Practical 2 (part 1) - Elasticity & Finite elements

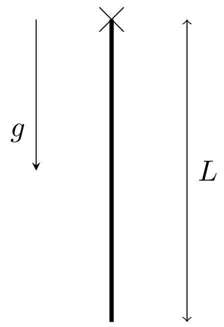
Florence Bertails-Descoubes, Mélina Skouras, Mickaël Ly

Part 2 of the practical on September 29, report due on **October 5 at 20h**.

Exercise 1 - 1D Linear elasticity

In this exercise, we are going to study the dynamics of a 1D elastic body assuming linear elasticity.

Let us consider an elastic rod of length L, cross-section area A and mass density ρ. It is fixed at one of its extremities and subjected to gravity. We assume that the behaviour of the rod is described by the linear elasticity model.



Remark 1: The equation of motion for a linear elastic material is not very different from the static equation you saw during class :

$$\begin{aligned} \nabla \cdot \sigma + b &= \rho \ddot{u} \\ \text{(With indices) } \forall i : \quad \sum_j \frac{\partial \sigma_{ij}}{\partial x_j} + b_i &= \rho \ddot{u}_i \end{aligned}$$

with $\ddot{\bullet}$ denoting the second time derivative, ρ the mass density, u the displacement field, σ the Cauchy stress tensor and b the body force density.

(Q1) Write the equation of motion for the 1D case.

Remark 2: (Reminder) Stress/strain relation for an isotropic material, using Cauchy strain:

$$\begin{aligned} \sigma &= 2\mu\epsilon + \lambda tr(\epsilon)\mathbb{I} \\ \epsilon &= \frac{1}{2}(\nabla u + \nabla u^{\intercal}) \end{aligned}$$

with λ and μ the *Lamé's coefficients.*
If you want to play with real values, you will find more easily tables for the *Young Modulus E* (or Y) and the *Poisson's ratio ν.*

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \mu = \frac{E}{2(1+\nu)}$$

To be continued next week.

Remark 3 - Finite elements:
When solving an PDE, most of the time the unknown is a function of time and space:

$$\begin{array}{ccc} \mathbb{R} \times \Omega \subset \mathbb{R}^n & \to & \mathbb{R}^m \\ t, x & \to & u(t,x) \end{array}$$

Last time, we used the *finite difference method* to discretize the equations in *time*. They can also be applied to discretize the equations in *space*, but most of the time, it is not straightforward.

The *finite element method* takes another approach, and *discretizes the function space*. Instead of looking for the function u in a space of infinite dimension, we are looking for an approximation of u in a *finite* subspace defined by p basis functions $\phi_i(x)$:

$$u(x,t) \approx \sum_{i=1}^{p} u_i(t)\phi_i(x),$$

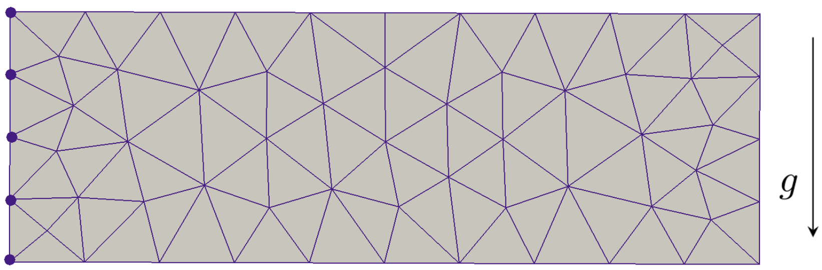
i.e., we are looking for the p functions $u_i(t)$ that approximate u, and which can then be discretized in time.

Exercise 2 - 2D Linear elasticity

In this exercise, we will study the behaviour of a 2D solid, assuming linear elasticity, that we will simulate using the software `FreeFEM` [1]. You will implement that in 1D in the second part of the practical to better understand how it works.

So, attached to this sheet, you can find a file called `linearElasticity2D.edp` [2], implementing the scenario described below. To run it, use the command `FreeFem++ linearElasticity2D.edp`. The goal is to modify this file to study the behaviour of the linear elasticity.

Here, we will focus on the static equilibrium of a cantilever beam.[3] The considered scenario consists of a 2D beam, fixed on one side, and submitted to gravity. We assume the material to be linearly elastic and isotropic, and we are interested in the static equilibrium shape.



Remark 4: You may not have all the keys yet to understand and explain the results you will get now. You should write your comments for the final report after next week class on the finite elements.

[1] `https://freefem.org`
[2] Mostly inspired by `https://doc.freefem.org/tutorials/elasticity.html`
[3] For the `FreeFEM` code for the dynamic case: `https://www.ljll.math.upmc.fr/~hecht/ftp/ff++/2016-Fields/FH-Lesson5-8-Fields.pdf`

(Q1) Starting from a high stiffness (Young modulus of the steel for example), decrease it and compare the different behaviours. What do you observe ?

(Q2) What is happening when the beam tends towards the rod (thickness towards 0) ?

(Q3) The scenario is configured to use what we call $\mathbb{P}^1$ elements.
Compare their behaviour (precision, convergence, computation time...) with other elements.
Line 19, replace `P1` by `P0`, `P2` or `P3`.

Remark 5: To study the convergence, as we do not have an analytic solution, you can take as the reference a very refined mesh. A code to evaluate the difference between two displacement field is at the end of the attached file.