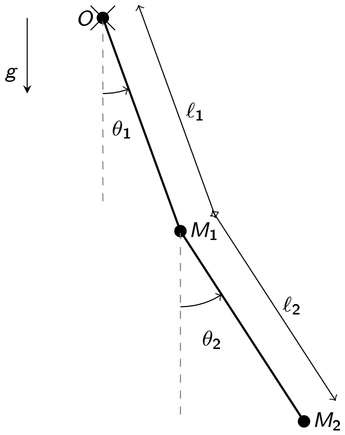

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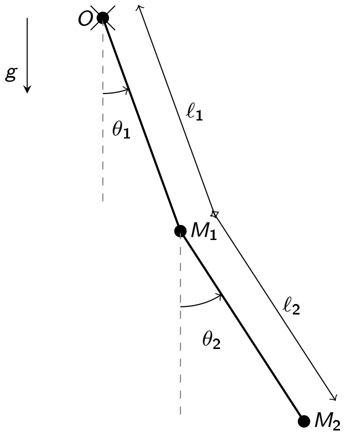

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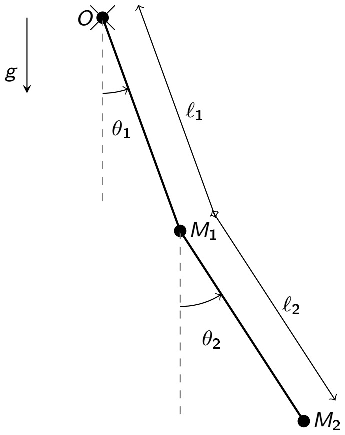

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Part II: Finite differences

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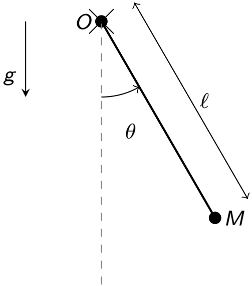

Computing the dynamics

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0 \qquad \text{with } \theta(0) = \theta_0 \text{ and } \dot{\theta}(0) = \lambda_0$$

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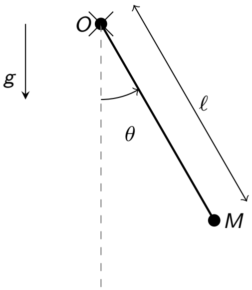

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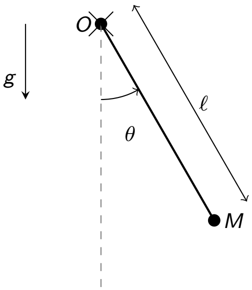

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- Nonlinear equation, no explicit solution
- $\rightarrow$ Recourse to numerical integration

Cauchy problem

We consider the following first-order differential equation with initial value,

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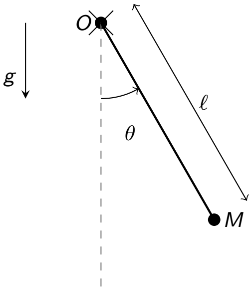
where f is a continuous function from $\mathbb{R}^p \times \mathbb{R}$ to $\mathbb{R}^p$, $p \geq 1$. Such a problem belongs to the class of Cauchy problems. The unknown of the problem is the function x from $\mathbb{R}$ to $\mathbb{R}^p$.

Cauchy problem

Exercise

Show that the simple pendulum equation of motion enters the formalism above.



Equation of motion (reminder)

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Existence and uniqueness of a solution

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A function $g : \mathbb{R}^p \to \mathbb{R}^p$ is Lipschitz continuous if there exists a constant $K \geq 0$ such that

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Exercise

- Show that a Lipschitz continuous function is necessarily continuous.
- Is the opposite true?

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If f is locally Lipschitz continuous with respect to its first (state) variable $x(t)$, then there exists a unique solution $\bar{x}$ to the Cauchy problem, and $\bar{x}$ is C^1 continuous.

Numerical scheme

Recall that the Cauchy problem $\begin{cases} \dot{x} = f(x(t), t) & t \in [a, b] \\ x(a) = x_0 \end{cases}$

admits the unique solution $\bar{x}(t)$ for $t \in [a, b]$ (under some regularity assumptions on f). In general, this solution has no explicit form, so we search for a numerical approximation of $\bar{x}(t)$.

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→ The greater m, the faster the method converges to the exact solution.

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Remark: Convergence is generally hard to prove directly. Instead, one usually prefers to prove consistency + stability w.r.t. errors instead.

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Consistency of a numerical scheme

Exercise

Show that `Explicit Euler` is a consistent scheme of first order.

Reminder: `Explicit Euler`: $x_{k+1} = x_k + hf(x_k, t_k)$

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We consider that at each time step, the computation of x_k may be altered by a perturbation ε_k on the increment, i.e., instead of computing the exact sequence x_k as previously, we compute $\tilde{x}_k$ such that

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- Means that a perturbation on the initial condition and on the increment Φ only yields a bounded perturbation on the numerical scheme, and so the scheme does not amplify errors.
- Easy criterion: stability w.r.t. errors is guaranteed when Φ satisfies some regularity properties: typically, when Φ is Lipschitz continuous w.r.t. x.

Convergence

Theorem

Convergence = consistency + stability

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Exercise

Show that `Explicit Euler` is a convergent scheme.

Other type of stability: Stability of the numerical solution (A-stability)

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Exercise

Derive the `Explicit Euler` scheme for the simple scalar Cauchy problem

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$\rightarrow$ To keep in mind: stability w.r.t. errors is useful for proving convergence. Stability as defined above is useful when considering integration of systems on moderate or large time intervals, or when using a large timestep (which is often useful in practice!).

Explicit vs. Implicit Euler

Exercise 1

Consider the linearized pendulum problem (valid for small angle θ),

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0 \qquad \text{with } \theta(0) = \theta_0 \text{ and } \dot{\theta}(0) = \lambda_0,$$

and express the condition on the time step h for `Explicit Euler` to be stable.

Explicit vs. Implicit Euler

Exercise 2

Same question for `Implicit Euler`.

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- References: G. Allaire, Analyse numérique et optimisation ("Numerical analysis and optimization"), E. Hairer et al., Geometric numerical integration.